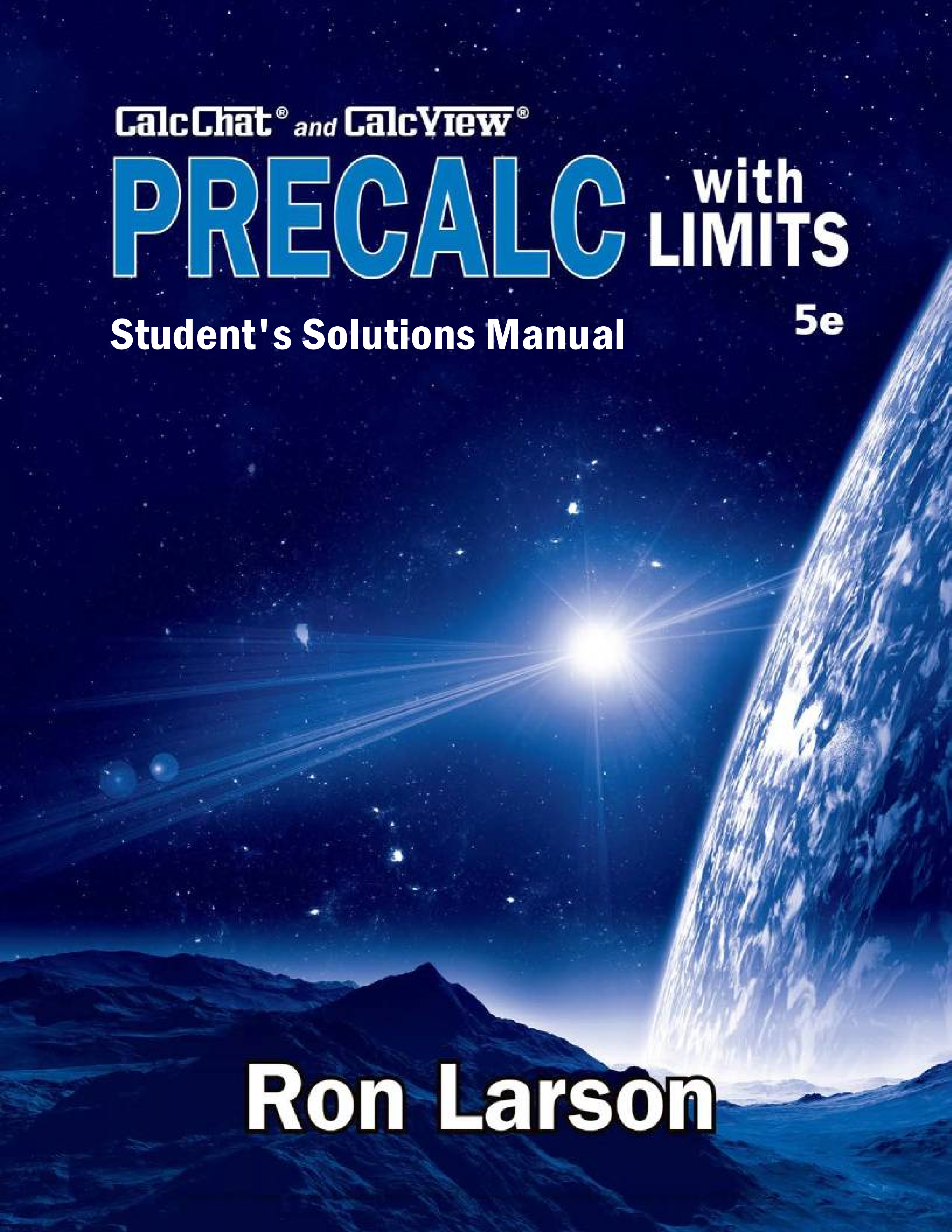

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Student's Solutions Manual

Ron Larson

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CHAPTER 1
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C H A P T E R 1
Functions and Their Graphs

Section 1.1 Rectangular Coordinates

1. Cartesian

3. The x-axis is the horizontal real number line.
Matches (c).

4. The y-axis is the vertical real number line.
Matches (f).

5. The origin is the point of intersection of the vertical and horizontal axes.
Matches (a).

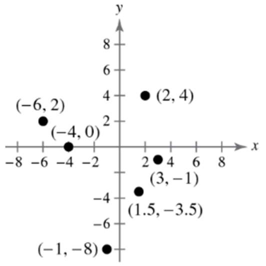
6. The quadrants are four regions of the coordinate plane.
Matches (d).

7. An x-coordinate is the directed distance from the y-axis. Matches (e).

8. A y-coordinate is the directed distance from the x-axis.
Matches (b).

9. $A: (2, 6), B: (-6, -2), C: (4, -4), D: (-3, 2)$

11.



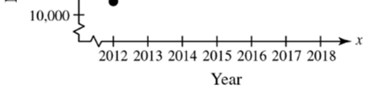
13. $(-3, 4)$

15. $x > 0$ and $y < 0$ in Quadrant IV.

17. $x = -4$ and $y > 0$ in Quadrant II.

19. $x + y = 0, x \neq 0, y \neq 0$ means $x = -y$ or $y = -x$.
This occurs in Quadrant II or IV.

21.



23.
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-2))^2 + (-6 - 6)^2} \\ &= \sqrt{(5)^2 + (-12)^2} \\ &= \sqrt{25 + 144} \\ &= 13 \text{ units} \end{aligned}$$

25.
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-5 - 1)^2 + (-1 - 4)^2} \\ &= \sqrt{(-6)^2 + (-5)^2} \\ &= \sqrt{36 + 25} \\ &= \sqrt{61} \text{ units} \end{aligned}$$

27.
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(-1 - \frac{4}{3}\right)^2} \\ &= \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{7}{3}\right)^2} \\ &= \sqrt{\frac{9}{4} + \frac{49}{9}} \\ &= \sqrt{\frac{277}{36}} \\ &= \frac{\sqrt{277}}{6} \text{ units} \end{aligned}$$

29. (a) $(1, 0), (13, 5)$

$$\begin{aligned} \text{Distance} &= \sqrt{(13 - 1)^2 + (5 - 0)^2} \\ &= \sqrt{12^2 + 5^2} = \sqrt{169} = 13 \end{aligned}$$

$(13, 5), (13, 0)$

$$\text{Distance} = |5 - 0| = |5| = 5$$

$(1, 0), (13, 0)$

$$\text{Distance} = |1 - 13| = |-12| = 12$$

(b) $5^2 + 12^2 = 25 + 144 = 169 = 13^2$

31. $d_1 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$

$d_2 = \sqrt{(4+1)^2 + (0+5)^2} = \sqrt{25+25} = \sqrt{50}$

$d_3 = \sqrt{(2+1)^2 + (1+5)^2} = \sqrt{9+36} = \sqrt{45}$

$\left(\sqrt{5}\right)^2 + \left(\sqrt{45}\right)^2 = \left(\sqrt{50}\right)^2$

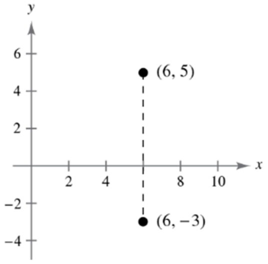
33. $d_1 = \sqrt{(1-3)^2 + (-3-2)^2} = \sqrt{4+25} = \sqrt{29}$

$d_2 = \sqrt{(3+2)^2 + (2-4)^2} = \sqrt{25+4} = \sqrt{29}$

$d_3 = \sqrt{(1+2)^2 + (-3-4)^2} = \sqrt{9+49} = \sqrt{58}$

$d_1 = d_2$

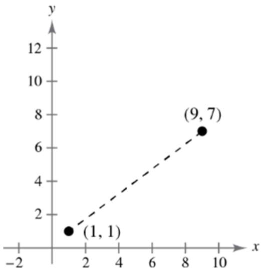
35. (a)



(b) $d = \sqrt{(5-(-3))^2 + (6-6)^2} = \sqrt{64} = 8$

(c) $\left(\dfrac{6+6}{2}, \dfrac{5+(-3)}{2}\right) = (6, 1)$

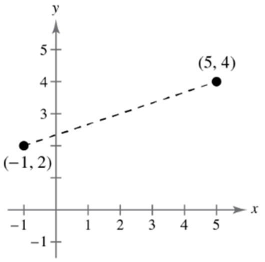
37. (a)



(b) $d = \sqrt{(9-1)^2 + (7-1)^2} = \sqrt{64+36} = 10$

(c) $\left(\dfrac{9+1}{2}, \dfrac{7+1}{2}\right) = (5, 4)$

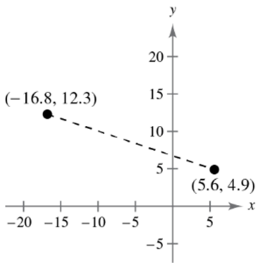
39. (a)



(b) $d = \sqrt{(5+1)^2 + (4-2)^2} + = \sqrt{36+4} = 2\sqrt{10}$

(c) $\left(\dfrac{-1+5}{2}, \dfrac{2+4}{2}\right) = (2, 3)$

41. (a)



(b) $d = \sqrt{(-16.8-5.6)^2 + (12.3-4.9)^2}$

$= \sqrt{501.76 + 54.76} = \sqrt{556.52}$

(c) $\left(\dfrac{-16.8+5.6}{2}, \dfrac{12.3+4.9}{2}\right) = (-5.6, 8.6)$

43. $d = \sqrt{(42-18)^2 + (50-12)^2}$

$= \sqrt{24^2 + 38^2}$

$= \sqrt{2020}$

$= 2\sqrt{505}$

≈ 45

The pass is about 45 yards.

45. $\text{midpoint} = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$= \left(\dfrac{2016+2018}{2}, \dfrac{485.9+514.4}{2}\right)$

$= (2017, 500.15)$

In 2017, the sales were about \$500.15 billion.

47. $(-2+2,\ -4+5) = (0, 1)$

$(2+2,\ -3+5) = (4, 2)$

$(-1+2,\ -1+5) = (1, 4)$

49. $(-7+4, -2+8) = (-3, 6)$

$(-2+4, 2+8) = (2, 10)$

$(-2+4, -4+8) = (2, 4)$

$(-7+4, -4+8) = (-3, 4)$

51. True. Because $x < 0$ and $y > 0$, $2x < 0$ and $-3y < 0$, which is located in Quadrant III.

53. True. Two sides of the triangle have lengths $\sqrt{149}$ and the third side has a length of $\sqrt{18}$.

55. The *y*-coordinate of a point on the *x*-axis is 0. The *x*-coordinates of a point on the *y*-axis is 0.

57. Because $x_m = \dfrac{x_1 + x_2}{2}$ and $y_m = \dfrac{y_1 + y_2}{2}$ we have:

$$\begin{aligned} 2x_m &= x_1 + x_2 & \qquad 2y_m &= y_1 + y_2 \\ 2x_m - x_1 &= x_2 & \qquad 2y_m - y_1 &= y_2 \end{aligned}$$

So, $(x_2, y_2) = (2x_m - x_1, 2y_m - y_1)$.

59. Use the Midpoint Formula to prove the diagonals of the parallelogram bisect each other.

$$\left(\frac{b+a}{2}, \frac{c+0}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$$

$$\left(\frac{a+b+0}{2}, \frac{c+0}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$$

61. (a) **First Set**

$$d(A, B) = \sqrt{(2-2)^2 + (3-6)^2} = \sqrt{9} = 3$$

$$d(B, C) = \sqrt{(2-6)^2 + (6-3)^2} = \sqrt{16+9} = 5$$

$$d(A, C) = \sqrt{(2-6)^2 + (3-3)^2} = \sqrt{16} = 4$$

Because $3^2 + 4^2 = 5^2$, *A, B,* and *C* are the vertices of a right triangle.

Second Set

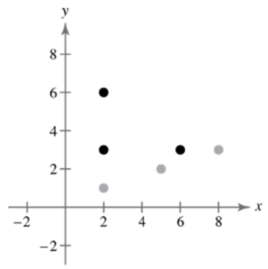
$$d(A, B) = \sqrt{(8-5)^2 + (3-2)^2} = \sqrt{10}$$

$$d(B, C) = \sqrt{(5-2)^2 + (2-1)^2} = \sqrt{10}$$

$$d(A, C) = \sqrt{(8-2)^2 + (3-1)^2} = \sqrt{40}$$

A, B, and *C* are the vertices of an isosceles triangle or are collinear: $\sqrt{10} + \sqrt{10} = 2\sqrt{10} = \sqrt{40}$.

(b)



First set: Not collinear

Second set: Collinear.

(c) A set of three points is collinear when the sum of two distances among the points is exactly equal to the third distance.

63. $4x - 6$

(a) $4(-1) - 6 = -4 - 6 = -10$

(b) $4(0) - 6 = 0 - 6 = -6$

65. (a) When $x = -3$, $2x^3 = 2(-3)^3 = 2(-27) = -54$.

(b) When $x = 0$, $2x^3 = 2(0)^3 = 0$.

67. $(-x)^2 - 2 = x^2 - 2$

69. $-x^2 + (-x)^2 = -x^2 + x^2 = 0$

71. (a)
$$\begin{aligned} P &= R - C \\ &= 135x - (93x + 35{,}000) \\ &= 42x - 35{,}000 \end{aligned}$$

(b) $P = 42(5000) - 35{,}000 = \$175{,}000$

Section 1.2 Graphs of Equations

1. solution or solution point

3. intercepts

5. origin

7. Two other approaches to solve problems mathematically are algebraic and graphical.

9. (a)
$$\begin{aligned} (2, 0): (2)^2 - 3(2) + 2 &\overset{?}{=} 0 \\ 4 - 6 + 2 &\overset{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

Yes, the point *is* on the graph.

(b)
$$\begin{aligned} (-2, 8): (-2)^2 - 3(-2) + 2 &\overset{?}{=} 8 \\ 4 + 6 + 2 &\overset{?}{=} 8 \\ 12 &\neq 8 \end{aligned}$$

No, the point *is not* on the graph.

11. (a)
$$\begin{aligned} (1, 5): 5 &\overset{?}{=} 4 - |1 - 2| \\ 5 &\overset{?}{=} 4 - 1 \\ 5 &\neq 3 \end{aligned}$$

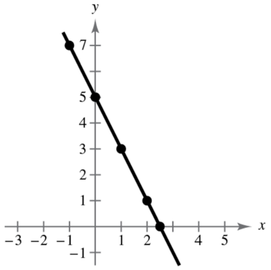
No, the point *is not* on the graph.

(b)
$$\begin{aligned} (6, 0): 0 &\overset{?}{=} 4 - |6 - 2| \\ 0 &\overset{?}{=} 4 - 4 \\ 0 &= 0 \end{aligned}$$

Yes, the point *is* on the graph.

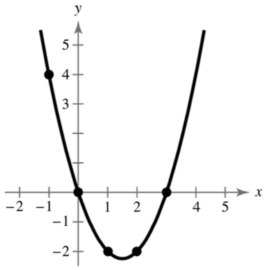
13. $y = -2x + 5$

x	-1	0	1	2	$\frac{5}{2}$
y	7	5	3	1	0
(x, y)	$(-1, 7)$	$(0, 5)$	$(1, 3)$	$(2, 1)$	$\left(\frac{5}{2}, 0\right)$



15. $y = x^2 - 3x$

x	-1	0	1	2	3
y	4	0	-2	-2	0
(x, y)	$(-1, 4)$	$(0, 0)$	$(1, -2)$	$(2, -2)$	$(3, 0)$



17. x-intercept: $(-2, 0)$

y-intercept: $(0, 2)$

19. x-intercept: $(3, 0)$

y-intercept: $(0, 9)$

21. $y = 5x - 6$

Let $y = 0$.

$$0 = 5x - 6$$
$$6 = 5x$$
$$\frac{6}{5} = x.$$

x-intercept: $\left(\frac{6}{5}, 0\right)$

Let $x = 0$.

$$y = 5(0) - 6$$
$$y = 0 - 6$$
$$y = -6$$

y-intercept: $(0, -6)$

23. $y = \sqrt{x + 4}$

Let $y = 0$.

$$0 = \sqrt{x + 4}$$
$$0 = x + 4$$
$$-4 = x$$

x-intercept: $(-4, 0)$

Let $x = 0$.

$$y = \sqrt{0 + 4}$$
$$y = \sqrt{4}$$
$$y = 2$$

y-intercept: $(0, 2)$

25. $y = |3x - 7|$

Let $y = 0$.

$$0 = |3x - 7|$$
$$0 = 3x - 7$$
$$7 = 3x$$
$$\frac{7}{3} = x$$

x-intercept: $\left(\frac{7}{3}, 0\right)$

Let $x = 0$.

$$y = |3(0) - 7|$$
$$y = |-7|$$
$$y = 7$$

y-intercept: $(0, 7)$

27. $y = 2x^3 - 4x^2$

Let $y = 0$.

$$0 = 2x^3 - 4x^2$$
$$0 = 2x^2(x - 2)$$
$$x = 0 \quad \text{or} \quad x = 2$$

x-intercepts: $(0, 0), (2, 0)$

Let $x = 0$.

$$y = 2(0)^3 - 4(0)^2$$
$$y = 0 - 0$$
$$y = 0$$

y-intercept: $(0, 0)$

29. $x^2 - y = 0$

$(-x)^2 - y = 0 \Rightarrow x^2 - y = 0 \Rightarrow$ y-axis symmetry

$x^2 - (-y) = 0 \Rightarrow x^2 + y = 0 \Rightarrow$ No x-axis symmetry

$(-x)^2 - (-y) = 0 \Rightarrow x^2 + y = 0 \Rightarrow$ No origin symmetry

31. $y = x^3$

$y = (-x)^3 \Rightarrow y = -x^3 \Rightarrow$ No y-axis symmetry

$-y = x^3 \Rightarrow y = -x^3 \Rightarrow$ No x-axis symmetry

$-y = (-x)^3 \Rightarrow -y = -x^3 \Rightarrow y = x^3 \Rightarrow$ Origin symmetry

33. $y = \dfrac{x}{x^2 + 1}$

$y = \dfrac{-x}{(-x)^2 + 1} \Rightarrow y = \dfrac{-x}{x^2 + 1} \Rightarrow$ No y-axis symmetry

$-y = \dfrac{x}{x^2 + 1} \Rightarrow y = \dfrac{-x}{x^2 + 1} \Rightarrow$ No x-axis symmetry

$-y = \dfrac{-x}{(-x)^2 + 1} \Rightarrow -y = \dfrac{-x}{x^2 + 1} \Rightarrow y = \dfrac{x}{x^2 + 1} \Rightarrow$ Origin symmetry

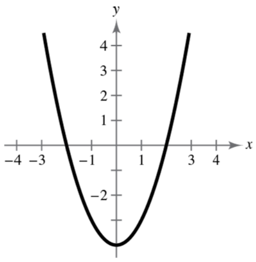
35. $xy^2 + 10 = 0$

$(-x)y^2 + 10 = 0 \Rightarrow -xy^2 + 10 = 0 \Rightarrow$ No y-axis symmetry

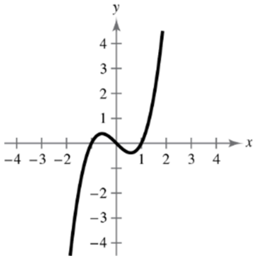
$x(-y)^2 + 10 = 0 \Rightarrow xy^2 + 10 = 0 \Rightarrow$ x-axis symmetry

$(-x)(-y)^2 + 10 = 0 \Rightarrow -xy^2 + 10 = 0 \Rightarrow$ No origin symmetry

37.



39.

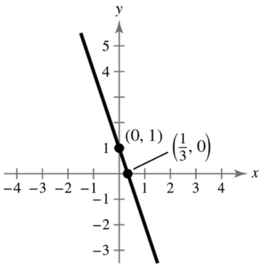


41. $y = -3x + 1$

x-intercept: $\left(\frac{1}{3}, 0\right)$

y-intercept: $(0, 1)$

No symmetry



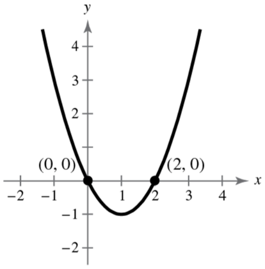
43. $y = x^2 - 2x$

x-intercepts: $(0, 0), (2, 0)$

y-intercept: $(0, 0)$

No symmetry

x	−1	0	1	2	3
y	3	0	−1	0	3



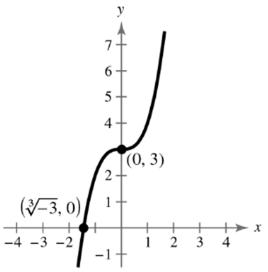
45. $y = x^3 + 3$

x-intercept: $\left(\sqrt[3]{-3}, 0\right)$

y-intercept: $(0, 3)$

No symmetry

x	-2	-1	0	1	2
y	-5	2	3	4	11



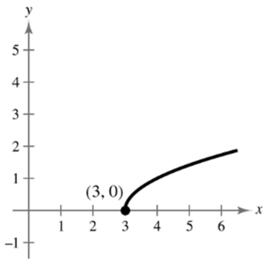
47. $y = \sqrt{x - 3}$

x-intercept: $(3, 0)$

y-intercept: none

No symmetry

x	3	4	7	12
y	0	1	2	3



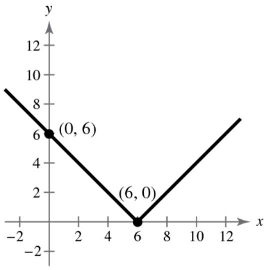
49. $y = |x - 6|$

x-intercept: $(6, 0)$

y-intercept: $(0, 6)$

No symmetry

x	-2	0	2	4	6	8	10
y	8	6	4	2	0	2	4



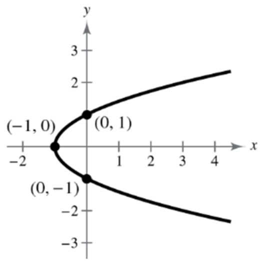
51. $x = y^2 - 1$

x-intercept: $(-1, 0)$

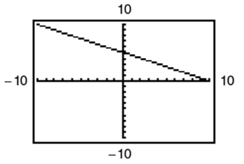
y-intercepts: $(0, -1), (0, 1)$

x-axis symmetry

x	-1	0	3
y	0	±1	±2

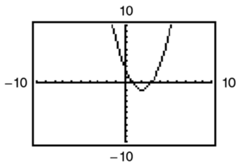


53. $y = 5 - \frac{1}{2}x$



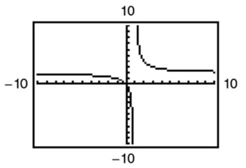
Intercepts: $(10, 0), (0, 5)$

55. $y = x^2 - 4x + 3$



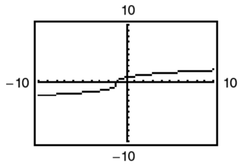
Intercepts: $(3, 0), (1, 0), (0, 3)$

57. $y = \dfrac{2x}{x - 1}$



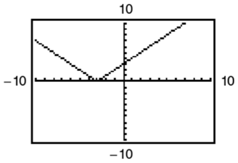
Intercept: $(0, 0)$

59. $y = \sqrt[3]{x + 1}$



Intercepts: $(-1, 0), (0, 1)$

61. $y = |x + 3|$



Intercepts: $(-3, 0), (0, 3)$

63. Center: $(0, 0)$; Radius: 3

$$(x - 0)^2 + (y - 0)^2 = 3^2$$
$$x^2 + y^2 = 9$$

65. Center: $(-4, 5)$; Radius: 2

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-4)]^2 + [y - 5]^2 = 2^2$$
$$(x + 4)^2 + (y - 5)^2 = 4$$

67. Center: $(3, 8)$; Solution point: $(-9, 13)$

$$\begin{aligned} r &= \sqrt{(x - h)^2 + (y - k)^2} \\ &= \sqrt{(-9 - 3)^2 + (13 - 8)^2} \\ &= \sqrt{(-12)^2 + (5)^2} \\ &= \sqrt{144 + 25} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 3)^2 + (y - 8)^2 = 13^2$$
$$(x - 3)^2 + (y - 8)^2 = 169$$

69. Endpoints of a diameter: $(3, 2), (-9, -8)$

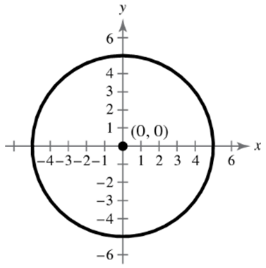
$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-9 - 3)^2 + (-8 - 2)^2} \\ &= \frac{1}{2}\sqrt{(-12)^2 + (-10)^2} \\ &= \frac{1}{2}\sqrt{144 + 100} \\ &= \frac{1}{2}\sqrt{244} = \frac{1}{2}(2)\sqrt{61} = \sqrt{61} \end{aligned}$$

$$(h, k): \left(\frac{3 + (-9)}{2} \cdot \frac{2 + (-8)}{2}\right) = \left(\frac{-6}{2} \cdot \frac{-6}{2}\right)$$
$$= (-3, -3)$$

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-3)]^2 + [y - (-3)]^2 = \left(\sqrt{61}\right)^2$$
$$(x + 3)^2 + (y + 3)^2 = 61$$

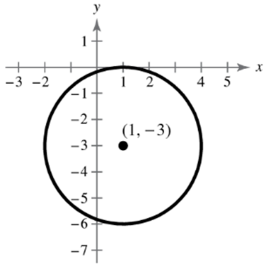
71. $x^2 + y^2 = 25$

Center: $(0, 0)$, Radius: 5



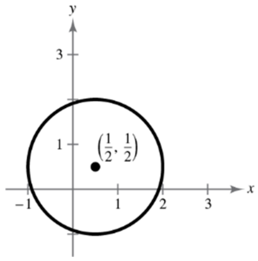
73. $(x - 1)^2 + (y + 3)^2 = 9$

Center: $(1, -3)$, Radius: 3

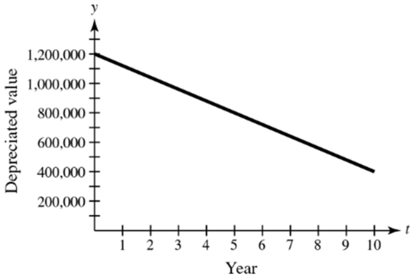


75. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{4}$

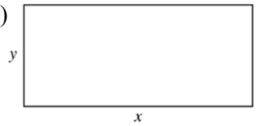
Center: $\left(\frac{1}{2}, \frac{1}{2}\right)$, Radius: $\frac{3}{2}$



77. $y = 1{,}200{,}000 - 80{,}000t,\ 0 \le t \le 10$



79. (a)



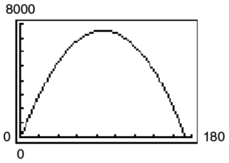
(b) $2x + 2y = \frac{1040}{3}$

$$2y = \frac{1040}{3} - 2x$$

$$y = \frac{520}{3} - x$$

$$A = xy = x\left(\frac{520}{3} - x\right)$$

(c)



(d) When $x = y = 86\frac{2}{3}$ yards, the area is a maximum of $7511\frac{1}{9}$ square yards.

(e) A regulation NFL playing field is 120 yards long and $53\frac{1}{3}$ yards wide. The actual area is 6400 square yards.

81. False. The line $y = x$ is symmetric with respect to the origin.

83. The test is for symmetry with respect to the x-axis. The statement should read: The graph of $x = 3y^2$ is symmetric with respect to the x-axis because

85. $y = ax^2 + bx^3$

(a) $y = a(-x)^2 + b(-x)^3$

$$= ax^2 - bx^3$$

To be symmetric with respect to the y-axis; a can be any non-zero real number, b must be zero.

Sample answer: $a = 1, b = 0$

(b) $-y = a(-x)^2 + b(-x)^3$

$$-y = ax^2 - bx^3$$

$$y = -ax^2 + bx^3$$

To be symmetric with respect to the origin; a must be zero, b can be any non-zero real number.

Sample answer: $a = 0, b = 1$

87. $3(7x + 1) = 3(7x) + 3(1) = 21x + 3$

89. $6(x - 1) + 4 = 6(x) - 6(1) = 6x - 6 + 4 = 6x - 2$

91. The least common denominator is $3(4) = 12$.

93. The least common denominator is $x - 4$.

95. $7\sqrt{72} - 5\sqrt{18} = 7\sqrt{2 \cdot 36} - 5\sqrt{2 \cdot 9}$

$$= 7(6)\sqrt{2} - 5(3)\sqrt{2}$$

$$= (42 - 15)\sqrt{2}$$

$$= 27\sqrt{2}$$

97. $7^{3/2} \cdot 7^{11/2} = 7^{3/2 + 11/2} = 7^7 = 823{,}543$

99. $(9x - 4) + (2x^2 - x + 15) = 2x^2 + 8x + 11$

101. $(2x + 9)(x - 7) = 2x^2 + 9x - 14x - 63$

$$= 2x^2 - 5x - 63$$

Section 1.3 Linear Equations in Two Variables

1. linear

3. point-slope

5. rate or rate of change

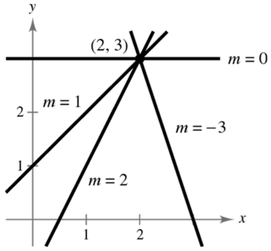
7. They are perpendicular to each other.

9. (a) $m = \frac{2}{3}$. Because the slope is positive, the line rises. Matches L_2.

(b) m is undefined. The line is vertical. Matches L_3.

(c) $m = -2$. The line falls. Matches L_1.

11.



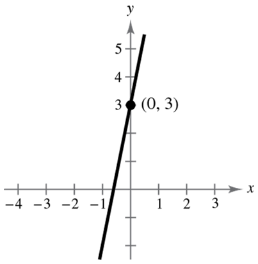
13. Two points on the line: $(0, 0)$ and $(4, 6)$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6}{4} = \frac{3}{2}$$

15. $y = 5x + 3$

Slope: $m = 5$

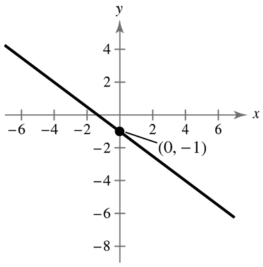
y-intercept: $(0, 3)$



17. $y = -\frac{3}{4}x - 1$

Slope: $m = -\frac{3}{4}$

y-intercept: $(0, -1)$

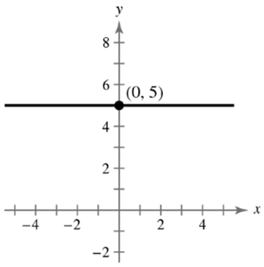


19. $y - 5 = 0$

$y = 5$

Slope: $m = 0$

y-intercept: $(0, 5)$

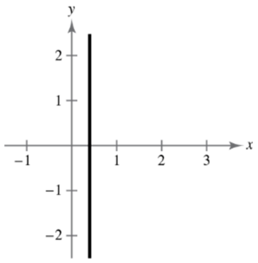


21. $5x - 2 = 0$

$x = \frac{2}{5}$, vertical line

Slope: undefined

y-intercept: none



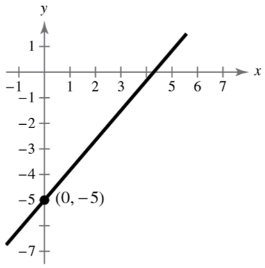
23. $7x - 6y = 30$

$-6y = -7x + 30$

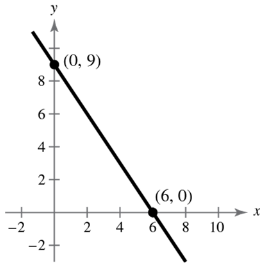
$y = \frac{7}{6}x - 5$

Slope: $m = \frac{7}{6}$

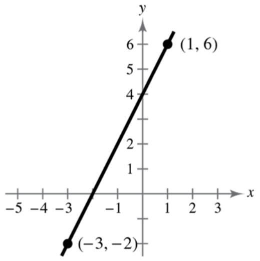
y-intercept: $(0, -5)$



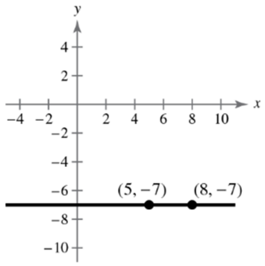
25. $m = \dfrac{0 - 9}{6 - 0} = \dfrac{-9}{6} = -\dfrac{3}{2}$



27. $m = \dfrac{6 - (-2)}{1 - (-3)} = \dfrac{8}{4} = 2$

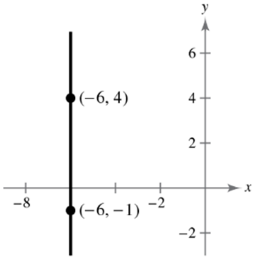


29. $m = \dfrac{-7 - (-7)}{8 - 5} = \dfrac{0}{3} = 0$

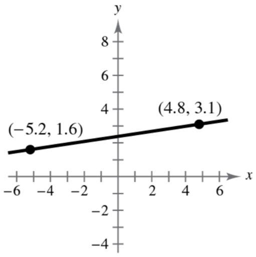


31. $m = \dfrac{4 - (-1)}{-6 - (-6)} = \dfrac{5}{0}$

m is undefined.



33. $m = \dfrac{1.6 - 3.1}{-5.2 - 4.8} = \dfrac{-1.5}{-10} = 0.15$



35. Point: $(5, 7)$, Slope: $m = 0$

Because $m = 0$, y does not change. Three other points are $(-1, 7)$, $(0, 7)$, and $(4, 7)$.

37. Point: $(-5, 4)$, Slope: $m = 2$

Because $m = 2 = \frac{2}{1}$, y increases by 2 for every one unit increase in x. Three additional points are $(-4, 6)$, $(-3, 8)$, and $(-2, 10)$.

39. Point: $(4, 5)$, Slope: $m = -\frac{1}{3}$

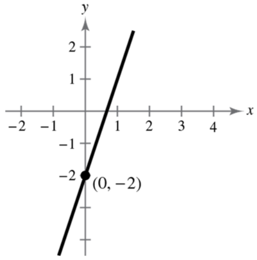
Because $m = -\frac{1}{3}$, y decreases by 1 unit for every three units increase in x. Three additional points are $(-2, 7)$, $\left(0, -\frac{19}{4}\right)$, and $(1, 6)$.

41. Point: $(-4, 3)$, Slope is undefined.

Because m is undefined, x does not change. Three points are $(-4, 0)$, $(-4, 5)$, and $(-4, 2)$.

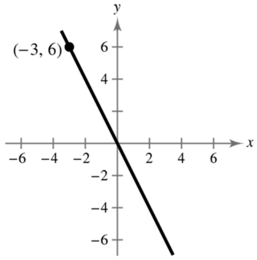
43. Point: $(0, -2)$; $m = 3$

$$\begin{aligned} y + 2 &= 3(x - 0) \\ y &= 3x - 2 \end{aligned}$$



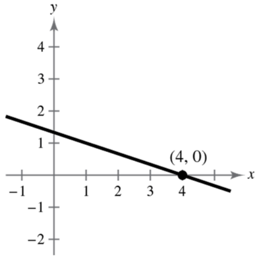
45. Point: $(-3, 6)$; $m = -2$

$$\begin{aligned} y - 6 &= -2(x + 3) \\ y &= -2x \end{aligned}$$



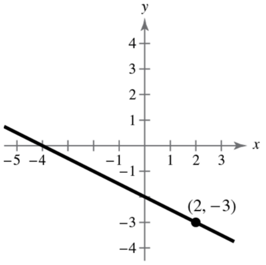
47. Point: $(4, 0)$; $m = -\frac{1}{3}$

$$\begin{aligned} y - 0 &= -\tfrac{1}{3}(x - 4) \\ y &= -\tfrac{1}{3}x + \tfrac{4}{3} \end{aligned}$$



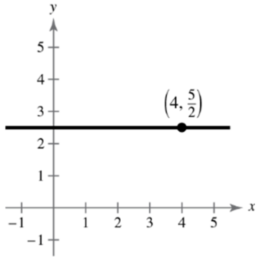
49. Point: $(2, -3)$; $m = -\dfrac{1}{2}$

$$\begin{aligned} y - (-3) &= -\frac{1}{2}(x - 2) \\ y + 3 &= -\frac{1}{2}x + 1 \\ y &= -\frac{1}{2}x - 2 \end{aligned}$$



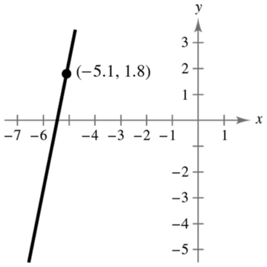
51. Point: $\left(4, \frac{5}{2}\right)$; $m = 0$

$$y - \tfrac{5}{2} = 0(x - 4)$$
$$y - \tfrac{5}{2} = 0$$
$$y = \tfrac{5}{2}$$



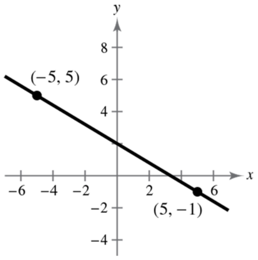
53. Point: $(-5.1, 1.8)$; $m = 5$

$$y - 1.8 = 5(x - (5.1))$$
$$y = 5x + 27.3$$



55. $(5, -1), (-5, 5)$

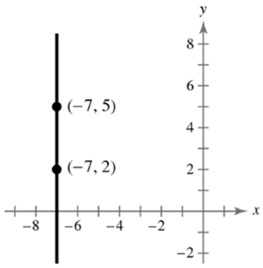
$$y + 1 = \frac{5 + 1}{-5 - 5}(x - 5)$$
$$y = -\frac{3}{5}(x - 5) - 1$$
$$y = -\frac{3}{5}x + 2$$



57. $(-7, 2), (-7, 5)$

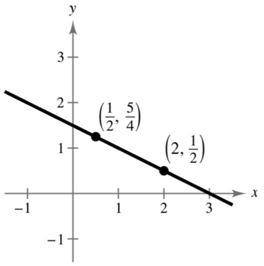
$$m = \frac{5 - 2}{-7 - (-7)} = \frac{3}{0}$$

m is undefined.



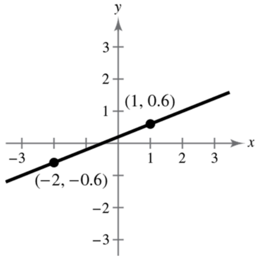
59. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$

$$y - \frac{1}{2} = \frac{\frac{5}{4} - \frac{1}{2}}{\frac{1}{2} - 2}(x - 2)$$
$$y = -\frac{1}{2}(x - 2) + \frac{1}{2}$$
$$y = -\frac{1}{2}x + \frac{3}{2}$$



61. $(1, 0.6), (-2, -0.6)$

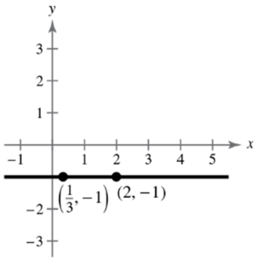
$$y - 0.6 = \frac{-0.6 - 0.6}{-2 -1}(x - 1)$$
$$y = 0.4(x - 1) + 0.6$$
$$y = 0.4x + 0.2$$



63. $(2, -1), \left(\frac{1}{3}, -1\right)$

$$y + 1 = \frac{-1 - (-1)}{\frac{1}{3} - 2}(x - 2)$$
$$y + 1 = 0$$
$$y = -1$$

The line is horizontal.



65. $L_1: y = -\frac{2}{3}x - 3$

$m_1 = -\frac{2}{3}$

$L_2: y = -\frac{2}{3}x - 1$

$m_2 = -\frac{2}{3}$

The slopes are equal, so the lines are parallel.

67. $L_1: y = \frac{1}{2}x - 3$

$m_1 = \frac{1}{2}$

$L_2: y = -\frac{1}{2}x + 1$

$m_2 = -\frac{1}{2}$

The lines are neither parallel nor perpendicular.

69. $L_1: (0, -1), (5, 9)$

$$m_1 = \frac{9+1}{5-0} = 2$$

$L_2: (0, 3), (4, 1)$

$$m_2 = \frac{1-3}{4-0} = -\frac{1}{2}$$

The slopes are negative reciprocals, so the lines are perpendicular.

71. $L_1: (-6, -3), (2, -3)$

$$m_1 = \frac{-3-(-3)}{2-(-6)} = \frac{0}{8} = 0$$

$L_2: \left(3, -\frac{1}{2}\right), \left(6, -\frac{1}{2}\right)$

$$m_2 = \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)}{6-3} = \frac{0}{3} = 0$$

L_1 and L_2 are both horizontal lines, so they are parallel.

73. $4x - 2y = 3$

$$y = 2x - \tfrac{3}{2}$$

Slope: $m = 2$

(a) $(2, 1), m = 2$

$$y - 1 = 2(x - 2)$$
$$y = 2x - 3$$

(b) $(2, 1), m = -\frac{1}{2}$

$$y - 1 = -\tfrac{1}{2}(x - 2)$$
$$y = -\tfrac{1}{2}x + 2$$

75. $3x + 4y = 7$

$$y = -\tfrac{3}{4}x + \tfrac{7}{4}$$

Slope: $m = -\frac{3}{4}$

(a) $\left(-\frac{2}{3}, \frac{7}{8}\right), m = -\frac{3}{4}$

$$y - \tfrac{7}{8} = -\tfrac{3}{4}\left(x - \left(-\tfrac{2}{3}\right)\right)$$
$$y = -\tfrac{3}{4}x + \tfrac{3}{8}$$

(b) $\left(-\frac{2}{3}, \frac{7}{8}\right), m = \frac{4}{3}$

$$y - \tfrac{7}{8} = \tfrac{4}{3}\left(x - \left(-\tfrac{2}{3}\right)\right)$$
$$y = \tfrac{4}{3}x + \tfrac{127}{72}$$

77. $y + 5 = 0$

$$y = -5$$

Slope: $m = 0$

(a) $(-2, 4), m = 0$

$$y = 4$$

(b) $(-2, 4)$, m is undefined.

$$x = -2$$

79. $x - y = 4$

$$y = x - 4$$

Slope: $m = 1$

(a) $(2.5, 6.8), m = 1$

$$y - 6.8 = 1(x - 2.5)$$
$$y = x + 4.3$$

(b) $(2.5, 6.8), m = -1$

$$y - 6.8 = (-1)(x - 2.5)$$
$$y = -x + 9.3$$

81.

$$\frac{x}{3} + \frac{y}{5} = 1$$
$$(15)\left(\frac{x}{3} + \frac{y}{5}\right) = 1(15)$$
$$5x + 3y - 15 = 0$$

83.

$$\frac{x}{-1/6} + \frac{y}{-2/3} = 1$$
$$6x + \frac{3}{2}y = -1$$
$$12x + 3y + 2 = 0$$

85.

$$\frac{x}{c} + \frac{y}{c} = 1, c \neq 0$$
$$x + y = c$$
$$1 + 2 = c$$
$$3 = c$$
$$x + y = 3$$
$$x + y - 3 = 0$$

87. (a) $m = 135$. The sales are increasing 135 units per year.

(b) $m = 0$. There is no change in sales during the year.

(c) $m = -40$. The sales are decreasing 40 units per year.

89. $y = \frac{6}{100}x$

$$y = \tfrac{6}{100}(200) = 12 \text{ feet}$$

91. Using the points $(0, 32)$ and $(100, 212)$, where the first coordinate represents a temperature in degrees Celsius and the second coordinate represents a temperature in degrees Fahrenheit, you have

$$m = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}.$$

Since the point $(0, 32)$ is the F- intercept, $b = 32$, the equation is $F = 1.8C + 32$ or $C = \frac{5}{9}F - \frac{160}{9}$.

93. Using the points $(0, 830)$ and $(5, 0)$, where the first coordinate represents the year t and the second coordinate represents the value V, you have

$$V = \frac{0 - 830}{5 - 0}t + 830 = -166t + 830, \; 0 \le t \le 5.$$

95. (a) Total Cost = cost for fuel and maintainance + cost for operator + purchase cost

$$C = 9.5t + 11.5t + 42{,}000$$
$$C = 21t + 42{,}000$$

(b) Revenue = Rate per hour · Hours

$$R = 45t$$

(c) $P = R - C$

$$P = 45t - (21t + 42{,}000)$$
$$P = 24t - 42{,}000$$

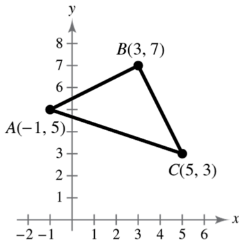
(d) Let $P = 0$, and solve for t.

$$0 = 24t - 42{,}000$$
$$42{,}000 = 24t$$
$$1750 = t$$

The equipment must be used 1750 hours to yield a profit of 0 dollars.

97. False. The slope with the greatest magnitude corresponds to the steepest line.

99. Find the slope of the line segments between the points A and B, and B and C.



$$m_{AB} = \frac{7 - 5}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

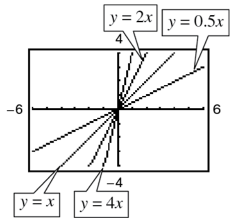
$$m_{BC} = \frac{3 - 7}{5 - 3} = \frac{-4}{2} = -2$$

Since the slopes are negative reciprocals, the line segments are perpendicular and therefore intersect to form a right angle. So, the triangle is a right triangle.

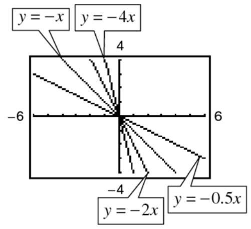
101. (a) The slope is $\frac{0 - (-1)}{2 - 0} = \frac{1}{2}$, not 2.

(b) The y-intercept is $(0, -3)$, not $(0, 4)$.

103. The line $y = 4x$ rises most quickly.



The line $y = -4x$ falls most quickly.



The greater the magnitude of the slope (the absolute value of the slope), the faster the line rises or falls.

105. $y = x^2 - x$

When $x = -2$: $y = (-2)^2 - (-2) = 4 + 2 = 6$

When $x = 0$: $y = 0^2 - 0 = 0$

When $x = 3$: $y = 3^2 - 3 = 9 - 3 = 6$

When $x = 6$: $y = 6^2 - 6 = 36 - 6 = 30$

107. $2f = \dfrac{(7 - x^3)}{5} \Rightarrow f = \dfrac{(7 - x^3)}{10}$

(a) When $x = -2$: $f = \dfrac{(7 - (-2)^3)}{10} = \dfrac{(7 + 8)}{10} = \dfrac{15}{10} = \dfrac{3}{2}$

(b) When $x = 0$: $f = \dfrac{(7 - 0^3)}{10} = \dfrac{7}{10}$

(c) When $x = 3$: $f = \dfrac{(7 - 3^3)}{10} = \dfrac{(7 - 27)}{10} = \dfrac{-20}{10} = -2$

(d) When $x = 6$: $f = \dfrac{(7 - 6^3)}{10} = \dfrac{(7 - 216)}{10} = \dfrac{-209}{10}$

109.
$$\begin{aligned} 2x^3 - 5x^2 - 2x + 5 &= 0 \\ x^2(2x - 5) - (2x - 5) &= 0 \\ (x^2 - 1)(2x - 5) &= 0 \\ x &= \pm 1, \frac{5}{2} \end{aligned}$$

111.
$$\begin{aligned} x^6 + 4x^3 - 5 &= 0 \\ (x^3 + 5)(x^3 - 1) &= 0 \\ x &= -\sqrt[3]{5}, 1 \end{aligned}$$

113.
$$\begin{aligned} 3\sqrt{x} - 5\sqrt{18} &= 0 \\ 3\sqrt{x} &= 5(3)\sqrt{2} \\ \sqrt{x} &= 5\sqrt{2} \\ x &= 25(2) = 50 \end{aligned}$$

115.
$$\begin{aligned} x &= \frac{2}{x} + 1 \\ x^2 &= 2 + x \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= 2, -1 \end{aligned}$$

117.
$$\begin{aligned} x + |x - 9| &= 7 \\ |x - 9| &= 7 - x \end{aligned}$$

If $x - 9 \geq 0$, then $x - 9 = 7 - x \Rightarrow 2x = 16 \Rightarrow x = 8$, which is extraneous.

If $x - 9 < 0$, then $-(x - 9) = 7 - x$, which is extraneous.

No real solution.

119.
$$\begin{aligned} \frac{\left[(x - 1)^2 + 1\right] - (x^2 + 1)}{x} &= \frac{(x^2 - 2x + 1 + 1) - (x^2 + 1)}{x} \\ &= \frac{-2x + 1}{x} \end{aligned}$$

121. $$\begin{aligned}\frac{4 - x^2 - 3}{x - 1} + x^2 + x + 4 &= \frac{4 - x^2 - 3 + (x - 1)(x^2 + x + 4)}{x - 1}\\ &= \frac{4 - x^2 - 3 + (x^3 + x^2 + 4x - x^2 - x - 4)}{x - 1}\\ &= \frac{4 - x^2 - 3 + x^3 + 3x - 4}{x - 1}\\ &= \frac{x^3 - x^2 + 3x - 3}{x - 1}\\ &= \frac{x^2(x - 1) + 3(x - 1)}{x - 1}\\ &= \frac{(x - 1)(x^2 + 3)}{x - 1}\\ &= x^2 + 3,\ x \neq 1\end{aligned}$$

Alternate solution:

$$\begin{aligned}\frac{4 - x^2 - 3}{x - 1} + x^2 + x + 4 &= \frac{(1 - x)(1 + x)}{x - 1} + x^2 + x + 4\\ &= -1 - x + x^2 + x + 4,\ x \neq 1\\ &= x^2 + 3,\ x \neq 1\end{aligned}$$

Section 1.4 Functions

1. independent; dependent

3. A relation is any set of ordered pairs. A function is a relation in which no two ordered pairs have the same first component and different second components.

5. The domain of a piece-wise function must be explicitly described, so that it can determine which equation is used to evaluate the function.

7. Yes, the relationship is a function. Each domain value is matched with exactly one range value.

9. No, it does not represent a function. The input values of 10 and 7 are each matched with two output values.

11. (a) Each element of A is matched with exactly one element of B, so it does represent a function.

(b) Each element of A is matched with exactly one element of B, so it does represent a function.

(c) The element 2 in A is not matched with an element of B, so the relation does not represent a function.

13. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

No, y *is not* a function of x.

15. $y = \sqrt{16 - x^2}$

Yes, y *is* a function of x.

17. $y = 4 - |x|$

Yes, y *is* a function of x.

19. $y = -75$ or $y = -75 + 0x$

Yes, y *is* a function of x.

21. $g(t) = 4t^2 - 3t + 5$

(a) $g(2) = 4(2)^2 - 3(2) + 5 = 16 - 6 + 5 = 15$

(b) $g(-1) = 4(-1)^2 - 3(-1) + 5 = 4 + 3 + 5 = 12$

(c) $$\begin{aligned}g(t + 2) &= 4(t + 2)^2 - 3(t + 2) + 5\\ &= 4(t^2 + 4t + 4) - 3t - 6 + 5\\ &= 4t^2 + 13t + 15\end{aligned}$$

23. $f(y) = 3 - \sqrt{y}$

(a) $f(4) = 3 - \sqrt{4} = 1$

(b) $f(0.25) = 3 - \sqrt{0.25} = 2.5$

(c) $f(4x^2) = 3 - \sqrt{4x^2} = 3 - 2|x|$

25. $q(x) = \dfrac{1}{x^2 - 9}$

(a) $q(0) = \dfrac{1}{0^2 - 9} = -\dfrac{1}{9}$

(b) $q(3) = \dfrac{1}{3^2 - 9}$ is undefined.

(c) $q(y + 3) = \dfrac{1}{(y + 3)^2 - 9} = \dfrac{1}{y^2 + 6y}$

27. $f(x) = \dfrac{|x|}{x}$

(a) $f(2) = \dfrac{|2|}{2} = 1$

(b) $f(-2) = \dfrac{|-2|}{-2} = -1$

(c) $f(x-1) = \dfrac{|x-1|}{x-1} = \begin{cases} -1, & \text{if } x < 1 \\ 1, & \text{if } x > 1 \end{cases}$

29. $f(x) = \begin{cases} 2x+1, & x < 0 \\ 2x+2, & x \ge 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

31. $f(x) = -x^2 + 5$

$f(-2) = -(-2)^2 + 5 = 1$

$f(-1) = -(-1)^2 + 5 = 4$

$f(0) = -(0)^2 + 5 = 5$

$f(1) = -(1)^2 + 5 = 4$

$f(2) = -(2)^2 + 5 = 1$

x	-2	-1	0	1	2
$f(x)$	1	4	5	-4	1

33. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \le 0 \\ (x-2)^2, & x > 0 \end{cases}$

$f(-2) = -\frac{1}{2}(-2) + 4 = 5$

$f(-1) = -\frac{1}{2}(-1) + 4 = 4\frac{1}{2} = \frac{9}{2}$

$f(0) = -\frac{1}{2}(0) + 4 = 4$

$f(1) = (1-2)^2 = 1$

$f(2) = (2-2)^2 = 0$

x	-2	-1	0	1	2
$f(x)$	5	$\frac{9}{2}$	4	1	0

35.
$$\begin{aligned} 15 - 3x &= 0 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

37.
$$\begin{aligned} \frac{3x-4}{5} &= 0 \\ 3x - 4 &= 0 \\ x &= \frac{4}{3} \end{aligned}$$

39.
$$\begin{aligned} f(x) &= x^2 - 81 \\ x^2 - 81 &= 0 \\ x^2 &= 81 \\ x &= \pm 9 \end{aligned}$$

41.
$$\begin{aligned} x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x+1)(x-1) &= 0 \\ x = 0, x = -1, \text{ or } x &= 1 \end{aligned}$$

43.
$$\begin{aligned} f(x) &= g(x) \\ x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$
$$\begin{aligned} x - 2 &= 0 \quad & x + 1 &= 0 \\ x &= 2 \quad & x &= -1 \end{aligned}$$

45.
$$\begin{aligned} f(x) &= g(x) \\ x^4 - 2x^2 &= 2x^2 \\ x^4 - 4x^2 &= 0 \\ x^2(x^2 - 4) &= 0 \\ x^2(x+2)(x-2) &= 0 \end{aligned}$$
$$\begin{aligned} x^2 = 0 &\Rightarrow x = 0 \\ x + 2 = 0 &\Rightarrow x = -2 \\ x - 2 = 0 &\Rightarrow x = 2 \end{aligned}$$

47. $f(x) = 5x^2 + x - 1$

Because $f(x)$ is a polynomial, the domain is all real numbers x.

49. $g(y) = \sqrt{y+6}$

Domain: $y + 6 \ge 0$

$y \ge -6$

The domain is all real numbers y such that $y \ge -6$.

51. $g(x) = \dfrac{1}{x} - \dfrac{3}{x+2}$

The domain is all real numbers x except $x = 0$, $x = -2$.

53. $f(s) = \dfrac{\sqrt{s-1}}{s-4}$

Domain: $s - 1 \ge 0 \Rightarrow s \ge 1$ and $s \ne 4$

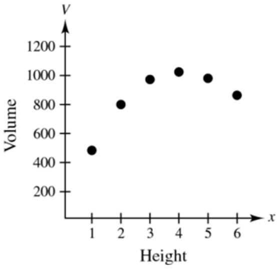
The domain consists of all real numbers s, such that $s \ge 1$ and $s \ne 4$.

55. (a)

Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

The volume is maximum when $x = 4$ and $V = 1024$ cubic centimeters.

(b)



V is a function of x.

(c) $V = x(24 - 2x)^2$

Domain: $0 < x < 12$

57. $y = -\frac{1}{10}x^2 + 3x + 6$

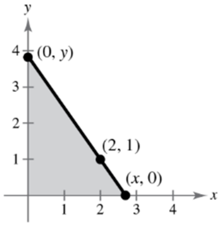
$y(25) = -\frac{1}{10}(25)^2 + 3(25) + 6 = 18.5$ feet

If the child holds a glove at a height of 5 feet, then the ball *will* be over the child's head because it will be at a height of 18.5 feet.

59. $A = \dfrac{1}{2}bh = \dfrac{1}{2}xy$

Because $(0, y)$, $(2, 1)$, and $(x, 0)$ all lie on the same line, the slopes between any pair are equal.

$$\frac{1-y}{2-0} = \frac{0-1}{x-2}$$
$$\frac{1-y}{2} = \frac{-1}{x-2}$$
$$y = \frac{2}{x-2} + 1$$
$$y = \frac{x}{x-2}$$



So, $A = \dfrac{1}{2}x\left(\dfrac{x}{x-2}\right) = \dfrac{x^2}{2(x-2)}$.

The domain of A includes x-values such that $x^2/\left[2(x-2)\right] > 0$. By solving this inequality, the domain is $x > 2$.

61. $p(t) = \begin{cases} 1.76t + 58.3, & 12 \le t < 16 \\ 0.90t + 71.5, & 16 \le t \le 18 \end{cases}$

For 2012 through 2015, use the first equation.

2012: $p(12) = 1.76(12) + 58.3 = 79.42\%$

2013: $p(13) = 1.76(13) + 58.3 = 81.18\%$

2014: $p(14) = 1.76(14) + 58.3 = 82.94\%$

2015: $p(15) = 1.76(15) + 58.3 = 84.70\%$

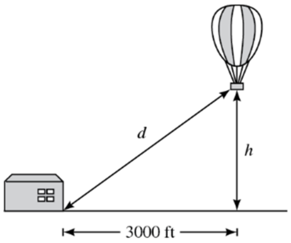
For 2016 through 2019, use the second equation.

2016: $p(16) = 0.90(16) + 71.5 = 85.90\%$

2017: $p(17) = 0.90(17) + 71.5 = 86.80\%$

2018: $p(18) = 0.90(18) + 71.5 = 87.70\%$

63. (a)



(b) $(3000)^2 + h^2 = d^2$

$$h = \sqrt{d^2 - (3000)^2}$$

Domain: $d \ge 3000$ (because both $d \ge 0$ and $d^2 - (3000)^2 \ge 0$)

65. (a) Cost = variable costs + fixed costs

$$C = 12.30x + 98{,}000$$

(b) Revenue = price per unit × number of units

$$R = 17.98x$$

(c) Profit = Revenue − Cost

$$\begin{aligned} P &= 17.98x - (12.30x + 98{,}000) \\ P &= 5.68x - 98{,}000 \end{aligned}$$

67. (a) $R = n(\text{rate}) = n[8.00 - 0.05(n - 80)],\ n \ge 80$

$$R = 12.00n - 0.05n^2 = 12n - \frac{n^2}{20} = \frac{240n - n^2}{20},\ n \ge 80$$

(b)

n	90	100	110	120	130	140	150
$R(n)$	\$675	\$700	\$715	\$720	\$715	\$700	\$675

The revenue is maximum when 120 people take the trip.

69.

$$\begin{aligned} f(x) &= x^2 - 2x + 4 \\ f(2 + h) &= (2 + h)^2 - 2(2 + h) + 4 \\ &= 4 + 4h + h^2 - 4 - 2h + 4 \\ &= h^2 + 2h + 4 \\ f(2) &= (2)^2 - 2(2) + 4 = 4 \\ f(2 + h) - f(2) &= h^2 + 2h \\ \frac{f(2 + h) - f(2)}{h} &= \frac{h^2 + 2h}{h} = h + 2,\ h \ne 0 \end{aligned}$$

71.

$$\begin{aligned} f(x) &= x^3 + 3x \\ f(x + h) &= (x + h)^3 + 3(x + h) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h \\ \frac{f(x + h) - f(x)}{h} &= \frac{(x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h) - (x^3 + 3x)}{h} \\ &= \frac{h(3x^2 + 3xh + h^2 + 3)}{h} \\ &= 3x^2 + 3xh + h^2 + 3,\ h \ne 0 \end{aligned}$$

73.

$$\begin{aligned} g(x) &= \frac{1}{x^2} \\ \frac{g(x) - g(3)}{x - 3} &= \frac{\frac{1}{x^2} - \frac{1}{9}}{x - 3} \\ &= \frac{9 - x^2}{9x^2(x - 3)} \\ &= \frac{-(x + 3)(x - 3)}{9x^2(x - 3)} \\ &= -\frac{x + 3}{9x^2},\ x \ne 3 \end{aligned}$$

75. $f(x) = \sqrt{5x}$

$$\frac{f(x) - f(5)}{x - 5} = \frac{\sqrt{5x} - 5}{x - 5},\ x \ne 5$$

77. False. The equation $y^2 = x^2 + 4$ is a relation between x and y. However, $y = \pm\sqrt{x^2 + 4}$ does not represent a function.

79. False. The range is $[-1, \infty)$.

81. By plotting the points, we have a parabola, so $g(x) = cx^2$. Because $(-4, -32)$ is on the graph, you have $-32 = c(-4)^2 \Rightarrow c = -2$. So, $g(x) = -2x^2$.

83. Because the function is undefined at 0, we have $r(x) = c/x$. Because $(-4, -8)$ is on the graph, you have $-8 = c/-4 \Rightarrow c = 32$. So, $r(x) = 32/x$.

85. The domain of $f(x) = \sqrt{x-1}$ includes $x = 1$, $x \geq 1$ and the domain of $g(x) = \dfrac{1}{\sqrt{x-1}}$ does not include $x = 1$ because you cannot divide by 0. The domain of $g(x) = \dfrac{1}{\sqrt{x-1}}$ is $x > 1$. So, the functions do not have the same domain.

87. No; x is the independent variable, f is the name of the function.

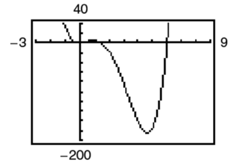
89.
$$\begin{aligned} x^3 - 3x^2 - x + 3 &= 0 \\ x^2(x-3) - (x-3) &= 0 \\ (x-3)(x^2-1) &= 0 \\ x &= 3, \pm 1 \end{aligned}$$

91.
$$\begin{aligned} 0 &= -2x^2 - 7x + 15 \\ 2x^2 + 7x - 15 &= 0 \\ (x+5)(2x-3) &= 0 \\ x &= -5, \tfrac{3}{2} \end{aligned}$$

93. x-intercepts: $(-2, 0)$, $(1, 0)$

y-intercept: $(0, -2)$

95.



x-intercepts: $(0, 0)$, $\left(\frac{3}{2}, 0\right)$, $(6, 0)$

97. $y = x^2 - 3.61x + 2.86$

Using a graphing utility, the x-intercepts are $(1.17, 0)$ and $(2.44, 0)$.

99. $\dfrac{(3^2+4)-(1^2+4)}{3-1} = \dfrac{13-5}{2} = \dfrac{8}{2} = 4$

101. $\dfrac{\frac{1}{3}-\frac{1}{2}}{6-4} = \dfrac{-\frac{1}{6}}{2} = \dfrac{-1}{12}$

103. $\dfrac{\sqrt{3^2+4^2}}{\frac{3}{4}-\frac{1}{3}} = \dfrac{\sqrt{25}}{\frac{5}{12}} = 5\left(\dfrac{12}{5}\right) = 12$

Section 1.5 Analyzing Graphs of Functions

1. zeros

3. average rate of change; secant

5. No. If a vertical line intersects the graph more than once, then it does not represent y as a function of x.

7. Domain: $(-2, 2]$; Range: $[-1, 8]$

(a) $f(-1) = -1$

(b) $f(0) = 0$

(c) $f(1) = -1$

(d) $f(2) = 8$

9. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$

(a) $f(2) = 0$

(b) $f(1) = 1$

(c) $f(3) = 2$

(d) $f(-1) = 3$

11. A vertical line intersects the graph at most once, so *y is* a function of x.

13. A vertical line intersects the graph more than once, so *y is not* a function of x.

15.
$$\begin{aligned} f(x) &= 2x^2 - 7x - 30 \\ 2x^2 - 7x - 30 &= 0 \\ (2x+5)(x-6) &= 0 \end{aligned}$$
$2x + 5 = 0$ or $x - 6 = 0$

$x = -\frac{5}{2}$ $\quad x = 6$

17.
$$\begin{aligned} f(x) &= \tfrac{1}{3}x^3 - 2x \\ \tfrac{1}{3}x^3 - 2x &= 0 \\ (3)\left(\tfrac{1}{3}x^3 - 2x\right) &= 0(3) \\ x^3 - 6x &= 0 \\ x\left(x^2 - 6\right) &= 0 \end{aligned}$$
$$\begin{aligned} x = 0 \quad \text{or} \quad x^2 - 6 &= 0 \\ x^2 &= 6 \\ x &= \pm\sqrt{6} \end{aligned}$$

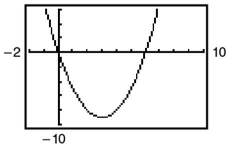
19.
$$\begin{aligned} f(x) &= x^3 - 4x^2 - 9x + 36 \\ x^3 - 4x^2 - 9x + 36 &= 0 \\ x^2(x - 4) - 9(x - 4) &= 0 \\ (x - 4)\left(x^2 - 9\right) &= 0 \end{aligned}$$
$$x - 4 = 0 \Rightarrow x = 4$$
$$x^2 - 9 = 0 \Rightarrow x = \pm 3$$

21.
$$\begin{aligned} f(x) &= \sqrt{2x} - 1 \\ \sqrt{2x} - 1 &= 0 \\ \sqrt{2x} &= 1 \\ 2x &= 1 \\ x &= \tfrac{1}{2} \end{aligned}$$

23.
$$\begin{aligned} f(x) &= \sqrt{x^2 - 1} \\ \sqrt{x^2 - 1} &= 0 \\ x^2 &= 1 \\ x &= 1, -1 \end{aligned}$$

25.
$$\begin{aligned} f(x) &= \frac{x + 3}{2x^2 - 6} \\ \frac{x + 3}{2x^2 - 6} &= 0 \\ x + 3 &= 0 \\ x &= -3 \end{aligned}$$

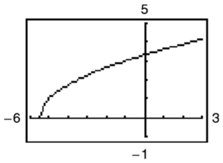
27. (a)



Zeros: $x = 0, 6$

(b)
$$\begin{aligned} f(x) &= x^2 - 6x \\ x^2 - 6x &= 0 \\ x(x - 6) &= 0 \end{aligned}$$
$$x = 0 \Rightarrow x = 0$$
$$x - 6 = 0 \Rightarrow x = 6$$

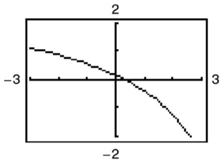
29. (a)



Zero: $x = -5.5$

(b)
$$\begin{aligned} f(x) &= \sqrt{2x + 11} \\ \sqrt{2x + 11} &= 0 \\ 2x + 11 &= 0 \\ x &= -\tfrac{11}{2} \end{aligned}$$

31. (a)



Zero: $x = 0.3333$

(b)
$$\begin{aligned} f(x) &= \frac{3x - 1}{x - 6} \\ \frac{3x - 1}{x - 6} &= 0 \\ 3x - 1 &= 0 \\ x &= \frac{1}{3} \end{aligned}$$

33. $f(x) = -\tfrac{1}{2}x^3$

The function is decreasing on $(-\infty, \infty)$.

35. $f(x) = \sqrt{x^2 - 1}$

The function is decreasing on $(-\infty, -1)$ and increasing on $(1, \infty)$.

37. $f(x) = |x + 1| + |x - 1|$

The function is increasing on $(1, \infty)$.

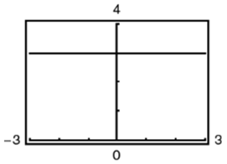
The function is constant on $(-1, 1)$.

The function is decreasing on $(-\infty, -1)$.

39. $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

The function is decreasing on $(-1, 0)$ and increasing on $(-\infty, -1)$ and $(0, \infty)$.

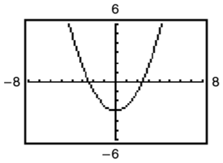
41. $f(x) = 3$



Constant on $(-\infty, \infty)$

x	−2	−1	0	1	2
$f(x)$	3	3	3	3	3

43. $g(x) = \frac{1}{2}x^2 - 3$

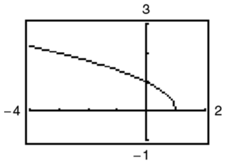


Decreasing on $(-\infty, 0)$.

Increasing on $(0, \infty)$.

x	−2	−1	0	1	2
$g(x)$	−1	$-\frac{5}{2}$	−3	$-\frac{5}{2}$	−1

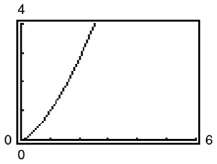
45. $f(x) = \sqrt{1 - x}$



Decreasing on $(-\infty, 1)$

x	−3	−2	−1	0	1
$f(x)$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

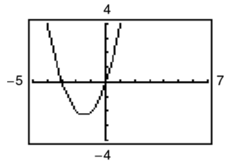
47. $f(x) = x^{3/2}$



Increasing on $(0, \infty)$

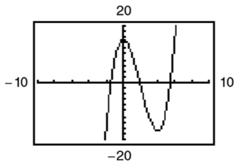
x	0	1	2	3	4
$f(x)$	0	1	2.8	5.2	8

49. $f(x) = x(x + 3)$



Relative minimum: $(-1.5, -2.25)$

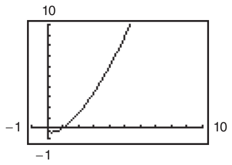
51. $h(x) = x^3 - 6x^2 + 15$



Relative minimum: $(4, -17)$

Relative maximum: $(0, 15)$

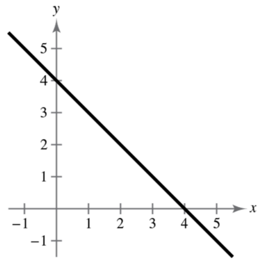
53. $h(x) = (x - 1)\sqrt{x}$



Relative minimum: $(0.33, -0.38)$

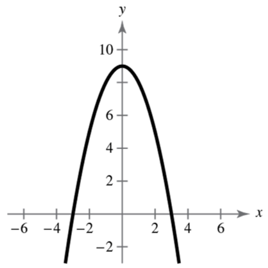
55. $f(x) = 4 - x$

$f(x) \geq 0$ on $(-\infty, 4]$



57. $f(x) = 9 - x^2$

$f(x) \geq 0$ on $[-3, 3]$



59. $f(x) = \sqrt{x-1}$

$$\begin{aligned} f(x) &\ge 0 \text{ on } [1, \infty) \\ \sqrt{x-1} &\ge 0 \\ x - 1 &\ge 0 \\ x &\ge 1 \end{aligned}$$

$[1, \infty)$

61. $f(x) = -2x + 15$

$$\frac{f(3) - f(0)}{3 - 0} = \frac{9 - 15}{3} = -2$$

The average rate of change from $x_1 = 0$ to $x_2 = 3$ is -2.

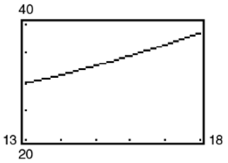
63. $f(x) = x^3 - 3x^2 - x$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{-6 - (-3)}{3} = \frac{-3}{3} = -1$$

The average rate of change from $x_1 = -1$ to $x_2 = 2$ is -1.

65. $y = 0.0729t^2 - 0.526t + 24.34, 13 \le t \le 18$

(a)



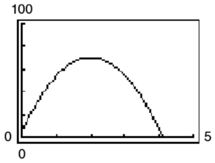
(b) $\dfrac{y(18) - y(13)}{18 - 13} \approx \dfrac{38.49 - 29.82}{5} \approx 1.73$

The amount of U.S. federal government spent on applied research increased by about \$1.73 billion each year from 2013 to 2018.

67. $s_0 = 6, v_0 = 64$

(a) $s = -16t^2 + 64t + 6$

(b)



(c) $\dfrac{s(3) - s(0)}{3 - 0} = \dfrac{54 - 6}{3} = 16$

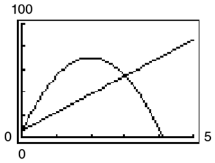
(d) The slope of the secant line is positive.

(e) $s(0) = 6, m = 16$

Secant line: $y - 6 = 16(t - 0)$

$y = 16t + 6$

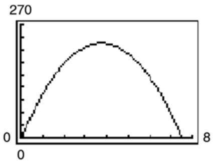
(f)



69. $v_0 = 120, s_0 = 0$

(a) $s = -16t^2 + 120t$

(b)



(c) The average rate of change from $t = 3$ to $t = 5$:

$\dfrac{s(5) - s(3)}{5 - 3} = \dfrac{200 - 216}{2} = -\dfrac{16}{2} = -8$ feet per second

(d) The slope of the secant line through $(3, s(3))$ and $(5, s(5))$ is negative.

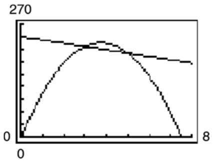
(e) The equation of the secant line: $m = -8$

Using $(5, s(5)) = (5, 200)$ we have

$y - 200 = -8(t - 5)$

$y = -8t + 240.$

(f)



71. $f(x) = x^6 - 2x^2 + 3$

$$\begin{aligned} f(-x) &= (-x)^6 - 2(-x)^2 + 3 \\ &= x^6 - 2x^2 + 3 \\ &= f(x) \end{aligned}$$

The function is even. y-axis symmetry.

73. $h(x) = x\sqrt{x+5}$

$$\begin{aligned} h(-x) &= (-x)\sqrt{-x+5} \\ &= -x\sqrt{5-x} \\ &\ne h(x) \\ &\ne -h(x) \end{aligned}$$

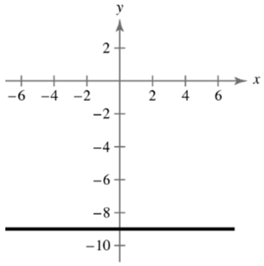
The function is neither odd nor even. No symmetry.

75. $f(s) = 4s^{3/2}$

$$\begin{aligned} &= 4(-s)^{3/2} \\ &\ne f(s) \\ &\ne -f(s) \end{aligned}$$

The function is neither odd nor even. No symmetry.

77.

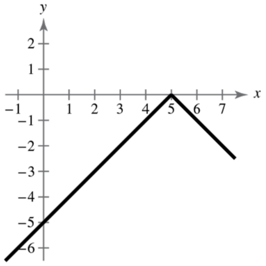


The graph of $f(x) = -9$ is symmetric to the y-axis, which implies $f(x)$ is even.

$$\begin{aligned} f(-x) &= -9 \\ &= f(x) \end{aligned}$$

The function is even.

79. $f(x) = -|x - 5|$

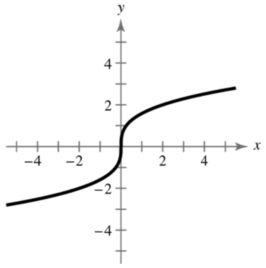


The graph displays no symmetry, which implies $f(x)$ is neither odd nor even.

$$\begin{aligned} f(x) &= -|(-x) - 5| \\ &= -|-x - 5| \\ &\neq f(x) \\ &\neq -f(x) \end{aligned}$$

The function is neither even nor odd.

81. $f(x) = \sqrt[3]{4x}$



The graph displays origin symmetry, which implies $f(x)$ is odd.

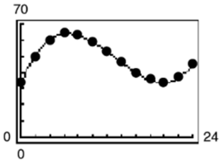
$$\begin{aligned} f(-x) &= \sqrt[3]{4(-x)} \\ &= \sqrt[3]{-4x} \\ &= -\sqrt[3]{4x} \\ &= -f(x) \end{aligned}$$

The function is odd.

83. $$\begin{aligned} h &= \text{top} - \text{bottom} \\ &= 3 - (4x - x^2) \\ &= 3 - 4x + x^2 \end{aligned}$$

85. $$\begin{aligned} L &= \text{right} - \text{left} \\ &= 2 - \sqrt[3]{2y} \end{aligned}$$

87. (a)



(b) The model is an excellent fit.

(c) The temperature was increasing from 6 A.M. until noon $(x = 0 \text{ to } x = 6)$. Then it decreases until 2 A.M. $(x = 6 \text{ to } x = 20)$. Then the temperature increases until 6 A.M. $(x = 20 \text{ to } x = 24)$.

(d) The maximum temperature according to the model is about 63.93°F. According to the data, it is 64°F. The minimum temperature according to the model is about 33.98°F. According to the data, it is 34°F.

(e) Answers may vary. Temperatures will depend upon the weather patterns, which usually change from day to day.

89. False. The function $f(x) = \sqrt{x^2 + 1}$ has a domain of all real numbers.

91. The error is that $-2x^3 - 5 \neq -(2x^3 - 5)$. The correct process is as follows.

$$\begin{aligned} f(x) &= 2x^3 - 5 \\ f(-x) &= 2(-x)^3 - 5 \\ &= -2x^3 - 5 \\ &= -(2x^3 + 5) \end{aligned}$$

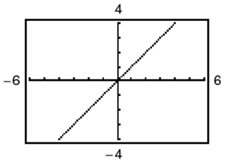
$f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$, so the function $f(x) = 2x^3 - 5$ is neither odd nor even.

93. $\left(-\frac{5}{3}, -7\right)$

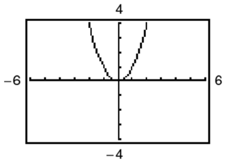
(a) If f is even, another point is $\left(\frac{5}{3}, -7\right)$.

(b) If f is odd, another point is $\left(\frac{5}{3}, 7\right)$.

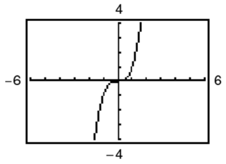
95. (a) $y = x$



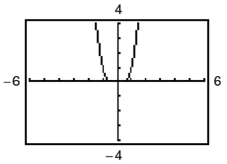
(b) $y = x^2$



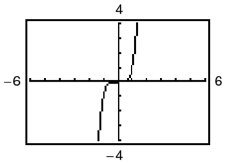
(c) $y = x^3$



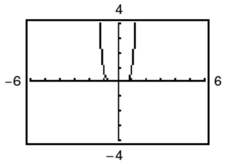
(d) $y = x^4$



(e) $y = x^5$



(f) $y = x^6$



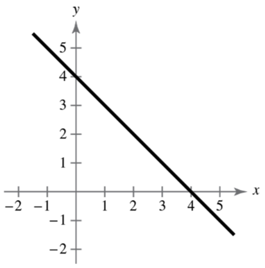
All the graphs pass through the origin. The graphs of the odd powers of x are symmetric with respect to the origin and the graphs of the even powers are symmetric with respect to the y-axis. As the powers increase, the graphs become flatter in the interval $-1 < x < 1$.

97. $(1, 3), (4, 0)$

Slope: $m = \dfrac{3 - 0}{1 - 4} = -1$

$y - 0 = -1(x - 4)$

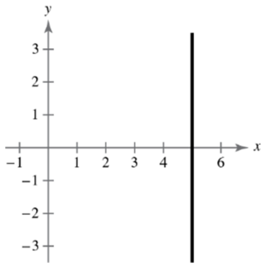
$y = -x + 4$



99. $(5, 0), (5, 1)$

Slope: $\dfrac{0 - 1}{5 - 5} = \dfrac{-1}{0} \Rightarrow m$ is undefined.

$x = 5$



101. $f(x) = 5x - 3$

(a) $f(-3) = 5(-3) - 3 = -15 - 3 = -18$

(b) $f(3) = 5(3) - 3 = 15 - 3 = 12$

(c) $f(x + 3) = 5(x + 3) - 3 = 5x + 15 - 3 = 5x + 12$

103. $f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$

(a) $f(0) = 2(0) + 3 = 3$

(b) $f(1) = 2(1) + 3 = 5$

(c) $f(2) = -2 + 4 = 2$

105. *Verbal Model:* (Sum) = (first number) + (second number)

Labels: Sum $= S$, first number $= n$, second number $= n + 1$

Equation: $S = n + (n + 1) = 2n + 1$

Section 1.6 A Library of Parent Functions

1. Greatest integer function

3. Reciprocal function

5. Square root function

7. Absolute value function

9. Linear function

11. (a) $f(1) = 4, f(0) = 6$

$(1, 4), (0, 6)$

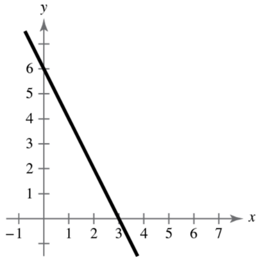
$$m = \frac{6 - 4}{0 - 1} = -2$$

$$y - 6 = -2(x - 0)$$

$$y = -2x + 6$$

$$f(x) = -2x + 6$$

(b)



13. (a) $f\left(\frac{1}{2}\right) = -\frac{5}{3}, f(6) = 2$

$\left(\frac{1}{2}, -\frac{5}{3}\right), (6, 2)$

$$m = \frac{2 - \left(-\frac{5}{3}\right)}{6 - \left(\frac{1}{2}\right)}$$

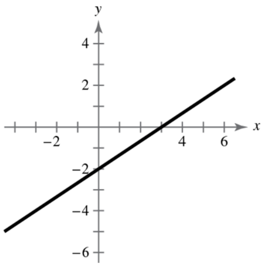
$$= \frac{\frac{11}{3}}{\frac{11}{2}} = \left(\frac{11}{3}\right) \cdot \left(\frac{2}{11}\right) = \frac{2}{3}$$

$$f(x) - 2 = \tfrac{2}{3}(x - 6)$$

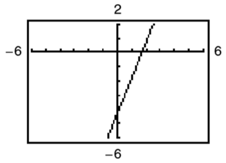
$$f(x) - 2 = \tfrac{2}{3}x - 4$$

$$f(x) = \tfrac{2}{3}x - 2$$

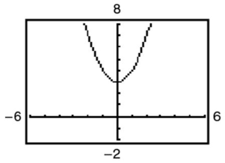
(b)



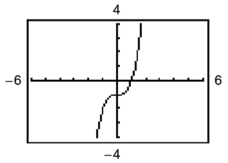
15. $f(x) = 2.5x - 4.25$



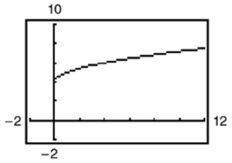
17. $g(x) = x^2 + 3$



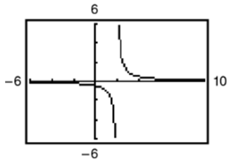
19. $f(x) = x^3 - 1$



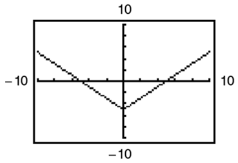
21. $f(x) = \sqrt{x} + 4$



23. $f(x) = \dfrac{1}{x - 2}$



25. $g(x) = |x| - 5$



27. $f(x) = \llbracket x \rrbracket$

(a) $f(2.1) = 2$

(b) $f(2.9) = 2$

(c) $f(-3.1) = -4$

(d) $f\left(\frac{7}{2}\right) = 3$

29. $k(x) = \llbracket 2x + 1 \rrbracket$

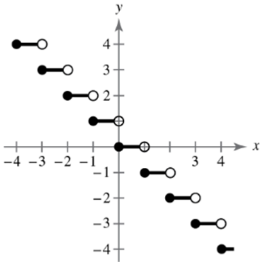
(a) $k\left(\frac{1}{3}\right) = \llbracket 2\left(\frac{1}{3}\right) + 1 \rrbracket = \llbracket \frac{5}{3} \rrbracket = 1$

(b) $k(-2.1) = \llbracket 2(-2.1) + 1 \rrbracket = \llbracket -3.1 \rrbracket = -4$

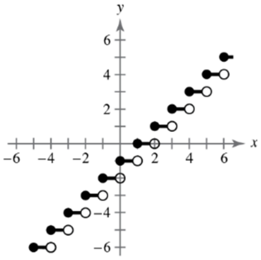
(c) $k(1.1) = \llbracket 2(1.1) + 1 \rrbracket = \llbracket 3.2 \rrbracket = 3$

(d) $k\left(\frac{2}{3}\right) = \llbracket 2\left(\frac{2}{3}\right) + 1 \rrbracket = \llbracket \frac{7}{3} \rrbracket = 2$

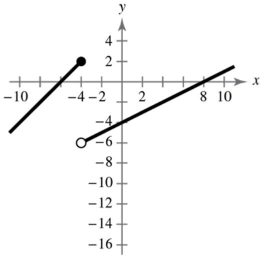
31. $g(x) = -\llbracket x \rrbracket$



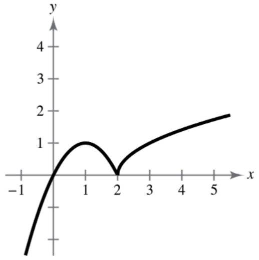
33. $g(x) = \llbracket x \rrbracket - 1$



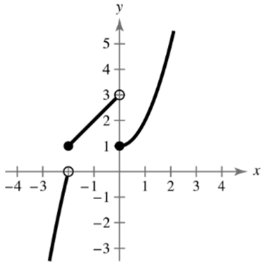
35. $g(x) = \begin{cases} x + 6, & x \le -4 \\ \frac{1}{2}x - 4, & x > -4 \end{cases}$



37. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \le 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$

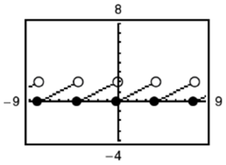


39. $h(x) = \begin{cases} 4 - x^2, & x < -2 \\ 3 + x, & -2 \le x < 0 \\ x^2 + 1, & x \ge 0 \end{cases}$



41. $s(x) = 2\left(\frac{1}{4}x - \llbracket \frac{1}{4}x \rrbracket\right)$

(a)



(b) Domain: $(-\infty, \infty)$; Range: $[0, 2)$

43. (a) $W(30) = 14(30) = 420$

$W(40) = 14(40) = 560$

$W(45) = 21(45 - 40) + 560 = 665$

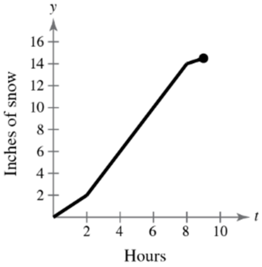
$W(50) = 21(50 - 40) + 560 = 770$

(b) $W(h) = \begin{cases} 14h, & 0 < h \le 36 \\ 21(h - 36) + 504, & h > 36 \end{cases}$

(c) $W(h) = \begin{cases} 16h, & 0 < h \le 40 \\ 24(h - 40) + 640, & h > 40 \end{cases}$

45. For the first two hours, the slope is 1. For the next six hours, the slope is 2. For the final hour, the slope is $\frac{1}{2}$.

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2t - 2, & 2 < t \le 8 \\ \frac{1}{2}t + 10, & 8 < t \le 9 \end{cases}$$



To find $f(t) = 2t - 2$, use $m = 2$ and $(2, 2)$.

$y - 2 = 2(t - 2) \Rightarrow y = 2t - 2$

To find $f(t) = \frac{1}{2}t + 10$, use $m = \frac{1}{2}$ and $(8, 14)$.

$y - 14 = \frac{1}{2}(t - 8) \Rightarrow y = \frac{1}{2}t + 10$

Total accumulation $= 14.5$ inches

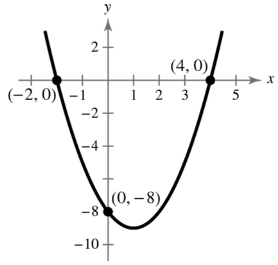
47. False. A piecewise-defined function is a function that is defined by two or more equations over a specified domain. That domain may or may not include *x*- and *y*-intercepts.

49. According to the graph, the domains should be $x \le 3$ and $x > 3$.

51. (a) Yes. The amount that you pay in sales tax will increase as the price of the item purchased increases.

(b) No. The length of time that you study the night before an exam does not necessarily determine your score on the exam.

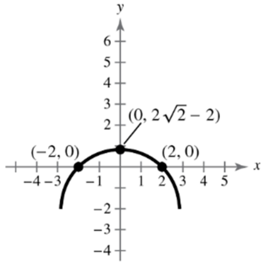
53. $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$



x-intercepts: $(4, 0), (-2, 0)$

y-intercept: $(0, -8)$

55. $p(x) = \sqrt{8 - x^2} - 2$



x-intercepts: $(2, 0), (-2, 0)$

y-intercept: $\left(0, 2\sqrt{2} - 2\right)$

57. $$\frac{\sqrt[3]{(-x)^2} + \sqrt[3]{-x}}{-x} = -\frac{\sqrt[3]{x^2} + \sqrt[3]{x}}{x} = \frac{-\left(x^{1/3} - 1\right)}{x^{2/3}}$$

Answers will vary.

59. $$\begin{aligned}(x - 3)^2 - 3(x - 3) + 2 &= x^2 - 6x + 9 - 3x + 9 + 2 \\ &= x^2 - 9x + 20 \\ &= (x - 5)(x - 4)\end{aligned}$$

Answers will vary.

Section 1.7 Transformations of Functions

1. $-f(x); f(-x)$

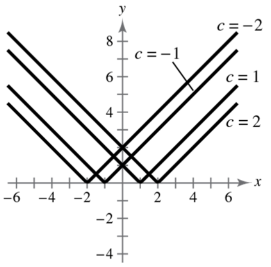
3. The three types of rigid transformations are horizontal shifts, vertical shifts, and reflections.

5. (a)

$f(x) = \lvert x \rvert + c$	Vertical shifts
$c = -2$: $f(x) = \lvert x \rvert - 2$	2 units down
$c = -1$: $f(x) = \lvert x \rvert - 1$	1 unit down
$c = 1$: $f(x) = \lvert x \rvert + 1$	1 unit up
$c = 2$: $f(x) = \lvert x \rvert + 2$	2 units up

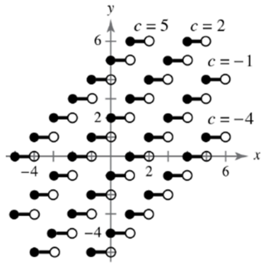
(b)

$f(x) = \lvert x - c \rvert$	Horizontal shifts
$c = -2$: $f(x) = \lvert x - (-2) \rvert = \lvert x + 2 \rvert$	2 units left
$c = -1$: $f(x) = \lvert x - (-1) \rvert = \lvert x + 1 \rvert$	1 unit left
$c = 1$: $f(x) = \lvert x - (1) \rvert = \lvert x - 1 \rvert$	1 unit right
$c = 2$: $f(x) = \lvert x - (2) \rvert = \lvert x - 2 \rvert$	2 units right



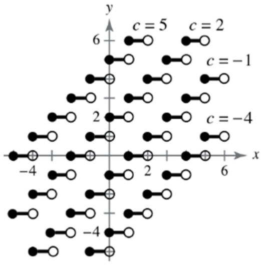
7. (a)

$f(x) = \llbracket x \rrbracket + c$	Vertical shifts
$c = -4$: $f(x) = \llbracket x \rrbracket - 4$	4 units down
$c = -1$: $f(x) = \llbracket x \rrbracket - 1$	1 unit down
$c = 2$: $f(x) = \llbracket x \rrbracket + 2$	2 units up
$c = 5$: $f(x) = \llbracket x \rrbracket + 5$	5 units up



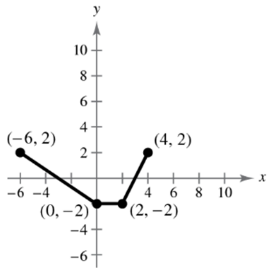
(b)

$f(x) = \llbracket x + c \rrbracket$	Horizontal shifts
$c = -4$: $f(x) = \llbracket x - (-4) \rrbracket = \llbracket x + 4 \rrbracket$	4 units left
$c = -1$: $f(x) = \llbracket x - (-1) \rrbracket = \llbracket x + 1 \rrbracket$	1 unit left
$c = 2$: $f(x) = \llbracket x - (2) \rrbracket = \llbracket x - 2 \rrbracket$	2 units right
$c = 5$: $f(x) = \llbracket x - (5) \rrbracket = \llbracket x - 5 \rrbracket$	5 units right



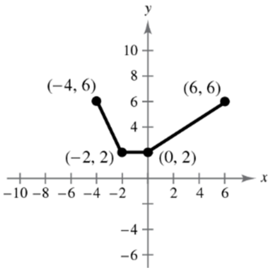
9. (a) $y = f(-x)$

Reflection in the y-axis



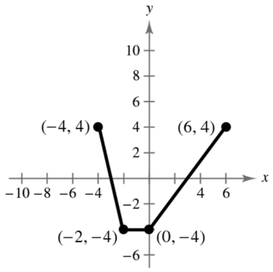
(b) $y = f(x) + 4$

Vertical shift 4 units upward



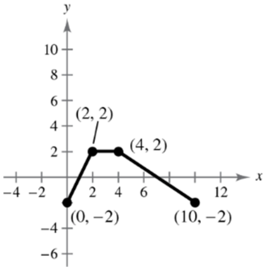
(c) $y = 2f(x)$

Vertical stretch (each y-value is multiplied by 2)



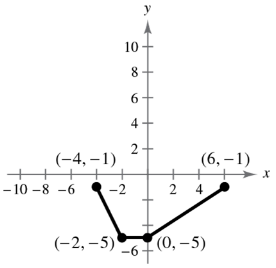
(d) $y = -f(x - 4)$

Reflection in the x-axis and a horizontal shift 4 units to the right



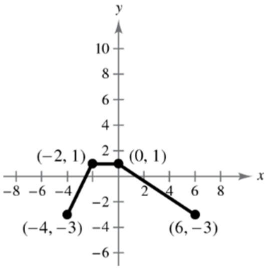
(e) $y = f(x) - 3$

Vertical shift 3 units downward



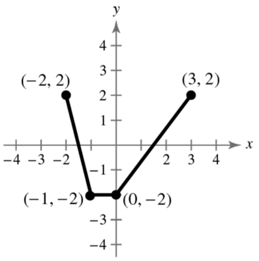
(f) $y = -f(x) - 1$

Reflection in the x-axis and a vertical shift 1 unit downward



(g) $y = f(2x)$

Horizontal shrink
(each x-value is divided by 2)



11. Parent function: $f(x) = x^2$

(a) Vertical shift 1 unit downward

$g(x) = x^2 - 1$

(b) Reflection in the x-axis, horizontal shift 1 unit to the left, and a vertical shift 1 unit upward

$g(x) = -(x + 1)^2 + 1$

13. Parent function: $f(x) = |x|$

(a) Reflection in the x-axis and a horizontal shift 3 units to the left

$g(x) = -|x + 3|$

(b) Horizontal shift 2 units to the right and a vertical shift 4 units downward

$g(x) = |x - 2| - 4$

15. Parent function: $f(x) = x^3$

Horizontal shift 2 units to the right

$y = (x - 2)^3$

17. Parent function: $f(x) = x^2$

Reflection in the x-axis

$y = -x^2$

19. Parent function: $f(x) = \sqrt{x}$

Reflection in the x-axis and a vertical shift 1 unit upward

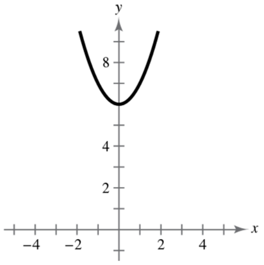
$y = -\sqrt{x} + 1$

21. $g(x) = x^2 + 6$

(a) Parent function: $f(x) = x^2$

(b) A vertical shift 6 units upward

(c)



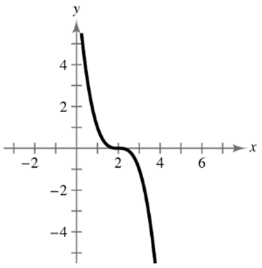
(d) $g(x) = f(x) + 6$

23. $g(x) = -(x - 2)^3$

(a) Parent function: $f(x) = x^3$

(b) Horizontal shift of 2 units to the right and a reflection in the x-axis

(c)



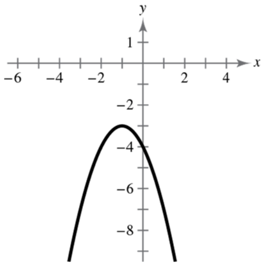
(d) $g(x) = -f(x - 2)$

25. $g(x) = -3 - (x + 1)^2$

(a) Parent function: $f(x) = x^2$

(b) Reflection in the x-axis, a vertical shift 3 units downward and a horizontal shift 1 unit left

(c)



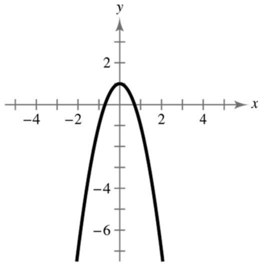
(d) $g(x) = -f(x + 1) - 3$

27. $g(x) = -2x^2 + 1$

(a) Parent function: $f(x) = x^2$

(b) A vertical stretch, reflection in the x-axis and a vertical shift 1 unit upward

(c)



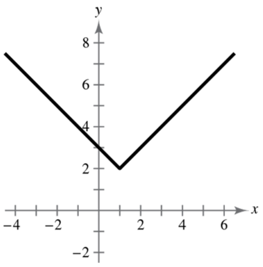
(d) $g(x) = -2f(x) + 1$

29. $g(x) = |x - 1| + 2$

(a) Parent function: $f(x) = |x|$

(b) A horizontal shift 1 unit right and a vertical shift 2 units upward

(c)



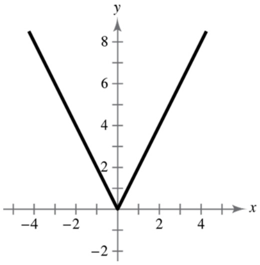
(d) $g(x) = f(x - 1) + 2$

31. $g(x) = |2x|$

(a) Parent function: $f(x) = |x|$

(b) A horizontal shrink

(c)



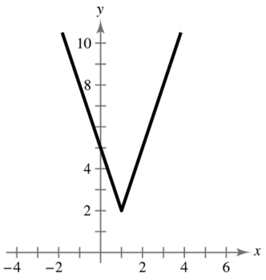
(d) $g(x) = f(2x)$

33. $g(x) = 3|x - 1| + 2$

(a) Parent function: $f(x) = |x|$

(b) A horizontal shift of 1 unit to the right, a vertical stretch, and a vertical shift 2 units upward

(c)



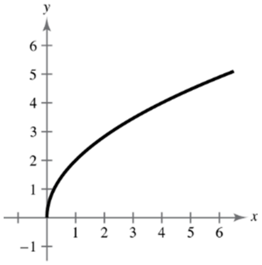
(d) $g(x) = 3f(x - 1) + 2$

35. $g(x) = 2\sqrt{x}$

(a) Parent function: $f(x) = \sqrt{x}$

(b) A vertical stretch (each y value is multiplied by 2)

(c)



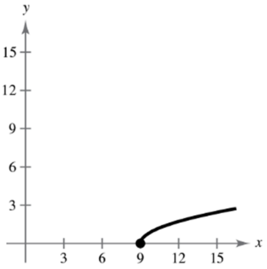
(d) $g(x) = 2f(x)$

37. $g(x) = \sqrt{x - 9}$

(a) Parent function: $f(x) = \sqrt{x}$

(b) horizontal shift 9 units to the right

(c)



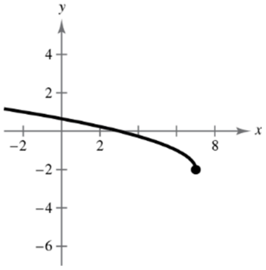
(d) $g(x) = f(x - 9)$

39. $g(x) = \sqrt{7 - x} - 2$ or $g(x) = \sqrt{-(x - 7) - 2}$

(a) Parent function: $f(x) = \sqrt{x}$

(b) Reflection in the y-axis, horizontal shift 7 units to the right, and a vertical shift 2 units downward

(c)



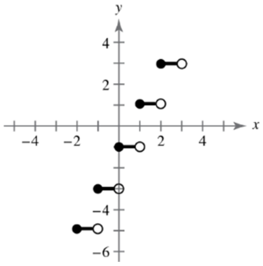
(d) $g(x) = f(7 - x) - 2$

41. $g(x) = 2[\![x]\!] - 1$

(a) Parent function: $f(x) = [\![x]\!]$

(b) A vertical shift of 1 unit downward and a vertical stretch (each y value is multiplied by 2)

(c)



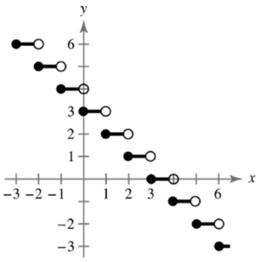
(d) $g(x) = 2f(x) - 1$

43. $g(x) = 3 - [\![x]\!]$

(a) Parent function: $f(x) = [\![x]\!]$

(b) Reflection in the x-axis and a vertical shift 3 units upward

(c)



(d) $g(x) = 3 - f(x)$

45. $g(x) = (x - 3)^2 - 7$

47. $f(x) = x^3$ moved 13 units to the right

$g(x) = (x - 13)^3$

49. $g(x) = -|x| + 12$

51. $f(x) = \sqrt{x}$ moved 6 units to the left and reflected in both the x- and y-axes

$g(x) = -\sqrt{-x + 6}$

53. $f(x) = x^2$

(a) Reflection in the x-axis and a vertical stretch (each y-value is multiplied by 3)

$g(x) = -3x^2$

(b) Vertical shift 3 units upward and a vertical stretch (each y-value is multiplied by 4)

$g(x) = 4x^2 + 3$

55. $f(x) = |x|$

(a) Reflection in the x-axis and a vertical shrink (each y-value is multiplied by $\frac{1}{2}$)

$g(x) = -\frac{1}{2}|x|$

(b) Vertical stretch (each y-value is multiplied by 3) and a vertical shift 3 units downward

$g(x) = 3|x| - 3$

57. Parent function: $f(x) = x^3$

Vertical stretch (each y-value is multiplied by 2)

$g(x) = 2x^3$

59. Parent function: $f(x) = x^2$

Reflection in the x-axis, vertical shrink (each y-value is multiplied by $\frac{1}{2}$)

$g(x) = -\frac{1}{2}x^2$

61. Parent function: $f(x) = \sqrt{x}$

Reflection in the y-axis, vertical shrink (each y-value is multiplied by $\frac{1}{2}$)

$g(x) = \frac{1}{2}\sqrt{-x}$

63. Parent function: $f(x) = x^3$

Reflection in the x-axis, horizontal shift 2 units to the right and a vertical shift 2 units upward

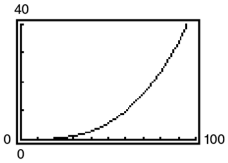
$g(x) = -(x - 2)^3 + 2$

65. Parent function: $f(x) = \sqrt{x}$

Reflection in the x-axis and a vertical shift 3 units downward

$g(x) = -\sqrt{x} - 3$

67. (a)



(b)

$$\begin{aligned} H(x) &= 0.00004636x^3 \\ H\left(\frac{x}{1.6}\right) &= 0.00004636\left(\frac{x}{1.6}\right)^3 \\ &= 0.00004636\left(\frac{x^3}{4.096}\right) \\ &= 0.0000113184x^3 = 0.00001132x^3 \end{aligned}$$

The graph of $H\left(\frac{x}{1.6}\right)$ is a horizontal stretch of the graph of $H(x)$.

69. False. $y = f(-x)$ is a reflection in the y-axis.

71. True. Because $|x| = |-x|$, the graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

73. $y = f(x + 2) - 1$

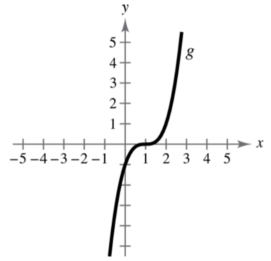
Horizontal shift 2 units to the left and a vertical shift 1 unit downward

$(0, 1) \to (0 - 2, 1 - 1) = (-2, 0)$

$(1, 2) \to (1 - 2, 2 - 1) = (-1, 1)$

$(2, 3) \to (2 - 2, 3 - 1) = (0, 2)$

75. Since the graph of $g(x)$ is a horizontal shift one unit to the right of $f(x) = x^3$, the equation should be $g(x) = (x - 1)^3$ and not $g(x) = (x + 1)^3$.



77. No. $g(x) = -x^4 - 2$. Yes. $h(x) = -(x - 3)^4$

79. $(2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$

81. $\begin{aligned}[t] (3x^2 + x - 1) - (1 - x) &= 3x^2 + x - 1 - 1 + x \\ &= 3x^2 + 2x - 2 \end{aligned}$

83. $x^2(x - 3) = x^3 - 3x^2$

85. $-2x(0.1x + 17) = (-2x)(0.1x) + (-2x)(17)$
$= -0.2x^2 - 34x$

87. $(3x + 5) \div (6x^2 + 10x) = \dfrac{3x + 5}{6x^2 + 10x} = \dfrac{3x + 5}{2x(3x + 5)}$
$= \dfrac{1}{2x}, x \neq \dfrac{-5}{3}$

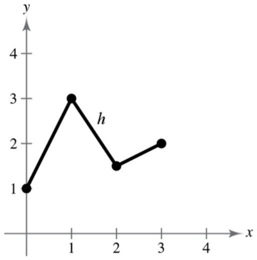
Section 1.8 Combinations of Functions: Composite Functions

1. addition; subtraction; multiplication; division

3. Since $(fg)(x) = 2x(x^2 + 1)$ and $f(x) = x^2 + 1$, $g(x) = 2x$, and $(fg)(x) = (gf)(x) = (2x)f(x)$.

5.

x	0	1	2	3
f	2	3	1	2
g	-1	0	$\frac{1}{2}$	0
$f + g$	1	3	$\frac{3}{2}$	2



7. $f(x) = x + 2, g(x) = x - 2$

(a) $(f + g)(x) = f(x) + g(x)$
$= (x + 2) + (x - 2)$
$= 2x$

(b) $(f - g)(x) = f(x) - g(x)$
$= (x + 2) - (x - 2)$
$= 4$

(c) $(fg)(x) = f(x) \cdot g(x)$
$= (x + 2)(x - 2)$
$= x^2 - 4$

(d) $\left(\dfrac{f}{x}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x + 2}{x - 2}$

Domain: all real numbers x except $x = 2$

9. $f(x) = x^2, g(x) = 4x - 5$

(a) $(f + g)(x) = f(x) + g(x)$
$= x^2 + (4x - 5)$
$= x^2 + 4x - 5$

(b) $(f - g)(x) = f(x) - g(x)$
$= x^2 - (4x - 5)$
$= x^2 - 4x + 5$

(c) $(fg)(x) = f(x) \cdot g(x)$
$= x^2(4x - 5)$
$= 4x^3 - 5x^2$

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
$= \dfrac{x^2}{4x - 5}$

Domain: all real numbers x except $x = \dfrac{5}{4}$

11. $f(x) = x^2 + 6, g(x) = \sqrt{1 - x}$

(a) $(f + g)(x) = f(x) + g(x) = x^2 + 6 + \sqrt{1 - x}$

(b) $(f - g)(x) = f(x) - g(x) = x^2 + 6 - \sqrt{1 - x}$

(c) $(fg)(x) = f(x) \cdot g(x) = (x^2 + 6)\sqrt{1 - x}$

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^2 + 6}{\sqrt{1 - x}} = \dfrac{(x^2 + 6)\sqrt{1 - x}}{1 - x}$

Domain: $x < 1$

13. $f(x) = \dfrac{x}{x+1},\ g(x) = x^3$

(a) $(f + g)(x) = \dfrac{x}{x+1} + x^3 = \dfrac{x + x^4 + x^3}{x+1}$

(b) $(f - g)(x) = \dfrac{x}{x+1} - x^3 = \dfrac{x - x^4 - x^3}{x+1}$

(c) $(fg)(x) = \dfrac{x}{x+1} \cdot x^3 = \dfrac{x^4}{x+1}$

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x}{x+1} \div x^3 = \dfrac{x}{x+1} \cdot \dfrac{1}{x^3}$

$= \dfrac{1}{x^2(x+1)}$

Domain: all real numbers x except $x = 0$ and $x = -1$

15. $(f + g)(2) = f(2) - g(2)$

$= (2 + 3) - (2^2 - 2)$

$= 7$

17. $(f - g)(3t) = f(3t) - g(3t)$

$= ((3t) + 3) - ((3t)^2 - 2)$

$= 3t + 3 - (9t^2 - 2)$

$= -9t^2 + 3t + 5$

19. $(fg)(6) = f(6)g(6)$

$= ((6) + 3)((6)^2 - 2)$

$= (9)(34)$

$= 306$

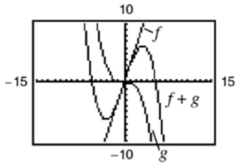
21. $(f/g)(5) = f(5)\ /\ g(5)$

$= ((5) + 3)\ /\ ((5)^2 - 2)$

$= \dfrac{8}{23}$

23. $f(x) = 3x,\ g(x) = -\dfrac{x^3}{10}$

$(f + g)(x) = 3x - \dfrac{x^3}{10}$

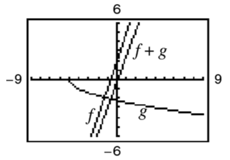


For $0 \le x \le 2$, $f(x)$ contributes most to the magnitude.

For $x > 6$, $g(x)$ contributes most to the magnitude.

25. $f(x) = 3x + 2,\ g(x) = -\sqrt{x + 5}$

$(f + g)x = 3x - \sqrt{x + 5} + 2$



For $0 \le x \le 2$, $f(x)$ contributes most to the magnitude.

For $x > 6$, $f(x)$ contributes most to the magnitude.

27. $f(x) = x + 8,\ g(x) = x - 3$

(a) $(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3) + 8 = x + 5$

(b) $(g \circ f)(x) = g(f(x)) = g(x + 8) = (x + 8) - 3 = x + 5$

(c) $(g \circ g)(x) = g(g(x)) = g(x - 3) = (x - 3) - 3 = x - 6$

29. $f(x) = \sqrt[3]{x - 1},\ g(x) = x^3 + 1$

(a) $(f \circ g)(x) = f(g(x))$

$= f(x^3 + 1)$

$= \sqrt[3]{(x^3 + 1) - 1}$

$= \sqrt[3]{x^3} = x$

(b) $(g \circ f)(x) = g(f(x))$

$= g\left(\sqrt[3]{x - 1}\right)$

$= \left(\sqrt[3]{x - 1}\right)^3 + 1$

$= (x - 1) + 1 = x$

(c) $(g \circ g)(x) = g(g(x))$

$= g(x^3 + 1)$

$= (x^3 + 1)^3 + 1$

$= x^9 + 3x^6 + 3x^3 + 2$

31. $f(x) = \sqrt{x + 4}$ Domain: $x \geq -4$

$g(x) = x^2$ Domain: all real numbers x

(a) $(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{x^2 + 4}$

Domain: all real numbers x

(b) $(g \circ f)(x) = g(f(x))$

$= g(\sqrt{x + 4}) = (\sqrt{x + 4})^2 = x + 4$

Domain: $x \geq -4$

33. $f(x) = |x|$ Domain: all real numbers x

$g(x) = x + 6$ Domain: all real numbers x

(a) $(f \circ g)(x) = f(g(x)) = f(x + 6) = |x + 6|$

Domain: all real numbers x

(b) $(g \circ f)(x) = g(f(x)) = g(|x|) = |x| + 6$

Domain: all real numbers x

35. $f(x) = \dfrac{1}{x}$ Domain: all real numbers x except $x = 0$

$g(x) = x + 3$ Domain: all real numbers x

(a) $(f \circ g)(x) = f(g(x)) = f(x + 3) = \dfrac{1}{x + 3}$

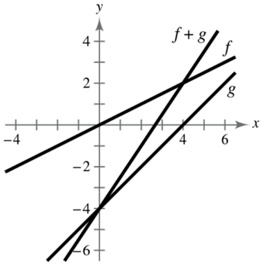
Domain: all real numbers x except $x = -3$

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = \dfrac{1}{x} + 3$

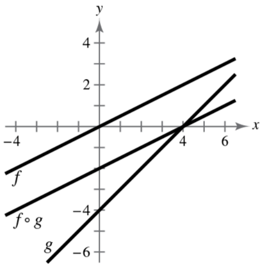
Domain: all real numbers x except $x = 0$

37. $f(x) = \frac{1}{2}x$, $g(x) = x - 4$

(a)



(b)



39. (a) $(f + g)(3) = f(3) + g(3) = 2 + 1 = 3$

(b) $\left(\dfrac{f}{g}\right)(2) = \dfrac{f(2)}{g(2)} = \dfrac{0}{2} = 0$

41. (a) $(f \circ g)(2) = f(g(2)) = f(2) = 0$

(b) $(g \circ f)(2) = g(f(2)) = g(0) = 4$

43. $h(x) = (2x^2 + 1)^2$

One possibility: Let $f(x) = x^2$ and $g(x) = 2x + 1$, then $(f \circ g)(x) = h(x)$.

45. $h(x) = \sqrt[3]{x^2 - 4}$

One possibility: Let $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - 4$, then $(f \circ g)(x) = h(x)$.

47. $h(x) = \dfrac{1}{x + 2}$

One possibility: Let $f(x) = 1/x$ and $g(x) = x + 2$, then $(f \circ g)(x) = h(x)$.

49. $h(x) = \dfrac{-x^2 + 3}{4 - x^2}$

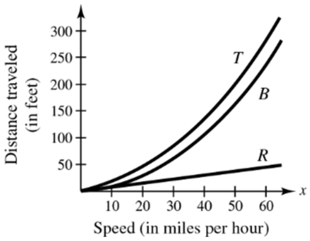
One possibility: Let $f(x) = \dfrac{x + 3}{4 + x}$ and $g(x) = -x^2$, then $(f \circ g)(x) = h(x)$.

51. $h(x) = \sqrt{\dfrac{1}{x^2 + 1}}$

One possibility: Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x^2 + 1}$, then $(f \circ g)(x) = f\left(\dfrac{1}{x^2 + 1}\right) = \sqrt{\dfrac{1}{x^2 + 1}}$.

53. (a) $T(x) = R(x) + B(x) = \frac{3}{4}x + \frac{1}{15}x^2$

(b)



(c) $B(x)$; As x increases, $B(x)$ increases at a faster rate.

55. (a) $c(t) = \dfrac{b(t) - d(t)}{p(t)} \times 100 =$ percent change

(b) $c(20)$ represents the percent change in the population due to births and deaths in the year 2020.

57. (a) $r(x) = \dfrac{x}{2}$

(b) $A(r) = \pi r^2$

(c) $(A \circ r)(x) = A(r(x)) = A\left(\dfrac{x}{2}\right) = \pi\left(\dfrac{x}{2}\right)^2$

$(A \circ r)(x)$ represents the area of the circular base of the tank on the square foundation with side length x.

59. False. $(f \circ g)(x) = 6x + 1$ and $(g \circ f)(x) = 6x + 6$

61. (a) Answer not unique. *Sample answer*:

$f(x) = x + 3,\ g(x) = x + 2$

$(f \circ g)(x) = f(g(x)) = (x + 2) + 3 = x + 5$

$(g \circ f)(x) = g(f(x)) = (x + 3) + 2 = x + 5$

(b) Answer not unique. *Sample answer*: $f(x) = x^2$, $g(x) = x^3$

$(f \circ g)(x) = f(g(x)) = (x^3)^2 = x^6$

$(g \circ f)(x) = g(f(x)) = (x^2)^3 = x^6$

63. Let $f(x)$ and $g(x)$ be two odd functions and define $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= [-f(x)][-g(x)] \quad \text{because } f \text{ and } g \text{ are odd} \\ &= f(x)g(x) \\ &= h(x). \end{aligned}$$

So, $h(x)$ is even.

Let $f(x)$ and $g(x)$ be two even functions and define $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= f(x)g(x) \quad \text{because } f \text{ and } g \text{ are even} \\ &= h(x). \end{aligned}$$

So, $h(x)$ is even.

65. Let $f(x)$ be an odd function, $g(x)$ be an even function, and define $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= [-f(x)]g(x) \quad \text{because } f \text{ is odd and } g \text{ is even} \\ &= -f(x)g(x) \\ &= -h(x). \end{aligned}$$

So, h is odd and the product of an odd function and an even function is odd.

67. $y^2 = x - 5$

$y^2 = (-x) - 5 \Rightarrow y^2 = -x - 5 \Rightarrow$ No y-axis symmetry

$(-y)^2 = x - 5 \Rightarrow y^2 = x - 5 \Rightarrow x$-axis symmetry

$(-y)^2 = (-x) - 5 \Rightarrow y^2 = -x - 5 \Rightarrow$ No origin symmetry

69. $y = x^2 + 1$

$y = (-x)^2 + 1 \Rightarrow y = x^2 + 1 \Rightarrow y$-axis symmetry

$-y = x^2 + 1 \Rightarrow y = -x^2 - 1 \Rightarrow$ No x-axis symmetry

$-y = (-x)^2 + 1 \Rightarrow -y = x^2 + 1 \Rightarrow$

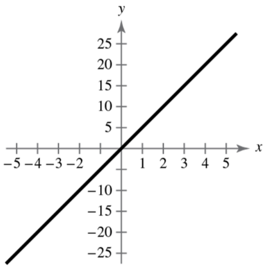
$y = -x^2 - 1 \Rightarrow$ No origin symmetry

71. $y = 5x$

$f(x) = 5x$

$f(-x) = 5(-x) = -5x = -f(x)$

Odd function

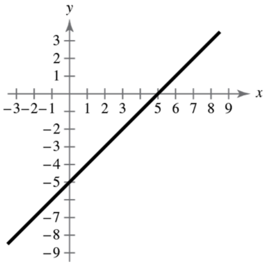


73. $y = x - 5$

$f(x) = x - 5$

$f(-x) = -x - 5$

Neither even nor odd



75. $2x + 3y = 5$

$3y = 5 - 2x$

$y = \frac{5}{3} - \frac{2}{3}x$

77. $x = \sqrt{y + 1}$

$x^2 = y + 1$

$y = x^2 - 1, x \geq 0$

79. $A = \pi(r + 2)^2 - \pi r^2$

$= \pi\left[(r + 2)^2 - r^2\right]$

$= \pi\left[r^2 + 4r + 4 - r^2\right]$

$= \pi(4r + 4)$

$= 4\pi(r + 1)$

Section 1.9 Inverse Functions

1. inverse

3. range; domain

5. To show that two function f and g are inverse functions, you must show that $f(g(x)) = x$ and $g(f(x)) = x$.

7. If a function is one-to-one, no two x-values in the domain can correspond to the same y-value in the range. Therefore, a horizontal line can intersect the graph at most once.

9. $f(x) = 6x$

$f^{-1}(x) = \frac{x}{6} = \frac{1}{6}x$

$f(f^{-1}(x)) = f\left(\frac{x}{6}\right) = 6\left(\frac{x}{6}\right) = x$

$f^{-1}(f(x)) = f^{-1}(6x) = \frac{6x}{6} = x$

11. $f(x) = 3x + 1$

$f^{-1}(x) = \frac{x - 1}{3}$

$f(f^{-1}(x)) = f\left(\frac{x - 1}{3}\right) = 3\left(\frac{x - 1}{3}\right) + 1 = x$

$f^{-1}(f(x)) = f^{-1}(3x + 1) = \frac{(3x + 1) - 1}{3} = x$

13. $f(x) = x^3 + 1$

$$f^{-1}(x) = \sqrt[3]{x - 1}$$

$$f\left(f^{-1}(x)\right) = f\left(\sqrt[3]{x - 1}\right) = \left(\sqrt[3]{x - 1}\right)^3 + 1 = (x - 1) + 1 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(x^3 + 1\right) = \sqrt[3]{\left(x^3 + 1\right) - 1} = \sqrt[3]{x^3} = x$$

15. $f(x) = x^2 - 4, x \geq 0$

$$f^{-1}(x) = \sqrt{x + 4}$$

$$f\left(f^{-1}(x)\right) = f\left(\sqrt{x + 4}\right) = \left(\sqrt{x + 4}\right)^2 - 4 = (x + 4) - 4 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(x^2 - 4\right) = \sqrt{\left(x^2 - 4\right) + 4} = \sqrt{x^2} = x$$

17. $(f \circ g)(x) = f\left(g(x)\right) = f(4x + 9) = \dfrac{4x + 9 - 9}{4} = \dfrac{4x}{4} = x$

$$(g \circ f)(x) = g\left(f(x)\right) = g\left(\frac{x - 9}{4}\right) = 4\left(\frac{x - 9}{4}\right) + 9 = x - 9 + 9 = x$$

19. $f\left(g(x)\right) = f\left(\sqrt[3]{4x}\right) = \dfrac{\left(\sqrt[3]{4x}\right)^3}{4}$

$$= \frac{4x}{4}$$

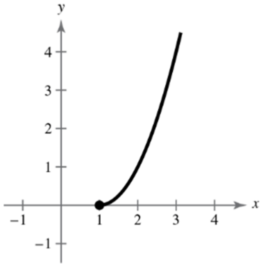
$$= x$$

$$g\left(f(x)\right) = g\left(\frac{x^3}{4}\right) = \sqrt[3]{4\left(\frac{x^3}{4}\right)}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

21.

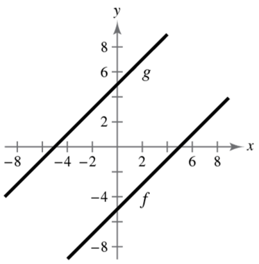


23. $f(x) = x - 5, g(x) = x + 5$

(a) $f\left(g(x)\right) = f(x + 5) = (x + 5) - 5 = x$

$g\left(f(x)\right) = g(x - 5) = (x - 5) + 5 = x$

(b)

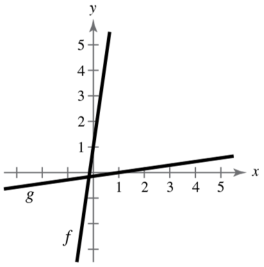


25. $f(x) = 7x + 1, g(x) = \dfrac{x - 1}{7}$

(a) $f\left(g(x)\right) = f\left(\dfrac{x - 1}{7}\right) = 7\left(\dfrac{x - 1}{7}\right) + 1 = x$

$g\left(f(x)\right) = g(7x + 1) = \dfrac{(7x + 1) - 1}{7} = x$

(b)

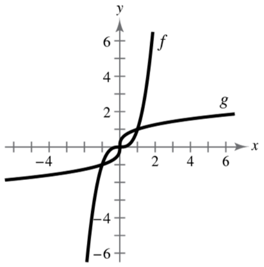


27. $f(x) = x^3, g(x) = \sqrt[3]{x}$

(a) $f\left(g(x)\right) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$

$g\left(f(x)\right) = g\left(x^3\right) = \sqrt[3]{\left(x^3\right)} = x$

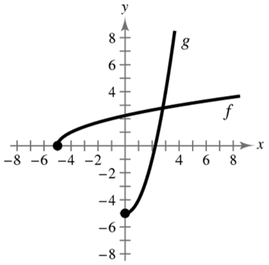
(b)



29. $f(x) = \sqrt{x + 5}, g(x) = x^2 - 5, x \geq 0$

(a) $$f(g(x)) = f(x^2 - 5), x \geq 0$$
$$= \sqrt{(x^2 - 5) + 5} = x$$
$$g(f(x)) = g(\sqrt{x + 5})$$
$$= (\sqrt{x + 5})^2 - 5 = x$$

(b)

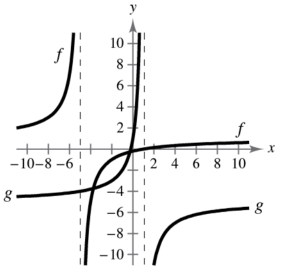


31. $f(x) = \dfrac{x - 1}{x + 5}, g(x) = -\dfrac{5x + 1}{x - 1}$

(a) $$f(g(x)) = f\left(-\frac{5x + 1}{x - 1}\right) = \frac{\left(-\dfrac{5x + 1}{x - 1} - 1\right)}{\left(-\dfrac{5x + 1}{x - 1} + 5\right)} \cdot \frac{x - 1}{x - 1} = \frac{-(5x + 1) - (x - 1)}{-(5x + 1) + 5(x - 1)} = \frac{-6x}{-6} = x$$

$$g(f(x)) = g\left(\frac{x - 1}{x + 5}\right) = -\frac{\left[5\left(\dfrac{x - 1}{x + 5}\right) + 1\right]}{\left[\dfrac{x - 1}{x + 5} - 1\right]} \cdot \frac{x + 5}{x + 5} = -\frac{5(x - 1) + (x + 5)}{(x - 1) - (x + 5)} = -\frac{6x}{-6} = x$$

(b)



33. No, $\{(-2, -1), (1, 0), (2, 1), (1, 2), (-2, 3), (-6, 4)\}$ does not represent a function. −2 and 1 are paired with two different values.

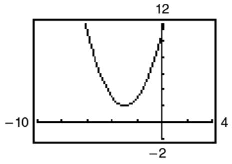
35.

x	3	5	7	9	11	13
$f^{-1}(x)$	−1	0	1	2	3	4

37. Yes, because no horizontal line crosses the graph of *f* at more than one point, *f has* an inverse.

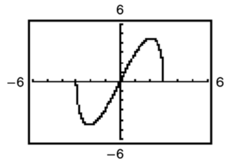
39. No, because some horizontal lines cross the graph of *f* twice, *f does not* have an inverse.

41. $g(x) = (x + 3)^2 + 2$



g does not pass the Horizontal Line Test, so *g does not* have an inverse.

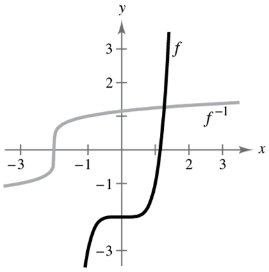
43. $f(x) = x\sqrt{9 - x^2}$



f does not pass the Horizontal Line Test, so *f does not* have an inverse.

45. (a)
$$\begin{aligned} f(x) &= x^5 - 2 \\ y &= x^5 - 2 \\ x &= y^5 - 2 \\ y &= \sqrt[5]{x + 2} \\ f^{-1}(x) &= \sqrt[5]{x + 2} \end{aligned}$$

(b)

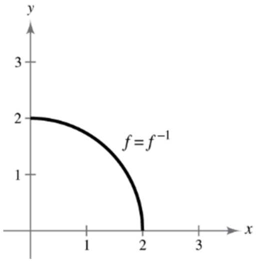


(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domains and ranges of f and f^{-1} are all real numbers.

47. (a)
$$\begin{aligned} f(x) &= \sqrt{4 - x^2},\ 0 \le x \le 2 \\ y &= \sqrt{4 - x^2} \\ x &= \sqrt{4 - y^2} \\ x^2 &= 4 - y^2 \\ y^2 &= 4 - x^2 \\ y &= \sqrt{4 - x^2} \\ f^{-1}(x) &= \sqrt{4 - x^2},\ 0 \le x \le 2 \end{aligned}$$

(b)

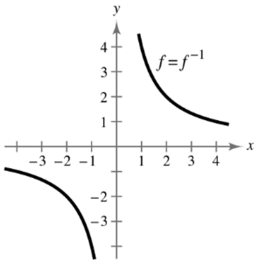


(c) The graph of f^{-1} is the same as the graph of f.

(d) The domains and ranges of f and f^{-1} are all real numbers x such that $0 \le x \le 2$.

49. (a)
$$\begin{aligned} f(x) &= \frac{4}{x} \\ y &= \frac{4}{x} \\ x &= \frac{4}{y} \\ xy &= 4 \\ y &= \frac{4}{x} \\ f^{-1}(x) &= \frac{4}{x} \end{aligned}$$

(b)

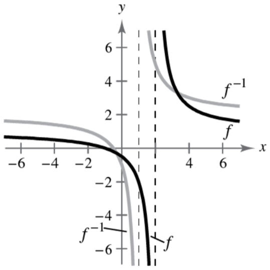


(c) The graph of f^{-1} is the same as the graph of f.

(d) The domains and ranges of f and f^{-1} are all real numbers except for 0.

51. (a)
$$\begin{aligned} f(x) &= \frac{x + 1}{x - 2} \\ y &= \frac{x + 1}{x - 2} \\ x &= \frac{y + 1}{y - 2} \\ x(y - 2) &= y + 1 \\ xy - 2x &= y + 1 \\ xy - y &= 2x + 1 \\ y(x - 1) &= 2x + 1 \\ y &= \frac{2x + 1}{x - 1} \\ f^{-1}(x) &= \frac{2x + 1}{x - 1} \end{aligned}$$

(b)

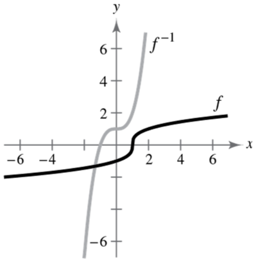


(c) The graph of f^{-1} is the reflection of graph of f in the line $y = x$.

(d) The domain of f and the range of f^{-1} is all real numbers except 2.
The range of f and the domain of f^{-1} is all real numbers except 1.

53. (a)
$$\begin{aligned} f(x) &= \sqrt[3]{x - 1} \\ y &= \sqrt[3]{x - 1} \\ x &= \sqrt[3]{y - 1} \\ x^3 &= y - 1 \\ y &= x^3 + 1 \\ f^{-1}(x) &= x^3 + 1 \end{aligned}$$

(b)



(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domains and ranges of f and f^{-1} are all real numbers.

55. $f(x) = x^4$

$$y = x^4$$
$$x = y^4$$
$$y = \pm\sqrt[4]{x}$$

This does not represent y as a function of x. f does not have an inverse.

57. $g(x) = \dfrac{x+1}{6}$

$$y = \frac{x+1}{6}$$
$$x = \frac{y+1}{6}$$
$$6x = y + 1$$
$$y = 6x - 1$$

This is a function of x, so g has an inverse.

$g^{-1}(x) = 6x - 1$

59. $p(x) = -4$

$$y = -4$$

Because $y = -4$ for all x, the graph is a horizontal line and fails the Horizontal Line Test. p does not have an inverse.

61. $f(x) = \sqrt{2x+3} \Rightarrow x \ge -\dfrac{3}{2}, y \ge 0$

$$y = \sqrt{2x+3}, x \ge -\frac{3}{2}, y \ge 0$$
$$x = \sqrt{2y+3}, y \ge -\frac{3}{2}, x \ge 0$$
$$x^2 = 2y + 3, x \ge 0, y \ge -\frac{3}{2}$$
$$y = \frac{x^2-3}{2}, x \ge 0, y \ge -\frac{3}{2}$$

This is a function of x, so f has an inverse.

$f^{-1}(x) = \dfrac{x^2-3}{2}, x \ge 0$

63. $f(x) = \dfrac{6x+4}{4x+5}$

$$y = \frac{6x+4}{4x+5}$$
$$x = \frac{6y+4}{4y+5}$$
$$x(4y+5) = 6y + 4$$
$$4xy + 5x = 6y + 4$$
$$4xy - 6y = -5x + 4$$
$$y(4x-6) = -5x + 4$$
$$y = \frac{-5x+4}{4x-6}$$
$$= \frac{5x-4}{6-4x}$$

This is a function of x, so f has an inverse.

$f^{-1}(x) = \dfrac{5x-4}{6-4x}$

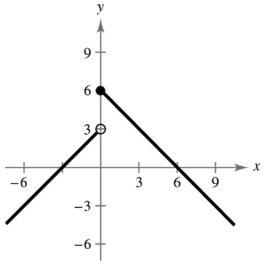
65. $f(x) = (x+3)^2, x \ge -3 \Rightarrow y \ge 0$

$$y = (x+3)^2, x \ge -3, y \ge 0$$
$$x = (y+3)^2, y \ge -3, x \ge 0$$
$$\sqrt{x} = y + 3, y \ge -3, x \ge 0$$
$$y = \sqrt{x} - 3, x \ge 0, y \ge -3$$

This is a function of x, so f has an inverse.

$f^{-1}(x) = \sqrt{x} - 3, x \ge 0$

67. $f(x) = \begin{cases} x + 3, & x < 0 \\ 6 - x, & x \ge 0 \end{cases}$



This graph fails the Horizontal Line Test, so f does not have an inverse.

69. (a)
$$\begin{aligned} y &= 10 + 0.75x \\ x &= 10 + 0.75y \\ x - 10 &= 0.75y \\ \frac{x - 10}{0.75} &= y \end{aligned}$$

So, $f^{-1}(x) = \dfrac{x - 10}{0.75}$.

$x =$ hourly wage, $y =$ number of units produced

(b) $y = \dfrac{24.25 - 10}{0.75} = 19$

So, 19 units are produced.

In Exercises 71 and 73, $f(x) = x + 4$, $f^{-1}(x) = x - 4$, $g(x) = 2x - 5$, $g^{-1}(x) = \dfrac{x + 5}{2}$.

71.
$$\begin{aligned} \left(g^{-1} \circ f^{-1}\right)(x) &= g^{-1}\left(f^{-1}(x)\right) \\ &= g^{-1}(x - 4) \\ &= \frac{(x - 4) + 5}{2} \\ &= \frac{x + 1}{2} \end{aligned}$$

73.
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(2x - 5) \\ &= (2x - 5) + 4 \\ &= 2x - 1 \end{aligned}$$

$$(f \circ g)^{-1}(x) = \frac{x + 1}{2}$$

Note: Comparing Exercises 71 and 73, $(f \circ g)^{-1}(x) = \left(g^{-1} \circ f^{-1}\right)(x)$.

In Exercises 75–79, $f(x) = \dfrac{1}{8}x - 3$, $f^{-1}(x) = 8(x + 3)$, $g(x) = x^3$, $g^{-1}(x) = \sqrt[3]{x}$.

75.
$$\begin{aligned} \left(f^{-1} \circ g^{-1}\right)(1) &= f^{-1}\left(g^{-1}(1)\right) \\ &= f^{-1}\left(\sqrt[3]{1}\right) \\ &= 8\left(\sqrt[3]{1} + 3\right) = 32 \end{aligned}$$

77.
$$\begin{aligned} \left(f^{-1} \circ f^{-1}\right)(4) &= f^{-1}\left(f^{-1}(4)\right) \\ &= f^{-1}(8[4 + 3]) \\ &= 8\left[8(4 + 3) + 3\right] \\ &= 8\left[8(7) + 3\right] \\ &= 8(59) = 472 \end{aligned}$$

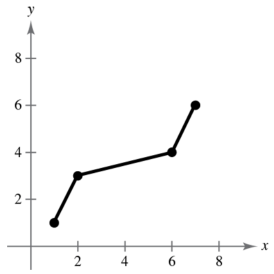
79.
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f\left(x^3\right) = \tfrac{1}{8}x^3 - 3 \\ y &= \tfrac{1}{8}x^3 - 3 \\ x &= \tfrac{1}{8}y^3 - 3 \\ x + 3 &= \tfrac{1}{8}y^3 \\ 8(x + 3) &= y^3 \\ \sqrt[3]{8(x + 3)} &= y \\ (f \circ g)^{-1}(x) &= 2\sqrt[3]{x + 3} \end{aligned}$$

81. False. $f(x) = x^2$ is even and does not have an inverse.

83. (a)

x	1	3	4	6
f	1	2	6	7

x	1	2	6	7
$f^{-1}(x)$	1	3	4	6



(b)

x	–4	–2	0	3
f	3	4	0	–1

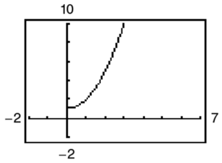
The graph does not pass the Horizontal Line Test, so $f^{-1}(x)$ does not exist.

85. Let $(f \circ g)(x) = y$. Then $x = (f \circ g)^{-1}(y)$. Also,

$$\begin{aligned} (f \circ g)(x) = y \Rightarrow f(g(x)) &= y \\ g(x) &= f^{-1}(y) \\ x &= g^{-1}\left(f^{-1}(y)\right) \\ x &= \left(g^{-1} \circ f^{-1}\right)(y). \end{aligned}$$

Because f and g are both one-to-one functions, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

87.



There is an inverse function $f^{-1}(x) = \sqrt{x - 1}$ because the domain of f is equal to the range of f^{-1} and the range of f is equal to the domain of f^{-1}.

89. $y = -(x - 5)^2 + 1$

$= -(x^2 - 10x + 25) + 1$

$= -(x^2 - 10x + 24)$

$= -(x - 6)(x - 4)$

Alternate solution:

$y = 1 - (x - 5)^2$

$= [1 - (x - 5)][1 + (x - 5)]$

$= (6 - x)(x - 4)$

$= -(x - 6)(x - 4)$

91. $y = -\left(x - \frac{13}{2}\right)^2 + \frac{25}{4}$

$= \frac{25}{4} - \left(x - \frac{13}{2}\right)^2$

$= \left[\frac{5}{2} + \left(x - \frac{13}{2}\right)\right]\left[\frac{5}{2} - \left(x - \frac{13}{2}\right)\right]$

$= (x - 4)(9 - x)$

$= -(x - 4)(x - 9)$

93. The function is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

95. The function is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

97. Relative maximum at $(2, 0)$

Relative minimum at $(0, -3)$

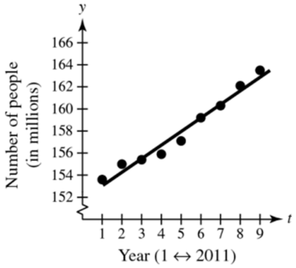
Section 1.10 Mathematical Modeling and Variation

1. sum of squared differences

3. correlation coefficient

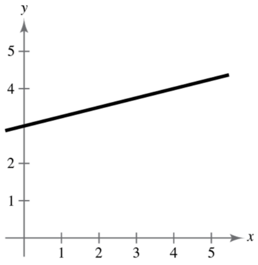
5. You can say that z varies jointly as x and y, or z is jointly proportional to x and y.

7. (a) Model: $y = 1.23t + 151.8,\ 1 \le t \le 9$



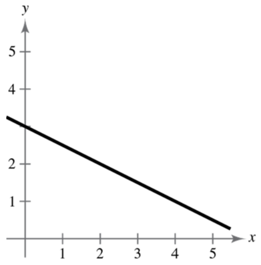
(b) The model is a good fit for the data.

9.



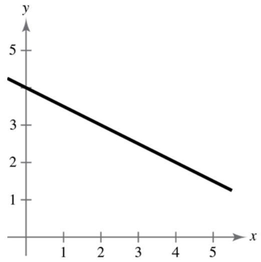
Using the point $(0, 3)$ and $(4, 4)$, $y = \frac{1}{4}x + 3$.

11.



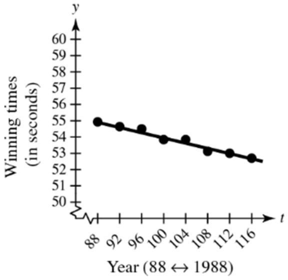
Using the points $(2, 2)$ and $(4, 1)$, $y = -\frac{1}{2}x + 3$.

13.



The line appears to pass through $(2, 3)$ and $(4, 2)$ so its equation is $y = -\frac{1}{2}x + 4$.

15. (a) and (b)



Use the points (88, 54.93) and (116, 52.70).

$$m = \frac{52.70 - 54.93}{116 - 88} \approx -0.080$$

$$y - 52.70 = -0.080(t - 116)$$

$$y - 52.70 = -0.080t + 9.28$$

$$y = -0.080t + 61.98$$

(c) $y = -0.083t + 62.30$

(d) The models are similar.

17. $y = kx$

$14 = k(2)$

$7 = k$

$y = 7x$

19. $y = kx$

$3 = k(-24)$

$-\frac{1}{8} = k$

$y = -\frac{1}{8}x$

21. $y = kx$

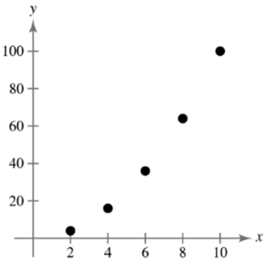
$8\pi = k(4)$

$2\pi = k$

$y = 2\pi x$

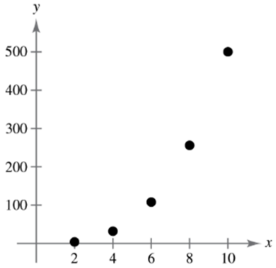
23. $k = 1$

x	2	4	6	8	10
$y = kx^2$	4	16	36	64	100



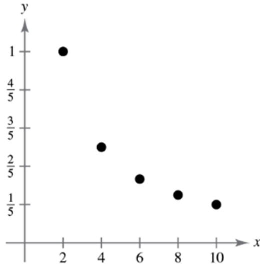
25. $k = \frac{1}{2}$

x	2	4	6	8	10
$y = \frac{1}{2}x^3$	4	32	108	256	500



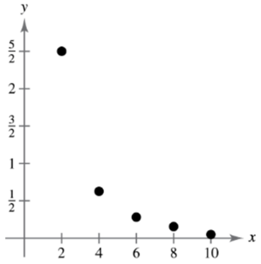
27. $k = 2, n = 1$

x	2	4	6	8	10
$y = \frac{2}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



29. $k = 10$

x	2	4	6	8	10
$y = \frac{k}{x^2}$	$\frac{5}{2}$	$\frac{5}{8}$	$\frac{5}{18}$	$\frac{5}{32}$	$\frac{1}{10}$



31. $y = \frac{k}{x}$

$1 = \frac{k}{5}$

$5 = k$

$y = \frac{5}{x}$

This equation checks with the other points given in the table.

33. $y = kx$

$-7 = k(10)$

$-\frac{7}{10} = k$

$y = -\frac{7}{10}x$

This equation checks with the other points given in the table.

35. $A = kr^2$

37. $y = \frac{k}{x^2}$

39. $F = \frac{kg}{r^2}$

41. $R = k(T - T_e)$

43. $P = kVI$

45. y is directly proportional to the square of x.

47. $A = \frac{1}{2}bh$

The area of a triangle is jointly proportional to its base and height.

49. $y = kx$

$54 = k(3)$

$18 = k$

$y = 18x$

51. $y = \frac{k}{x}$

$3 = \frac{k}{25}$

$75 = k$

$y = \frac{75}{x}$

53. $z = kxy$

$64 = k(4)(8)$

$2 = k$

$z = 2xy$

55. $P = \frac{kx}{y^2}$

$\frac{28}{3} = \frac{k(42)}{9^2}$

$\frac{28}{3} \cdot \frac{81}{42} = k$

$\frac{2 \cdot 27}{3} = k$

$18 = k$

$P = \frac{18x}{y^2}$

57. $I = kP$

$113.75 = k(3250)$

$0.035 = k$

$I = 0.035P$

59. $y = kx$

$33 = k(13)$

$\frac{33}{13} = k$

$y = \frac{33}{13}x$

When $x = 10$ inches, $y \approx 25.4$ centimeters.

When $x = 20$ inches, $y \approx 50.8$ centimeters.

61. $d = kF$

$0.12 = k(220)$

$\frac{3}{5500} = k$

$d = \frac{3}{5500}F$

$0.16 = \frac{3}{5500}F$

$\frac{880}{3} = F$

The required force is $293\frac{1}{3}$ newtons.

63. $d = kF$

$1.9 = k(25) \Rightarrow k = 0.076$

$d = 0.076F$

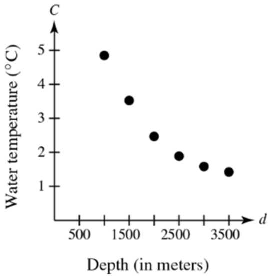
When the distance compressed is 3 inches, we have

$3 = 0.076F$

$F \approx 39.47.$

No child over 39.47 pounds should use the toy.

65. (a)



(b) The data shows an inverse variation model fits.

(c) $4.85 = \frac{k_1}{1000}$ $\quad 3.525 = \frac{k_2}{1500}$ $\quad 2.468 = \frac{k_3}{2000}$ $\quad 1.888 = \frac{k_4}{2500}$ $\quad 1.583 = \frac{k_5}{3000}$ $\quad 1.422 = \frac{k_6}{3500}$

$4850.0 = k_1$ $\quad 5287.5 = k_2$ $\quad 4936.0 = k_3$ $\quad 4720.0 = k_5$ $\quad 4749.0 = k_5$ $\quad 4977.0 = k_6$

Mean: $k = \frac{4850 + 5287.5 + 4936 + 4720 + 4977}{6} \approx 4919.92$, Model: $C = \frac{4919.92}{d}$

(d)
$$C = \frac{4919.92}{d}$$
$$3 = \frac{4919.92}{d}$$
$$d = \frac{4919.92}{3} \approx 1639.97 \text{ meters}$$

The temperature is 3° C at approximately 1640 meters.

67.
$$d = kv^2$$
$$0.02 = k\left(\frac{1}{4}\right)^2$$
$$k = 0.32$$
$$d = 0.32v^2$$
$$0.12 = 0.32v^2$$
$$v^2 = \frac{0.12}{0.32} = \frac{3}{8}$$
$$v = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}}{4} \approx 0.61 \text{ mi/hr}$$

69. $f_0 = k\frac{\sqrt{T_0}}{l_0\sqrt{p}}$ where f_0 = original frequency, T_0 = original tension, l_0 = original length of string, and p = mass density

$f_0 = 100$

(a) The frequency of a string with four times the tension would double the frequency in order to maintain the direct proportion.

Let $f_{\text{new}} = \frac{k \cdot \sqrt{T_{\text{new}}}}{l_{\text{new}} \cdot \sqrt{p}}$ and $T_{\text{new}} = 4T_0$, $l_{\text{new}} = l_0$

$$f_{\text{new}} = \frac{k \cdot \sqrt{4T_0}}{l_0 \cdot \sqrt{p}} = 2 \cdot \frac{k \cdot \sqrt{T_0}}{l_0 \cdot \sqrt{p}} = 2 \cdot f_0 = 2 \cdot 100 = 200Hz$$

(b) The frequency of a string with two times the length would half the frequency in order to maintain the inverse proportion.

Let $f_{\text{new}} = \dfrac{k \cdot \sqrt{T_{\text{new}}}}{l_{\text{new}} \cdot \sqrt{p}}$ and $l_{\text{new}} = 2l_0$, $\text{T}_{\text{new}} = T_0$

$$f_{\text{new}} = \frac{k \cdot \sqrt{T_0}}{2l_0 \cdot \sqrt{p}} = \frac{1}{2} \cdot \frac{k \cdot \sqrt{T_0}}{l_0 \cdot \sqrt{p}} = \frac{1}{2} f_0 = \frac{1}{2} \cdot 100 = 50Hz$$

(c) The frequency of a string with four times the tension would double the frequency, and two times the length would half the frequency in order to maintain the inverse proportion. Therefore the frequency would remain the same,

Let $f_{\text{new}} = \dfrac{k \cdot \sqrt{T_{\text{new}}}}{l_{\text{new}} \cdot \sqrt{p}}$ and $T_{\text{new}} = 4T_0$, $l_{\text{new}} = 2l_0$

$$f_{\text{new}} = \frac{k \cdot \sqrt{4T_0}}{2l_0 \cdot \sqrt{p}} = \frac{2 \cdot k\sqrt{T_0}}{2 \cdot l_0 \cdot \sqrt{p}} = \frac{2}{2} f_0 = 1 \cdot 100 = 100Hz$$

71. True. If $y = k_1 x$ and $x = k_2 z$, then $y = k_1(k_2 z) = (k_1 k_2)z$.

73. False. In the equation $S = 4\pi r^2$, π is a constant, not a variable.

75. True. If $f(x) = (x - 2)p(x)$, then $f(2) = (2 - 2)p(2) = 0$.

77.
$$\frac{x-1}{5x-4} - \frac{2x}{5x-4} = \frac{x-1-2x}{5x-4} = \frac{-x-1}{5x-4} = \frac{x+1}{4-5x}$$

79.
$$\frac{5}{x-1} + \frac{2x}{x+3} = \frac{5(x+3) + 2x(x-1)}{(x-1)(x+3)} = \frac{2x^2 + 3x + 15}{(x-1)(x+3)}$$

81. $\dfrac{x+3}{2x+1} \cdot \dfrac{4}{3(x+3)} = \dfrac{4}{3(2x+1)}, \quad x \neq -3$

83. $x^3 + y = 8 \Rightarrow y = 8 - x^3$ is a function of x.

Domain: all real numbers x

85. $1 - y^2 = x$ is not a function of x.

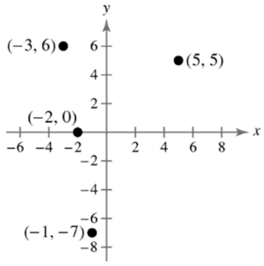
For example, for $x = 0$, $y = \pm 1$.

87. $y = \sqrt{x + 5}$ is a function of x.

Domain: all real numbers x such that $x \geq -5$

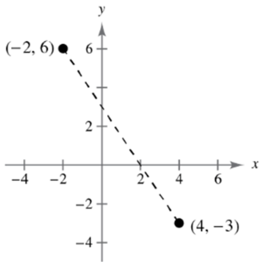
Review Exercises for Chapter 1

1.



3. $x > 0$ and $y = -2$ in Quadrant IV.

5. (a)

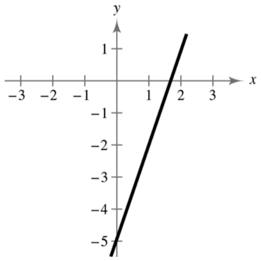


(b) $d = \sqrt{(-2-4)^2 + (6+3)^2} = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}$

(c) Midpoint: $\left(\dfrac{-2+4}{2}, \dfrac{6-3}{2}\right) = \left(1, \dfrac{3}{2}\right)$

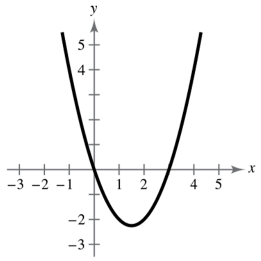
7. $y = 3x - 5$

x	-2	-1	0	1	2
y	-11	-8	-5	-2	1



9. $y = x^2 - 3x$

x	-1	0	1	2	3	4
y	4	0	-2	-2	0	4



11. $y = 2x + 7$

x-intercept: Let $y = 0$.

$$0 = 2x + 7$$
$$x = -\tfrac{7}{2}$$
$$\left(-\tfrac{7}{2}, 0\right)$$

y-intercept: Let $x = 0$.

$$y = 2(0) + 7$$
$$y = 7$$
$$(0, 7)$$

13. $y = (x - 3)^2 - 4$

x-intercepts: $0 = (x - 3)^2 - 4 \Rightarrow (x - 3)^2 = 4$

$$\Rightarrow x - 3 = \pm 2$$
$$\Rightarrow x = 3 \pm 2$$
$$\Rightarrow x = 5 \text{ or } x = 1$$
$$(5, 0), (1, 0)$$

y-intercept: $y = (0 - 3)^2 - 4$

$y = 9 - 4$

$y = 5$

$(0, 5)$

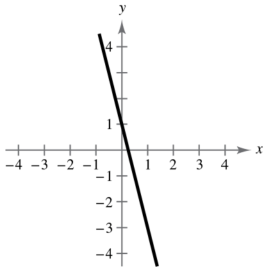
15. $y = -4x + 1$

Intercepts: $\left(\tfrac{1}{4}, 0\right), (0, 1)$

$y = -4(-x) + 1 \Rightarrow y = 4x + 1 \Rightarrow$ No y-axis symmetry

$-y = -4x + 1 \Rightarrow y = 4x - 1 \Rightarrow$ No x-axis symmetry

$-y = -4(-x) + 1 \Rightarrow y = -4x - 1 \Rightarrow$ No origin symmetry



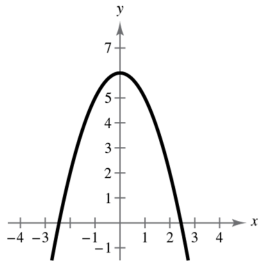
17. $y = 6 - x^2$

Intercepts: $\left(\pm\sqrt{6}, 0\right), (0, 6)$

$y = 6 - \left(-x^2\right) \Rightarrow y = 6 - x^2 \Rightarrow$ y-axis symmetry

$-y = 6 - x^2 \Rightarrow y = -6 + x^2 \Rightarrow$ No x-axis symmetry

$-y = 6 - (-x)^2 \Rightarrow y = -6 + x^2 \Rightarrow$ No origin symmetry



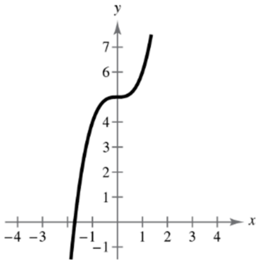
19. $y = x^3 + 5$

Intercepts: $\left(\sqrt[3]{-5}, 0\right), (0, 5)$

$y = (-x)^3 + 5 \Rightarrow y = -x^3 + 5 \Rightarrow$ No y-axis symmetry

$-y = x^3 + 5 \Rightarrow y = -x^3 - 5 \Rightarrow$ No x-axis symmetry

$-y = (-x)^3 + 5 \Rightarrow y = x^3 - 5 \Rightarrow$ No origin symmetry



21. $y = \sqrt{x + 5}$

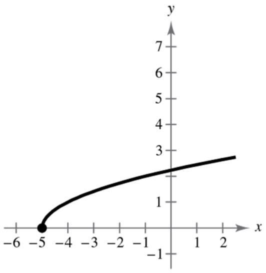
Domain: $[-5, \infty)$

Intercepts: $(-5, 0), \left(0, \sqrt{5}\right)$

$y = \sqrt{-x + 5} \Rightarrow$ No y-axis symmetry

$-y = \sqrt{x + 5} \Rightarrow y = -\sqrt{x + 5} \Rightarrow$ No x-axis symmetry

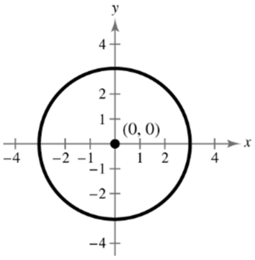
$-y = \sqrt{-x + 5} \Rightarrow y = -\sqrt{-x + 5} \Rightarrow$ No origin symmetry



23. $x^2 + y^2 = 9$

Center: $(0, 0)$

Radius: 3

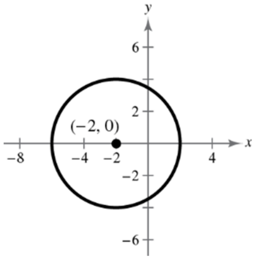


25. $(x + 2)^2 + y^2 = 16$

$(x - (-2))^2 + (y - 0)^2 = 4^2$

Center: $(-2, 0)$

Radius: 4



27. Endpoints of a diameter: $(0, 0)$ and $(4, -6)$

Center: $\left(\dfrac{0 + 4}{2}, \dfrac{0 + (-6)}{2}\right) = (2, -3)$

Radius: $r = \sqrt{(2 - 0)^2 + (-3 - 0)^2} = \sqrt{4 + 9} = \sqrt{13}$

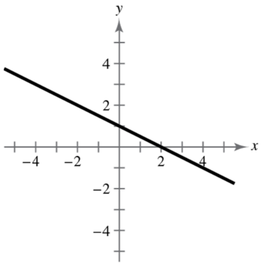
Standard form: $(x - 2)^2 + (y - (-3))^2 = \left(\sqrt{13}\right)^2$

$$(x - 2)^2 + (y + 3)^2 = 13$$

29. $y = -\frac{1}{2}x + 1$

Slope: $m = -\frac{1}{2}$

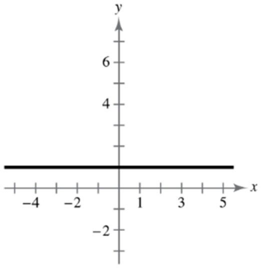
y-intercept: $(0, 1)$



31. $y = 1$

Slope: $m = 0$

y-intercept: $(0, 1)$

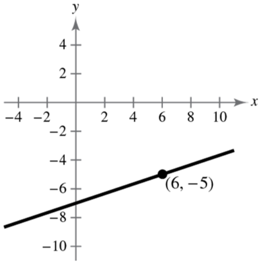


33. $(5, -2), (-1, 4)$

$$m = \frac{4 - (-2)}{-1 - 5} = \frac{6}{-6} = -1$$

35. $(6, -5), m = \frac{1}{3}$

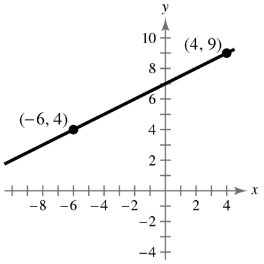
$$y - (-5) = \tfrac{1}{3}(x - 6)$$
$$y + 5 = \tfrac{1}{3}x - 2$$
$$y = \tfrac{1}{3}x - 7$$



37. $(-6, 4), (4, 9)$

$$m = \frac{9 - 4}{4 - (-6)} = \frac{5}{10} = \frac{1}{2}$$

$$y - 4 = \frac{1}{2}(x - (-6))$$
$$y - 4 = \frac{1}{2}(x + 6)$$
$$y - 4 = \frac{1}{2}x + 3$$
$$y = \frac{1}{2}x + 7$$



39. Point: $(3, -2)$

$$5x - 4y = 8$$
$$y = \tfrac{5}{4}x - 2$$

(a) Parallel slope: $m = \frac{5}{4}$

$$y - (-2) = \tfrac{5}{4}(x - 3)$$
$$y + 2 = \tfrac{5}{4}x - \tfrac{15}{4}$$
$$y = \tfrac{5}{4}x - \tfrac{23}{4}$$

(b) Perpendicular slope: $m = -\frac{4}{5}$

$$y - (-2) = -\tfrac{4}{5}(x - 3)$$
$$y + 2 = -\tfrac{4}{5}x + \tfrac{12}{5}$$
$$y = -\tfrac{4}{5}x + \tfrac{2}{5}$$

41. *Verbal Model:* Sale price = (List price) − (Discount)

Labels: Sale price $= S$

List price $= L$

Discount $= 20\%$ of $L = 0.2L$

Equation: $S = L - 0.2L$

$S = 0.8L$

43. $16x - y^4 = 0$

$$y^4 = 16x$$
$$y = \pm 2\sqrt[4]{x}$$

No, y is not a function of x. Some x-values correspond to two y-values.

45. $y = \sqrt{1 - x}$

Yes, the equation represents y as a function of x. Each x-value, $x \leq 1$, corresponds to only one y-value.

47. $f(x) = x^2 + 1$

(a) $f(2) = (2)^2 + 1 = 5$

(b) $f(-4) = (-4)^2 + 1 = 17$

(c) $f(t^2) = (t^2)^2 + 1 = t^4 + 1$

(d) $f(t + 1) = (t + 1)^2 + 1$
$= t^2 + 2t + 2$

49. $f(x) = \sqrt{25 - x^2}$

Domain: $25 - x^2 \geq 0$

$(5 + x)(5 - x) \geq 0$

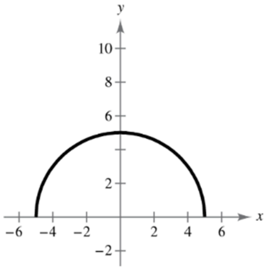
Critical numbers: $x = \pm 5$

Test intervals: $(-\infty, -5), (-5, 5), (5, \infty)$

Test: Is $25 - x^2 \geq 0$?

Solution set: $-5 \leq x \leq 5$

Domain: all real numbers x such that $-5 \leq x \leq 5$, or $[-5, 5]$



51. $v(t) = -32t + 48$

$v(1) = 16$ feet per second

53. $f(x) = 2x^2 + 3x - 1$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\left[2(x+h)^2 + 3(x+h) - 1\right] - \left(2x^2 + 3x - 1\right)}{h}\\ &= \frac{2x^2 + 4xh + 2h^2 + 3x + 3h - 1 - 2x^2 - 3x + 1}{h}\\ &= \frac{h(4x + 2h + 3)}{h}\\ &= 4x + 2h + 3, \quad h \neq 0\end{aligned}$$

55. $y = (x - 3)^2$

A vertical line intersects the graph no more than once, so *y is* a function of *x*.

57. $f(x) = 3x^2 - 16x + 21$

$$\begin{aligned}3x^2 - 16x + 21 &= 0\\ (3x - 7)(x - 3) &= 0\end{aligned}$$

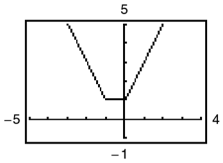
$$\begin{aligned}3x - 7 = 0 \quad &\text{or} \quad x - 3 = 0\\ x = \tfrac{7}{3} \quad &\text{or} \quad x = 3\end{aligned}$$

59. $f(x) = |x| + |x + 1|$

f is increasing on $(0, \infty)$.

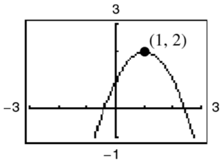
f is decreasing on $(-\infty, -1)$.

f is constant on $(-1, 0)$.



61. $f(x) = -x^2 + 2x + 1$

Relative maximum: $(1, 2)$



63. $f(x) = -x^2 + 8x - 4$

$$\frac{f(4) - f(0)}{4 - 0} = \frac{12 - (-4)}{4} = 4$$

The average rate of change of *f* from $x_1 = 0$ to $x_2 = 4$ is 4.

65. $f(x) = x^4 - 20x^2$

$$f(-x) = (-x)^4 - 20(-x)^2 = x^4 - 20x^2 = f(x)$$

The function is even, so the graph has *y*-axis symmetry.

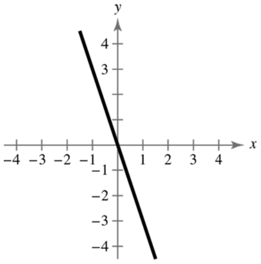
67. (a) $f(2) = -6, f(-1) = 3$

Points: $(2, -6), (-1, 3)$

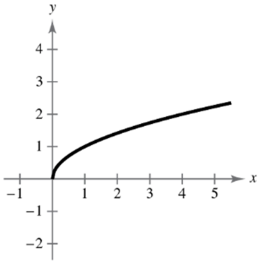
$$m = \frac{3 - (-6)}{-1 - 2} = \frac{9}{-3} = -3$$

$$\begin{aligned}y - (-6) &= -3(x - 2)\\ y + 6 &= -3x + 6\\ y &= -3x\\ f(x) &= -3x\end{aligned}$$

(b)



69. $g(x) = \sqrt{x}$

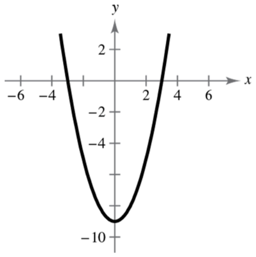


71. (a) $f(x) = x^2$

(b) $h(x) = x^2 - 9$

Vertical shift 9 units downward

(c)



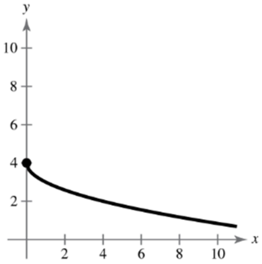
(d) $h(x) = f(x) - 9$

73. (a) $f(x) = \sqrt{x}$

(b) $h(x) = -\sqrt{x} + 4$

Reflection in the x-axis and a vertical shift 4 units upward

(c)



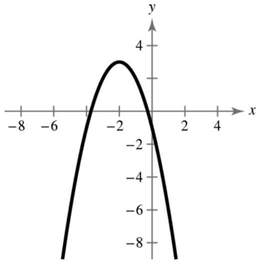
(d) $h(x) = -f(x) + 4$

75. (a) $f(x) = x^2$

(b) $h(x) = -(x + 2)^2 + 3$

Reflection in the x-axis, a horizontal shift 2 units to the left, and a vertical shift 3 units upward

(c)



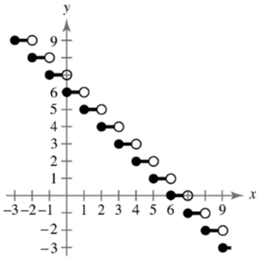
(d) $h(x) = -f(x + 2) + 3$

77. (a) $f(x) = [\![x]\!]$

(b) $h(x) = -[\![x]\!] + 6$

Reflection in the x-axis and a vertical shift 6 units upward

(c)



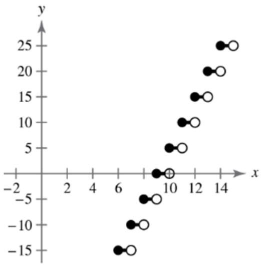
(d) $h(x) = -f(x) + 6$

79. (a) $f(x) = [\![x]\!]$

(b) $h(x) = 5[\![x - 9]\!]$

Horizontal shift 9 units to the right and a vertical stretch (each y-value is multiplied by 5)

(c)



(d) $h(x) = 5f(x - 9)$

81. $f(x) = x^2 + 3, g(x) = 2x - 1$

(a) $(f + g)(x) = (x^2 + 3) + (2x - 1) = x^2 + 2x + 2$

(b) $(f - g)(x) = (x^2 + 3) - (2x - 1) = x^2 - 2x + 4$

(c) $(fg)(x) = (x^2 + 3)(2x - 1) = 2x^3 - x^2 + 6x - 3$

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2 + 3}{2x - 1}$, Domain: $x \neq \dfrac{1}{2}$

83. $f(x) = \frac{1}{3}x - 3,\ g(x) = 3x + 1$

The domains of f and g are all real numbers.

(a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(3x + 1)\\ &= \tfrac{1}{3}(3x + 1) - 3\\ &= x + \tfrac{1}{3} - 3\\ &= x - \tfrac{8}{3}\end{aligned}$$

Domain: all real numbers

(b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(\tfrac{1}{3}x - 3\right)\\ &= 3\left(\tfrac{1}{3}x - 3\right) + 1\\ &= x - 9 + 1\\ &= x - 8\end{aligned}$$

Domain: all real numbers

In Exercise 85 use the following functions. $f(x) = x - 100,\ g(x) = 0.95x$

85. $(f \circ g)(x) = f(0.95x) = 0.95x - 100$ represents the sale price if first the 5% discount is applied and then the \$100 rebate.

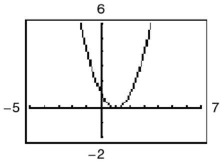
87.
$$\begin{aligned}f(x) &= 3x + 8\\ y &= 3x + 8\\ x &= 3y + 8\\ x - 8 &= 3y\\ y &= \frac{x - 8}{3}\\ y &= \frac{1}{3}(x - 8)\end{aligned}$$

So, $f^{-1}(x) = \dfrac{1}{3}(x - 8) = \dfrac{x - 8}{3}.$

$$f(f^{-1}(x)) = f\left(\frac{1}{3}(x - 8)\right) = 3\left(\frac{1}{3}(x - 8)\right) + 8 = x - 8 + 8 = x$$

$$f^{-1}(f(x)) = f^{-1}(3x + 8) = \frac{1}{3}(3x + 8 - 8) = \frac{1}{3}(3x) = x$$

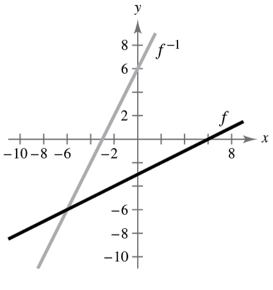
89. $f(x) = (x - 1)^2$



The function is not one-to-one. Some horizontal lines intersect the graph more than once.

91. (a)
$$\begin{aligned}f(x) &= \tfrac{1}{2}x - 3\\ y &= \tfrac{1}{2}x - 3\\ x &= \tfrac{1}{2}y - 3\\ x + 3 &= \tfrac{1}{2}y\\ 2(x + 3) &= y\\ f^{-1}(x) &= 2x + 6\end{aligned}$$

(b)



(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

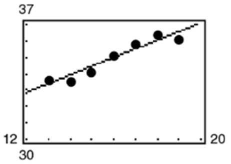
(d) The domains and ranges of f and f^{-1} are the set of all real numbers.

93. $f(x) = 2(x - 4)^2$ is increasing on $(4, \infty)$.

Let $f(x) = 2(x - 4)^2,\ x > 4$ and $y > 0$.

$$\begin{aligned}y &= 2(x - 4)^2\\ x &= 2(y - 4)^2,\ x > 0,\ y > 4\\ \frac{x}{2} &= (y - 4)^2\\ \sqrt{\frac{x}{2}} &= y - 4\\ \sqrt{\frac{x}{2}} + 4 &= y\\ f^{-1}(x) &= \sqrt{\frac{x}{2}} + 4,\ x > 0\end{aligned}$$

95. (a) and (b)



$C = 0.53t + 26.5,\ 13 \le t \le 19$

The model fits the data well.

97. $d = \dfrac{kx}{p}$

When $p = 4$ and $x = 25{,}000$, $d = 1250$.

$$1250 = \frac{k(25{,}000)}{4} \Rightarrow k = \frac{1}{5} = 0.2$$

When $p = 5.50$,

$$d = \frac{(0.2)(25{,}000)}{5.50} \approx 909 \text{ pastries.}$$

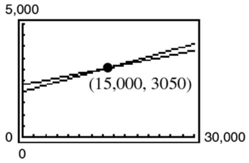
99. True. If $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, then the domain of g is all real numbers, which is equal to the range of f and vice versa.

Problem Solving for Chapter 1

1. (a) $W_1 = 0.07S + 2000$

(b) $W_2 = 0.05S + 2300$

(c)



Point of intersection: $(15{,}000, 3050)$

Both jobs pay the same, \$3050, if you sell \$15,000 per month.

(d) No. If you think you can sell \$20,000 per month, keep your current job with the higher commission rate. For sales over \$15,000 it pays more than the other job.

3. (a) Let $f(x)$ and $g(x)$ be two even functions.

Then define $h(x) = f(x) \pm g(x)$.

$$\begin{aligned} h(-x) &= f(-x) \pm g(-x) \\ &= f(x) \pm g(x) \text{ because } f \text{ and } g \text{ are even} \\ &= h(x) \end{aligned}$$

So, $h(x)$ is also even.

(b) Let $f(x)$ and $g(x)$ be two odd functions.

Then define $h(x) = f(x) \pm g(x)$.

$$\begin{aligned} h(-x) &= f(-x) \pm g(-x) \\ &= -f(x) \pm g(x) \text{ because } f \text{ and } g \text{ are odd} \\ &= -h(x) \end{aligned}$$

So, $h(x)$ is also odd. (If $f(x) \ne g(x)$)

(c) Let $f(x)$ be odd and $g(x)$ be even. Then define $h(x) = f(x) \pm g(x)$.

$$\begin{aligned} h(-x) &= f(-x) \pm g(-x) \\ &= -f(x) \pm g(x) \text{ because } f \text{ is odd and } g \text{ is even} \\ &\ne h(x) \\ &\ne -h(x) \end{aligned}$$

So, $h(x)$ is neither odd nor even.

5. $f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$

$$f(-x) = a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0 = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0 = f(x)$$

So, $f(x)$ is even.

7. (a)

April 11:	10 hours
April 12:	24 hours
April 13:	24 hours
April 14:	$23\frac{2}{3}$ hours
Total:	$81\frac{2}{3}$ hours

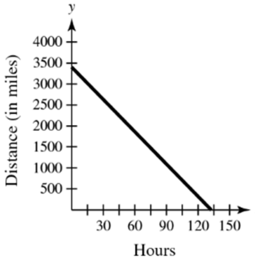
(b) Speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{2100}{81\frac{2}{3}} = \dfrac{180}{7} = 25\dfrac{5}{7}$ mph

(c) $D = -\dfrac{180}{7}t + 3400$

Domain: $0 \le t \le \dfrac{1190}{9}$

Range: $0 \le D \le 3400$

(d)



9. (a)–(d) Use $f(x) = 4x$ and $g(x) = x + 6$.

(a) $(f \circ g)(x) = f(x + 6) = 4(x + 6) = 4x + 24$

(b) $(f \circ g)^{-1}(x) = \dfrac{x - 24}{4} = \dfrac{1}{4}x - 6$

(c) $f^{-1}(x) = \dfrac{1}{4}x$

$g^{-1}(x) = x - 6$

(d) $(g^{-1} \circ f^{-1})(x) = g^{-1}\left(\dfrac{1}{4}x\right) = \dfrac{1}{4}x - 6$

(e) $f(x) = x^3 + 1$ and $g(x) = 2x$

$(f \circ g)(x) = f(2x) = (2x)^3 + 1 = 8x^3 + 1$

$(f \circ g)^{-1}(x) = \sqrt[3]{\dfrac{x - 1}{8}} = \dfrac{1}{2}\sqrt[3]{x - 1}$

$f^{-1}(x) = \sqrt[3]{x - 1}$

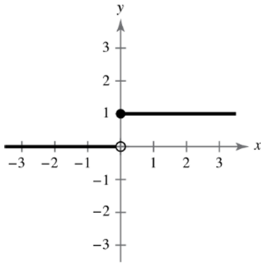
$g^{-1}(x) = \dfrac{1}{2}x$

$(g^{-1} \circ f^{-1})(x) = g^{-1}\left(\sqrt[3]{x - 1}\right) = \dfrac{1}{2}\sqrt[3]{x - 1}$

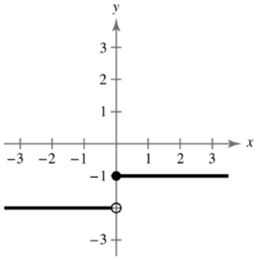
(f) Answers will vary.

(g) Conjecture: $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$

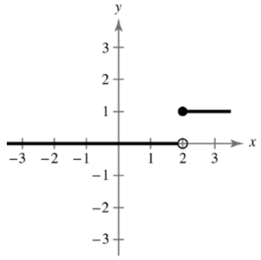
11. $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$



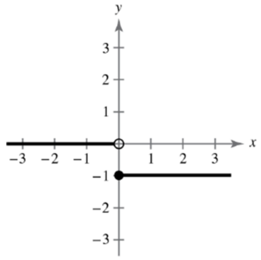
(a) $H(x) - 2$



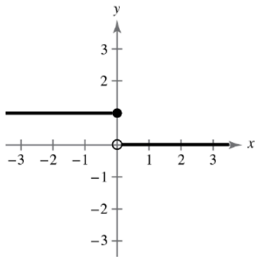
(b) $H(x - 2)$



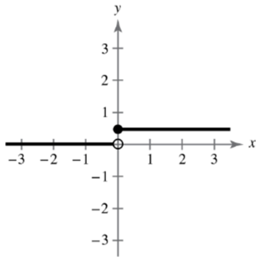
(c) $-H(x)$



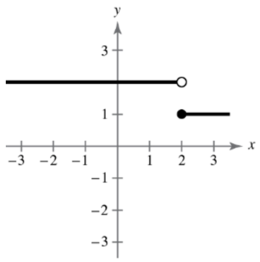
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



(f) $-H(x-2)+2$



13. $(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))) = (f \circ g \circ h)(x)$

$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = (f \circ g \circ h)(x)$

15.

x	$f(x)$	$f^{-1}(x)$
−4	—	2
−3	4	1
−2	1	0
−1	0	—
0	−2	−1
1	−3	−2
2	−4	—
3	—	—
4	—	−3

(a)

x	$f(f^{-1}(x))$
−4	$f(f^{-1}(-4)) = f(2) = -4$
−2	$f(f^{-1}(-2)) = f(0) = -2$
0	$f(f^{-1}(0)) = f(-1) = 0$
4	$f(f^{-1}(4)) = f(-3) = 4$

(b)

x	$(f + f^{-1})(x)$
−3	$f(-3) + f^{-1}(-3) = 4 + 1 = 5$
−2	$f(-2) + f^{-1}(-2) = 1 + 0 = 1$
0	$f(0) + f^{-1}(0) = -2 + (-1) = -3$
1	$f(1) + f^{-1}(1) = -3 + (-2) = -5$

(c)

x	$(f \cdot f^{-1})(x)$
−3	$f(-3)f^{-1}(-3) = (4)(1) = 4$
−2	$f(-2)f^{-1}(-2) = (1)(0) = 0$
0	$f(0)f^{-1}(0) = (-2)(-1) = 2$
1	$f(1)f^{-1}(1) = (-3)(-2) = 6$

(d)

x	$\lvert f^{-1}(x)\rvert$
−4	$\lvert f^{-1}(-4)\rvert = \lvert 2\rvert = 2$
−3	$\lvert f^{-1}(-3)\rvert = \lvert 1\rvert = 1$
0	$\lvert f^{-1}(0)\rvert = \lvert -1\rvert = 1$
4	$\lvert f^{-1}(4)\rvert = \lvert -3\rvert = 3$

Practice Test for Chapter 1

1. Find the equation of the line through $(2, 4)$ and $(3, -1)$.

2. Find the equation of the line with slope $m = 4/3$ and y-intercept $b = -3$.

3. Find the equation of the line through $(4, 1)$ perpendicular to the line $2x + 3y = 0$.

4. If it costs a company \$32 to produce 5 units of a product and \$44 to produce 9 units, how much does it cost to produce 20 units? (Assume that the cost function is linear.)

5. Given $f(x) = x^2 - 2x + 1$, find $f(x - 3)$.

6. Given $f(x) = 4x - 11$, find $\dfrac{f(x) - f(3)}{x - 3}$.

7. Find the domain and range of $f(x) = \sqrt{36 - x^2}$.

8. Which equations determine y as a function of x?
   (a) $6x - 5y + 4 = 0$
   (b) $x^2 + y^2 = 9$
   (c) $y^3 = x^2 + 6$

9. Sketch the graph of $f(x) = x^2 - 5$.

10. Sketch the graph of $f(x) = |x + 3|$.

11. Sketch the graph of $f(x) = \begin{cases} 2x + 1, & \text{if } x \geq 0, \\ x^2 - x, & \text{if } x < 0. \end{cases}$

12. Use the graph of $f(x) = |x|$ to graph the following:
    (a) $f(x + 2)$
    (b) $-f(x) + 2$

13. Given $f(x) = 3x + 7$ and $g(x) = 2x^2 - 5$, find the following:
    (a) $(g - f)(x)$
    (b) $(fg)(x)$

14. Given $f(x) = x^2 - 2x + 16$ and $g(x) = 2x + 3$, find $f(g(x))$.

15. Given $f(x) = x^3 + 7$, find $f^{-1}(x)$.

16. Which of the following functions have inverses?
    (a) $f(x) = |x - 6|$
    (b) $f(x) = ax + b, a \neq 0$
    (c) $f(x) = x^3 - 19$

17. Given $f(x) = \sqrt{\dfrac{3 - x}{x}}, 0 < x \le 3,$ find $f^{-1}(x)$.

Exercises 18–20, true or false?

18. $y = 3x + 7$ and $y = \frac{1}{3}x - 4$ are perpendicular.

19. $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

20. If a function has an inverse, then it must pass both the Vertical Line Test and the Horizontal Line Test.

CHAPTER 2

Polynomial and Rational Functions

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C H A P T E R 2
Polynomial and Rational Functions

Section 2.1 Quadratic Functions and Models

1. nonnegative integer; real

3. Yes, $f(x) = (x - 2)^2 + 3$ is in the form $f(x) = a(x - h)^2 + k$. The vertex is $(2, 3)$.

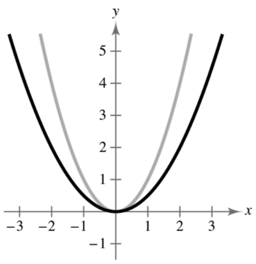
5. $f(x) = x^2 - 2$ opens upward and has vertex $(0, -2)$. Matches graph (b).

6. $f(x) = (x + 1)^2 - 2$ opens upward and has vertex $(-1, -2)$. Matches graph (a).

7. $f(x) = -(x - 4)^2$ opens downward and has vertex $(4, 0)$. Matches graph (c).

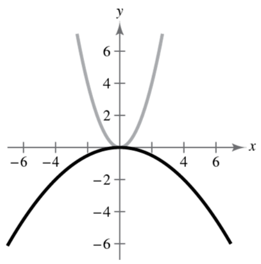
8. $f(x) = 4 - (x - 2)^2 = -(x - 2)^2 + 4$ opens downward and has vertex $(2, 4)$. Matches graph (d).

9. (a) $y = \frac{1}{2}x^2$



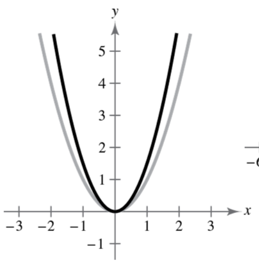
Vertical shrink

(b) $y = -\frac{1}{8}x^2$



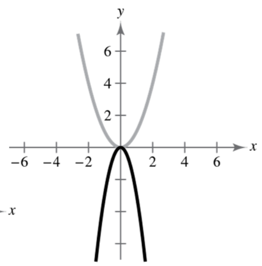
Vertical shrink and a reflection in the x-axis

(c) $y = \frac{3}{2}x^2$



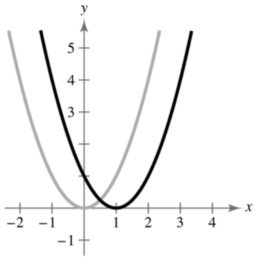
Vertical stretch

(d) $y = -3x^2$



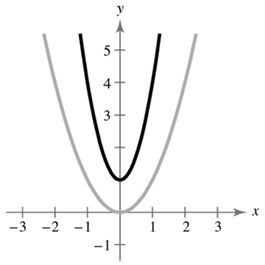
Vertical stretch and a reflection in the x-axis

11. (a) $y = (x - 1)^2$



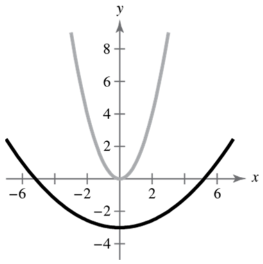
Horizontal shift one unit to the right

(b) $y = (3x)^2 + 1$



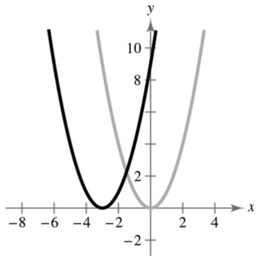
Horizontal shrink and a vertical shift one unit upward

(c) $y = \left(\frac{1}{3}x\right)^2 - 3$



Horizontal stretch and a vertical shift three units downward

(d) $y = (x + 3)^2$



Horizontal shift three units to the left

13. $$\begin{aligned} f(x) &= x^2 - 6x \\ &= \left(x^2 - 6x + 9\right) - 9 \\ &= (x - 3)^2 - 9 \end{aligned}$$

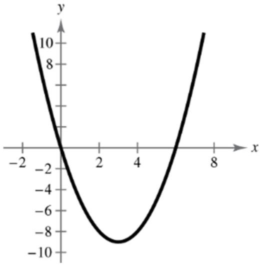
Vertex: $(3, -9)$

Axis of symmetry: $x = 3$

Find x-intercepts:

$$\begin{aligned} x^2 - 6x &= 0 \\ x(x - 6) &= 0 \\ x &= 0 \\ x - 6 = 0 \Rightarrow x &= 6 \end{aligned}$$

x-intercepts: $(0, 0), (6, 0)$



15. $h(x) = x^2 - 8x + 16 = (x - 4)^2$

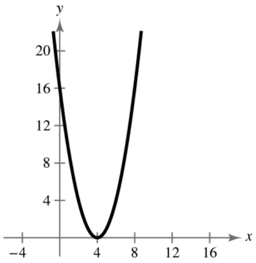
Vertex: $(4, 0)$

Axis of symmetry: $x = 4$

Find x-intercepts:

$$\begin{aligned} (x - 4)^2 &= 0 \\ x - 4 &= 0 \\ x &= 4 \end{aligned}$$

x-intercept: $(4, 0)$



17. $$\begin{aligned} f(x) &= x^2 - 6x + 2 \\ &= \left(x^2 - 6x + 9\right) - 9 + 2 \\ &= \left(x^2 - 6x + 9\right) - 7 \\ &= (x - 3)^2 - 7 \end{aligned}$$

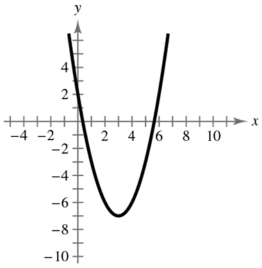
Vertex: $(3, -7)$

Axis of symmetry: $x = 3$

Find x-intercepts:

$$\begin{aligned} x^2 - 6x + 2 &= 0 \\ x^2 - 6x &= -2 \\ x^2 - 6x + 9 &= -2 + 9 \\ (x - 3)^2 &= 7 \\ x &= 3 \pm \sqrt{7} \end{aligned}$$

x-intercepts: $\left(3 \pm \sqrt{7}, 0\right)$



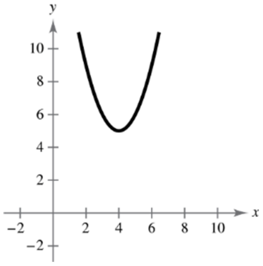
19. $$\begin{aligned} f(x) &= x^2 - 8x + 21 \\ &= \left(x^2 - 8x + 16\right) - 16 + 21 \\ &= (x - 4)^2 + 5 \end{aligned}$$

Vertex: $(4, 5)$

Axis of symmetry: $x = 4$

Find x-intercepts:

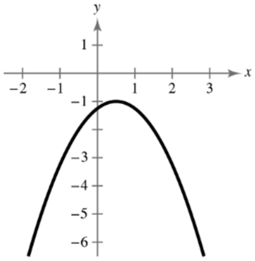
$$\begin{aligned} x^2 - 8x + 21 &= 0 \\ x^2 - 8x &= -21 \\ x^2 - 8x + 16 &= -21 + 16 \\ (x - 4)^2 &= -5 \\ x - 4 &= \pm\sqrt{-5} \\ x &= 4 \pm \sqrt{5}\,i \end{aligned}$$



Not a real number

No x-intercepts

21. $$\begin{aligned} f(x) &= -x^2 + x - \frac{5}{4} \\ &= -\left(x^2 - x + \frac{5}{4}\right) \\ &= -\left(x^2 - x + \frac{1}{4}\right) - \frac{5}{4} + \frac{1}{4} \\ &= -\left(x - \frac{1}{2}\right)^2 - 1 \end{aligned}$$



Vertex: $\left(\frac{1}{2}, -1\right)$

Axis of symmetry: $x = \frac{1}{2}$

Find x-intercepts:

$$-x^2 + x - \frac{5}{4} = 0$$

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 - 5}}{2} = \frac{-1 \pm \sqrt{-4}}{2} \\ &= \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i \end{aligned}$$

No x-intercepts

23. $f(x) = -x^2 + 2x + 5$

$= -(x^2 - 2x + 1) - (-1) + 5$

$= -(x - 1)^2 + 6$

Vertex: $(1, 6)$

Axis of symmetry: $x = 1$

Find x-intercepts:

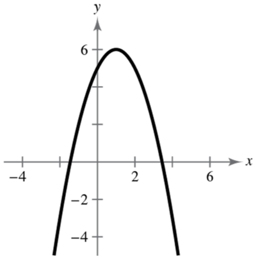
$-x^2 + 2x + 5 = 0$

$x^2 - 2x - 5 = 0$

$$x = \frac{2 \pm \sqrt{4 + 20}}{2}$$

$$= 1 \pm \sqrt{6}$$

x-intercepts: $(1 - \sqrt{6}, 0), (1 + \sqrt{6}, 0)$



25. $h(x) = 4x^2 - 4x + 21$

$$= 4\left(x^2 - x + \frac{1}{4}\right) - 4\left(\frac{1}{4}\right) + 21$$

$$= 4\left(x - \frac{1}{2}\right)^2 + 20$$

Vertex: $\left(\frac{1}{2}, 20\right)$

Axis of symmetry: $x = \frac{1}{2}$

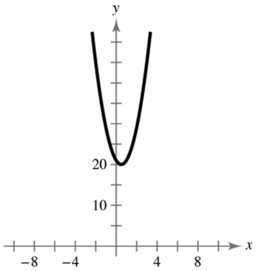
Find x-intercepts:

$4x^2 - 4x + 21 = 0$

$$x = \frac{4 \pm \sqrt{16 - 336}}{2(4)}$$

Not a real number

No x-intercepts

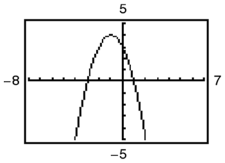


27. $f(x) = -(x^2 + 2x - 3) = -(x + 1)^2 + 4$

Vertex: $(-1, 4)$

Axis of symmetry: $x = -1$

x-intercepts: $(-3, 0), (1, 0)$

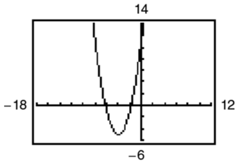


29. $g(x) = x^2 + 8x + 11 = (x + 4)^2 - 5$

Vertex: $(-4, -5)$

Axis of symmetry: $x = -4$

x-intercepts: $(-4 \pm \sqrt{5}, 0)$



31. $f(x) = -2x^2 + 12x - 18$

$= -2(x^2 - 6x + 9 - 9) - 18$

$= -2(x^2 - 6x + 9) + 18 - 18$

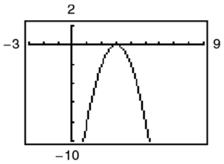
$= -2(x^2 - 6x + 9)$

$= -2(x - 3)^2$

Vertex: $(3, 0)$

Axis of symmetry: $x = 3$

x-intercept: $(3, 0)$

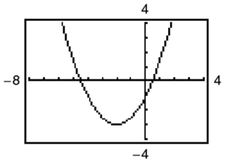


33. $g(x) = \frac{1}{2}(x^2 + 4x - 2) = \frac{1}{2}(x + 2)^2 - 3$

Vertex: $(-2, -3)$

Axis of symmetry: $x = -2$

x-intercepts: $(-2 \pm \sqrt{6}, 0)$



35. $(-2, -1)$ is the vertex.

$f(x) = a(x + 2)^2 - 1$

Because the graph passes through $(0, 3)$,

$3 = a(0 + 2)^2 - 1$

$3 = 4a - 1$

$4 = 4a$

$1 = a.$

So, $y = (x + 2)^2 - 1.$

37. $(-2, 5)$ is the vertex.

$f(x) = a(x + 2)^2 + 5$

Because the graph passes through $(0, 9)$,

$9 = a(0 + 2)^2 + 5$

$4 = 4a$

$1 = a.$

So, $f(x) = 1(x + 2)^2 + 5 = (x + 2)^2 + 5.$

39. $(1, -2)$ is the vertex.

$f(x) = a(x - 1)^2 - 2$

Because the graph passes through $(-1, 14)$,

$$\begin{aligned} 14 &= a(-1 - 1)^2 - 2 \\ 14 &= 4a - 2 \\ 16 &= 4a \\ 4 &= a. \end{aligned}$$

So, $f(x) = 4(x - 1)^2 - 2$.

41. $(5, 12)$ is the vertex.

$f(x) = a(x - 5)^2 + 12$

Because the graph passes through $(7, 15)$,

$$\begin{aligned} 15 &= a(7 - 5)^2 + 12 \\ 3 &= 4a \Rightarrow a = \tfrac{3}{4}. \end{aligned}$$

So, $f(x) = \frac{3}{4}(x - 5)^2 + 12$.

43. $\left(-\frac{1}{4}, \frac{3}{2}\right)$ is the vertex.

$f(x) = a\left(x + \frac{1}{4}\right)^2 + \frac{3}{2}$

Because the graph passes through $(-2, 0)$,

$$\begin{aligned} 0 &= a\left(-2 + \tfrac{1}{4}\right)^2 + \tfrac{3}{2} \\ -\tfrac{3}{2} &= \tfrac{49}{16}a \Rightarrow a = -\tfrac{24}{49}. \end{aligned}$$

So, $f(x) = -\frac{24}{49}\left(x + \frac{1}{4}\right)^2 + \frac{3}{2}$.

45. $y = x^2 - 2x - 3$

$$\begin{aligned} 0 &= x^2 - 2x - 3 \\ 0 &= (x - 3)(x + 1) \end{aligned}$$

$x = 3$ or $x = -1$

x-intercepts: $(3, 0), (-1, 0)$

47. $y = 2x^2 + 5x - 3$

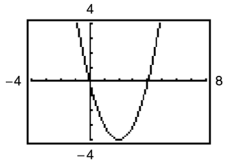
$$\begin{aligned} 0 &= 2x^2 + 5x - 3 \\ 0 &= (2x - 1)(x + 3) \end{aligned}$$

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$x + 3 = 0 \Rightarrow x = -3$

x-intercepts: $\left(\frac{1}{2}, 0\right), (-3, 0)$

49. $f(x) = x^2 - 4x$



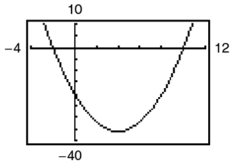
x-intercepts: $(0, 0), (4, 0)$

$$\begin{aligned} 0 &= x^2 - 4x \\ 0 &= x(x - 4) \end{aligned}$$

$x = 0$ or $x = 4$

The x-intercepts and the solutions of $f(x) = 0$ are the same.

51. $f(x) = x^2 - 8x - 20$



x-intercepts: $(-2, 0), (10, 0)$

$$\begin{aligned} 0 &= x^2 - 8x - 20 \\ 0 &= (x + 2)(x - 10) \end{aligned}$$

$x + 2 = 0 \Rightarrow x = -2$

$x - 10 = 0 \Rightarrow x = 10$

The x-intercepts and the solutions of $f(x) = 0$ are the same.

53. (a) $f(x) = [x - (-3)](x - 3)$

$= (x + 3)(x - 3)$

$= x^2 - 9$ opens upward

(b) $g(x) = -[x - (-3)](x - 3)$

$= -(x + 3)(x - 3)$

$= -x^2 + 9$ opens downward

55. (a) $f(x) = (x - 1)(x - 1)$

$= x^2 - 2x + 1$ opens upward

(b) $g(x) = -x^2 + 1$ opens downward

57. Let $x =$ the first number and $y =$ the second number. Then the sum is

$x + y = 110 \Rightarrow y = 110 - x.$

The product is $P(x) = xy = x(110 - x) = 110x - x^2$.

$$\begin{aligned} P(x) &= -x^2 + 110x \\ &= -\left(x^2 - 110x + 3025 - 3025\right) \\ &= -\left[(x - 55)^2 - 3025\right] \\ &= -(x - 55)^2 + 3025 \end{aligned}$$

The maximum value of the product occurs at the vertex of $P(x)$ and is 3025. This happens when $x = y = 55$.

59. Let $x =$ the first number and $y =$ the second number.

Then the sum is

$$x + 2y = 24 \Rightarrow y = \frac{24 - x}{2}.$$

The product is $P(x) = xy = x\left(\frac{24 - x}{2}\right).$

$$\begin{aligned} P(x) &= \frac{1}{2}\left(-x^2 + 24x\right) \\ &= -\frac{1}{2}\left(x^2 - 24x + 144 - 144\right) \\ &= -\frac{1}{2}\left[(x - 12)^2 - 144\right] = -\frac{1}{2}(x - 12)^2 + 72 \end{aligned}$$

The maximum value of the product occurs at the vertex of $P(x)$ and is 72. This happens when $x = 12$ and $y = (24 - 12)/2 = 6.$ So, the numbers are 12 and 6.

61. $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$

The vertex occurs at $-\frac{b}{2a} = \frac{-24/9}{2(-4/9)} = 3.$

The maximum height is

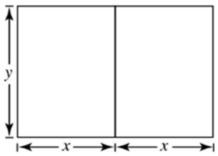
$$y(3) = -\frac{4}{9}(3)^2 + \frac{24}{9}(3) + 12 = 16 \text{ feet.}$$

63. $C = 800 - 10x + 0.25x^2 = 0.25x^2 - 10x + 800$

The vertex occurs at $x = -\frac{b}{2a} = -\frac{-10}{2(0.25)} = 20.$

The cost is minimum when $x = 20$ fixtures.

65. (a)



$$4x + 3y = 200 \Rightarrow y = \frac{1}{3}(200 - 4x) = \frac{4}{3}(50 - x)$$

$$A = 2xy = 2x\left[\frac{4}{3}(50 - x)\right] = \frac{8}{3}x(50 - x) = \frac{8x(50 - x)}{3}$$

(b) To find the dimensions that produce a maximum enclosed area, you can find the vertex.

To do this, either write the quadratic function in standard form or use $x = -\frac{b}{2a}$, so the coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$

$$\begin{aligned} A &= \frac{8}{3}x(50 - x) \\ &= -\frac{8}{3}\left(x^2 - 50x\right) \\ &= -\frac{8}{3}\left(x^2 - 50x + 625 - 625\right) \\ &= -\frac{8}{3}\left[(x - 25)^2 - 625\right] \\ &= -\frac{8}{3}(x - 25)^2 + \frac{5000}{3} \end{aligned}$$

So the vertex is $\left(25, \frac{5000}{3}\right)$ from the standard form, or is $x = -\frac{b}{2a} = -\frac{\frac{400}{3}}{2\left(-\frac{8}{3}\right)} = \frac{400}{16} = 25$ and $A(25) = \frac{5000}{3}.$

When $x = 25$ feet and $y = \frac{(200 - 4(25))}{3} = \frac{100}{3}$ feet.

The dimensions are $2x = 50$ feet by $33\frac{1}{3}$ feet, and the maximum enclosed area is $\frac{5000}{3} \approx 1666.67$ square feet.

67. True. The equation $-12x^2 - 1 = 0$ has no real solution, so the graph has no x-intercepts.

69. $f(x) = x^2 + bx - 25$, minimum value: –50

The minimum value, –50, is the y-coordinate of the vertex.

Find the x-coordinate:

$$x = -\frac{b}{2a} = -\frac{b}{2(1)} = -\frac{b}{2}$$

$$f(x) = x^2 + bx - 25$$

$$f\left(-\frac{b}{2}\right) = \left(-\frac{b}{2}\right)^2 + b\left(-\frac{b}{2}\right) - 25$$

$$-50 = \frac{b^2}{4} - \frac{b^2}{2} - 25$$

$$-25 = \frac{-b^2}{4}$$

$$100 = b^2$$

$$\pm 10 = b$$

71. $f(x) = ax^2 + bx + c$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$f\left(-\frac{b}{2a}\right) = a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right) + c$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$= \frac{b^2 - 2b^2 + 4ac}{4a} = \frac{4ac - b^2}{4a}$$

So, the vertex occurs at

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

73. $3x^4$

(a) Standard form: $3x^4$

(b) Degree: 4

Leading coefficient: 3

(c) Monomial

75. $x - 4x^2 + 1$

(a) Standard form: $-4x^2 + x + 1$

(b) Degree: 2

Leading coefficient: –4

(c) Trinomial

77. $-29x^3 + \frac{1}{4}x$

(a) Standard form: $-29x^3 + \frac{1}{4}x$

(b) Degree: 3

Leading coefficient: –29

(c) Binomial

79. $(y - 1)(y - 9) = y^2 - 9y - y + 9 = y^2 - 10y + 9$

81. $$\begin{aligned}(x^2 + x - 3)(x^2 - 4x - 2) &= (x^4 - 4x^3 - 2x^2) + (x^3 - 4x^2 - 2x) - 3(x^2 - 4x - 2)\\ &= x^4 - 3x^3 - 9x^2 + 10x + 6\end{aligned}$$

83. $y = (x + 2)^2$

$y = 0 \Rightarrow x = -2 \Rightarrow$ x-intercept: $(-2, 0)$

$x = 0 \Rightarrow y = 4 \Rightarrow$ y-intercept: $(0, 4)$

85. $y = |x - 4| - 2$

$y = 0 \Rightarrow |x - 4| = 2 \Rightarrow x = 2, 6 \Rightarrow$ x-intercepts: $(2, 0)$, $(6, 0)$

$x = 0 \Rightarrow y = 2 \Rightarrow$ y-intercept: $(0, 2)$

Section 2.2 Polynomial Functions of Higher Degree

1. continuous

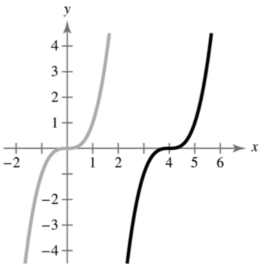
3. touches; crosses

5. No. If f is an even-degree fourth-degree polynomial function, its left and right end behavior is either that it rises left and right or falls left and right.

7. Because f is a polynomial, it is a continuous on $[x_1, x_2]$ and $f(x_1) < 0$ and $f(x_2) > 0$. Then $f(x) = 0$ for some value of x in $[x_1, x_2]$.

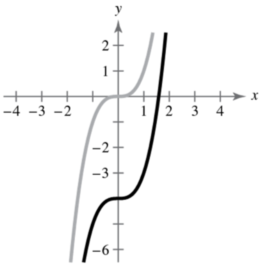
9. $y = x^3$

(a) $f(x) = (x - 4)^3$



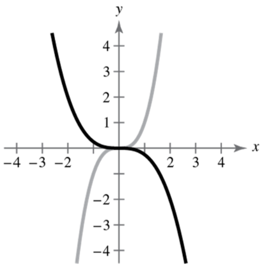
Horizontal shift four units to the right

(b) $f(x) = x^3 - 4$



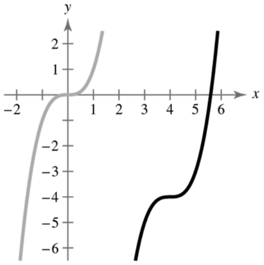
Vertical shift four units downward

(c) $f(x) = -\frac{1}{4}x^3$



Reflection in the x-axis and a vertical shrink (each y-value is multiplied by $\frac{1}{4}$)

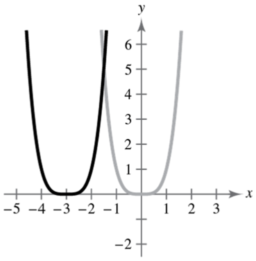
(d) $f(x) = (x - 4)^3 - 4$



Horizontal shift four units to the right and vertical shift four units downward

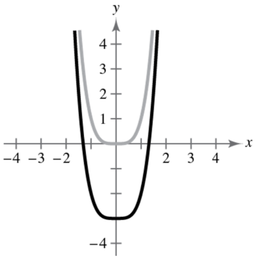
11. $y = x^4$

(a) $f(x) = (x+3)^4$



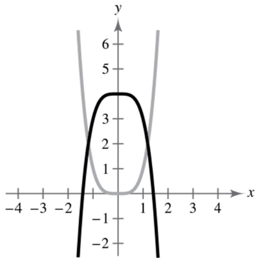
Horizontal shift three units to the left

(b) $f(x) = x^4 - 3$



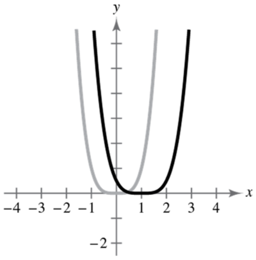
Vertical shift three units downward

(c) $f(x) = 4 - x^4$



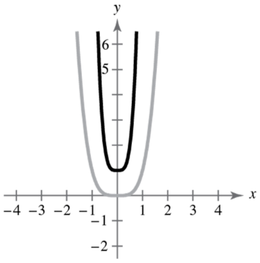
Reflection in the x-axis and then a vertical shift four units upward

(d) $f(x) = \frac{1}{2}(x-1)^4$



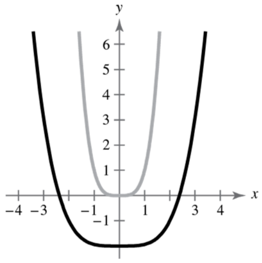
Horizontal shift one unit to the right and a vertical shrink (each y-value is multiplied by $\frac{1}{2}$)

(e) $f(x) = (2x)^4 + 1$



Vertical shift one unit upward and a horizontal shrink (each y-value is multiplied by 16)

(f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$



Vertical shift two units downward and a horizontal stretch (each y-value is multipied by $\frac{1}{16}$)

13. $f(x) = -2x^2 - 5x$ is a parabola with x-intercepts $(0, 0)$ and $\left(-\frac{5}{2}, 0\right)$ and opens downward. Matches graph (c).

14. $f(x) = 2x^3 - 3x + 1$ has intercepts $(0, 1), (1, 0), \left(-\frac{1}{2} - \frac{1}{2}\sqrt{3}, 0\right)$ and $\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}, 0\right)$. Matches graph (f).

15. $f(x) = -\frac{1}{4}x^4 + 3x^2$ has intercepts $(0, 0)$ and $\left(\pm 2\sqrt{3}, 0\right)$. Matches graph (a).

16. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$ has y-intercept $\left(0, -\frac{4}{3}\right)$. Matches graph (e).

17. $f(x) = x^4 + 2x^3$ has intercepts $(0, 0)$ and $(-2, 0)$. Matches graph (d).

18. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$ has intercepts $(0, 0), (1, 0), (-1, 0), (3, 0), (-3, 0)$. Matches graph (b).

19. $f(x) = 12x^3 + 4x$

Degree: 3

Leading coefficient: 12

The degree is odd and the leading coefficient is positive. The graph falls to the left and rises to the right.

21. $g(x) = 5 - \frac{7}{2}x - 3x^2$

Degree: 2

Leading coefficient: -3

The degree is even and the leading coefficient is negative. The graph falls to the left and falls to the right.

23. $g(x) = 6x - 9x^3 + x^2$

Degree: 3

Leading coefficient: -9

The degree is odd and the leading coefficient is negative. The graph rises to the left and falls to the right.

25. $f(x) = 9.8x^6 - 1.2x^3$

Degree: 6

Leading coefficient: 9.8

The degree is even and the leading coefficient is positive. The graph rises to the left and rises to the right.

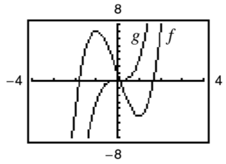
27. $f(s) = -\frac{7}{8}\left(s^3 + 5s^2 - 7s + 1\right)$

Degree: 3

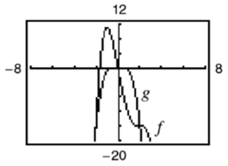
Leading coefficient: $-\frac{7}{8}$

The degree is odd and the leading coefficient is negative. The graph rises to the left and falls to the right.

29. $f(x) = 3x^3 - 9x + 1;\ g(x) = 3x^3$



31. $f(x) = -\left(x^4 - 4x^3 + 16x\right);\ g(x) = -x^4$



33. $f(x) = x^2 - 36$

(a) $0 = x^2 - 36$

$0 = (x + 6)(x - 6)$

$x + 6 = 0 \qquad x - 6 = 0$

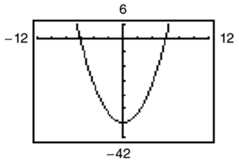
$x = -6 \qquad x = 6$

Zeros: ±6

(b) Each zero has a multiplicity of one (odd multiplicity).

(c) Turning points: 1 (the vertex of the parabola)

(d)



35. $h(t) = t^2 - 6t + 9$

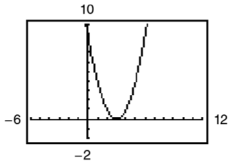
(a) $0 = t^2 - 6t + 9 = (t - 3)^2$

Zero: $t = 3$

(b) $t = 3$ has a multiplicity of 2 (even multiplicity).

(c) Turning points: 1 (the vertex of the parabola)

(d)



37. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$

(a) $0 = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$

$= \frac{1}{3}\left(x^2 + x - 2\right)$

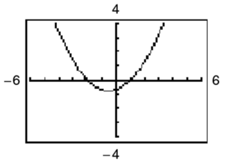
$= \frac{1}{3}(x + 2)(x - 1)$

Zeros: $x = -2, x = 1$

(b) Each zero has a multiplicity of 1 (odd multiplicity).

(c) Turning points: 1 (the vertex of the parabola)

(d)



39. $g(x) = 5x\left(x^2 - 2x - 1\right)$

(a) $0 = 5x\left(x^2 - 2x - 1\right)$

$0 = x\left(x^2 - 2x - 1\right)$

For $x^2 - 2x - 1 = 0, a = 1, b = -2, c = -1.$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

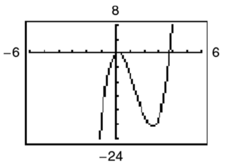
$$= 1 \pm \sqrt{2}$$

Zeros: $x = 0, x = 1 \pm \sqrt{2}$

(b) Each zero has a multiplicity of 1 (odd multiplicity).

(c) Turning points: 2

(d)



41. $f(x) = -3x^3 + 12x^2 - 3x$

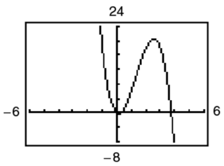
(a) $0 = -3x^3 + 12x^2 - 3x = -3x(x^2 - 4x + 1)$

Zeros: $x = 0,\ x = 2 \pm \sqrt{3}$ (by Quadratic Formula)

(b) Each zero has a multiplicity of 1 (odd multiplicity).

(c) Turning points: 2

(d)



43. $g(t) = t^5 - 6t^3 + 9t$

(a) $0 = t^5 - 6t^3 + 9t = t(t^4 - 6t^2 + 9) = t(t^2 - 3)^2$

$= t(t + \sqrt{3})^2(t - \sqrt{3})^2$

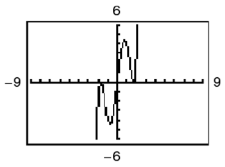
Zeros: $t = 0, t = \pm\sqrt{3}$

(b) $t = 0$ has a multiplicity of 1 (odd multiplicity).

$t = \pm\sqrt{3}$ each have a multiplicity of 2 (even multiplicity).

(c) Turning points: 4

(d)



45. $f(x) = 3x^4 + 9x^2 + 6$

(a) $0 = 3x^4 + 9x^2 + 6$

$0 = 3(x^4 + 3x^2 + 2)$

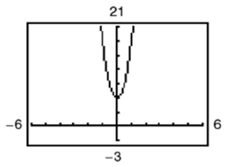
$0 = 3(x^2 + 1)(x^2 + 2)$

No real zeros

(b) No multiplicity

(c) Turning points: 1

(d)



47. $g(x) = x^3 + 3x^2 - 4x - 12$

(a) $0 = x^3 + 3x^2 - 4x - 12 = x^2(x + 3) - 4(x + 3)$

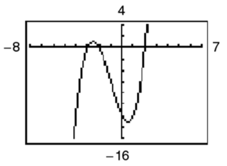
$= (x^2 - 4)(x + 3) = (x - 2)(x + 2)(x + 3)$

Zeros: $x = \pm 2, x = -3$

(b) Each zero has a multiplicity of 1 (odd multiplicity).

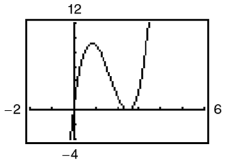
(c) Turning points: 2

(d)



49. $y = 4x^3 - 20x^2 + 25x$

(a)



(b) x-intercepts: $(0, 0), \left(\frac{5}{2}, 0\right)$

(c) $0 = 4x^3 - 20x^2 + 25x$

$0 = x(4x^2 - 20x + 25)$

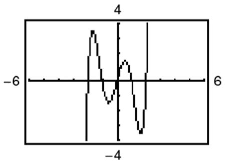
$0 = x(2x - 5)^2$

$x = 0, \frac{5}{2}$

(d) The solutions are the same as the x-coordinates of the x-intercepts.

51. $y = x^5 - 5x^3 + 4x$

(a)



(b) x-intercepts: $(0, 0), (\pm 1, 0), (\pm 2, 0)$

(c) $0 = x^5 - 5x^3 + 4x$

$0 = x(x^2 - 1)(x^2 - 4)$

$0 = x(x + 1)(x - 1)(x + 2)(x - 2)$

$x = 0, \pm 1, \pm 2$

(d) The solutions are the same as the x-coordinates of the x-intercepts.

53. $f(x) = (x - 0)(x - 7)$

$= x^2 - 7x$

Note: $f(x) = ax(x - 7)$ has zeros 0 and 7 for all real numbers $a \neq 0$.

55. $f(x) = (x - 0)(x + 2)(x + 4)$

$= x(x^2 + 6x + 8)$

$= x^3 + 6x^2 + 8x$

Note: $f(x) = ax(x + 2)(x + 4)$ has zeros $0, -2$, and -4 for all real numbers $a \neq 0$.

57. $f(x) = (x - 4)(x + 3)(x - 3)(x - 0)$

$= (x - 4)(x^2 - 9)x$

$= x^4 - 4x^3 - 9x^2 + 36x$

Note: $f(x) = a(x^4 - 4x^3 - 9x^2 + 36x)$ has zeros $4, -3, 3$, and 0 for all real numbers $a \neq 0$.

59. $f(x) = \left[x - (1 + \sqrt{2})\right]\left[x - (1 - \sqrt{2})\right]$

$= \left[(x - 1) - \sqrt{2}\right]\left[(x - 1) + \sqrt{2}\right]$

$= (x - 1)^2 - (\sqrt{2})^2$

$= x^2 - 2x + 1 - 2$

$= x^2 - 2x - 1$

Note: $f(x) = a(x^2 - 2x - 1)$ has zeros $1 + \sqrt{2}$ and $1 - \sqrt{2}$ for all real numbers $a \neq 0$.

61. $f(x) = (x - 2)\left[x - (2 + \sqrt{5})\right]\left[x - (2 - \sqrt{5})\right]$

$= (x - 2)\left[(x - 2) - \sqrt{5}\right]\left[(x - 2) + \sqrt{5}\right]$

$= (x - 2)\left[(x - 2)^2 - 5\right]$

$= (x - 2)\left[x^2 - 4x + 4 - 5\right]$

$= (x - 2)(x^2 - 4x - 1)$

$= x^3 - 6x^2 + 7x + 2$

Note: $f(x) = a(x^3 - 6x^2 + 7x + 2)$ has zeros 2, $2 + \sqrt{5}$, and $2 - \sqrt{5}$ for all real numbers $a \neq 0$.

63. $f(x) = (x + 3)(x + 3) = x^2 + 6x + 9$

Note: $f(x) = a(x^2 + 6x + 9)$, $a \neq 0$, has degree 2 and zero $x = -3$.

65. $f(x) = (x - 0)(x + 5)(x - 1)$

$= x(x^2 + 4x - 5)$

$= x^3 + 4x^2 - 5x$

Note: $f(x) = ax(x^2 + 4x - 5)$, $a \neq 0$, has degree 3 and zeros $x = 0, -5$, and 1.

67. $f(x) = (x - (-5))^2(x - 1)(x - 2) = x^4 + 7x^3 - 3x^2 - 55x + 50$

or $f(x) = (x - (-5))(x - 1)^2(x - 2) = x^4 + x^3 - 15x^2 + 23x - 10$

or $f(x) = (x - (-5))(x - 1)(x - 2)^2 = x^4 - 17x^2 + 36x - 20$

Note: Any nonzero scalar multiple of these functions would also have degree 4 and zeros $x = -5, 1$, and 2.

69. $f(t) = \frac{1}{4}(t^2 - 2t + 15) = \frac{1}{4}(t - 1)^2 + \frac{7}{2}$

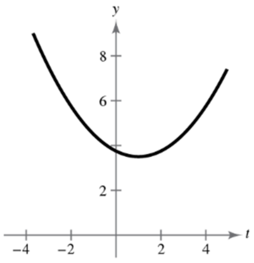
(a) Rises to the left; rises to the right

(b) No real zeros (no x-intercepts)

(c)

t	-1	0	1	2	3
$f(t)$	4.5	3.75	3.5	3.75	4.5

(d) The graph is a parabola with vertex $\left(1, \frac{7}{2}\right)$.



71. $f(x) = x^3 - 25x = x(x + 5)(x - 5)$

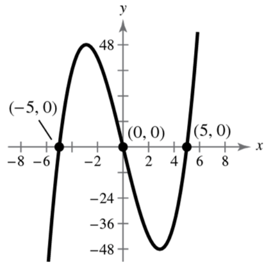
(a) Falls to the left; rises to the right

(b) Zeros: $0, -5, 5$

(c)

x	-2	-1	0	1	2
$f(x)$	42	24	0	-24	-42

(d)



73. $f(x) = -8 + \frac{1}{2}x^4 = \frac{1}{2}(x^4 - 16)$

$= \frac{1}{2}(x^2 + 4)(x - 2)(x + 2)$

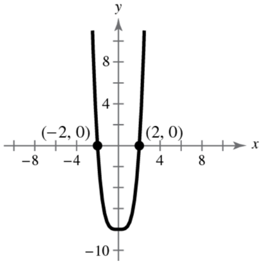
(a) Rises to the left; rises to the right

(b) Zeros $x = \pm 2$:

(c)

x	-2	-1	0	1	2
$f(x)$	0	$-\frac{15}{2}$	-8	$-\frac{15}{2}$	0

(d)



75. $f(x) = 3x^3 - 15x^2 + 18x = 3x(x - 2)(x - 3)$

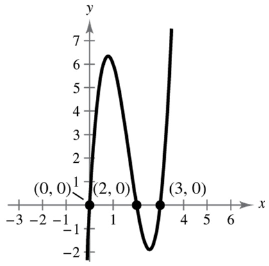
(a) Falls to the left; rises to the right

(b) Zeros: 0, 2, 3

(c)

x	0	1	2	2.5	3	3.5
$f(x)$	0	6	0	-1.875	0	7.875

(d)



77. $f(x) = -5x^2 - x^3 = -x^2(5 + x)$

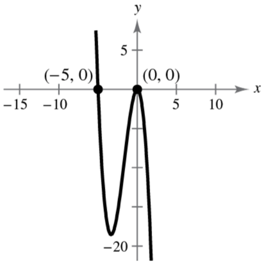
(a) Rises to the left; falls to the right

(b) Zeros: 0, −5

(c)

x	-5	-4	-3	-2	-1	0	1
$f(x)$	0	-16	-18	-12	-4	0	-6

(d)



79. $f(x) = 9x^2(x + 2)^3$

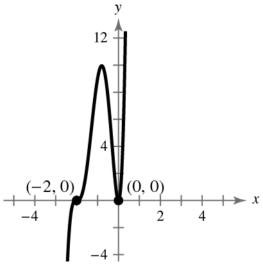
(a) Falls to the left, rises to the right

(b) Zeros: $x = 0, \ -2$

(c)

x	-3	-2	-1	0	1
$f(x)$	-81	0	9	0	243

(d)



81. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

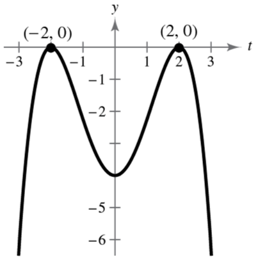
(a) Falls to the left; falls to the right

(b) Zeros: 2, −2

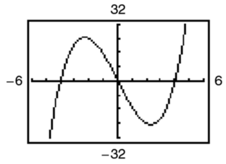
(c)

t	-3	-2	-1	0	1	2	3
$g(t)$	$-\frac{25}{4}$	0	$-\frac{9}{4}$	-4	$-\frac{9}{4}$	0	$-\frac{25}{4}$

(d)

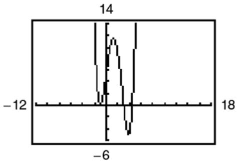


83. $f(x) = x^3 - 16x = x(x - 4)(x + 4)$



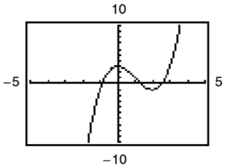
Zeros: 0 of multiplicity 1; 4 of multiplicity 1; and −4 of multiplicity 1

85. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$



Zeros: -1 of multiplicity 2; 3 of multiplicity 1; $\frac{9}{2}$ of multiplicity 1

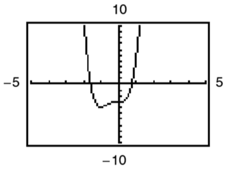
87. $f(x) = x^3 - 3x^2 + 3$



x	y
−3	−5
−2	−1
−1	−1
0	3
1	1
2	−1
3	3
4	19

The function has three zeros. They are in the intervals $[-1, 0]$, $[1, 2]$, and $[2, 3]$. They are $x \approx -0.879, 1.347, 2.532$.

89. $g(x) = 3x^4 + 4x^3 - 3$



x	y
−4	509
−3	132
−2	13
−1	−4
0	−3
1	4
2	77
3	348

The function has two zeros. They are in the intervals $[-2, -1]$ and $[0, 1]$. They are $x \approx -1.585, 0.779$.

91. (a) Volume $= l \cdot w \cdot h$

height $= x$

length $=$ width $= 36 - 2x$

So, $V(x) = (36 - 2x)(36 - 2x)(x) = x(36 - 2x)^2$.

(b) Domain: $0 < x < 18$

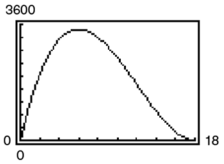
The length and width must be positive.

(c)

Box Height	Box Width	Box Volume, V
1	$36 - 2(1)$	$1[36 - 2(1)]^2 = 1156$
2	$36 - 2(2)$	$2[36 - 2(2)]^2 = 2048$
3	$36 - 2(3)$	$3[36 - 2(3)]^2 = 2700$
4	$36 - 2(4)$	$4[36 - 2(4)]^2 = 3136$
5	$36 - 2(5)$	$5[36 - 2(5)]^2 = 3380$
6	$36 - 2(6)$	$6[36 - 2(6)]^2 = 3456$
7	$36 - 2(7)$	$7[36 - 2(7)]^2 = 3388$

The volume is a maximum of 3456 cubic inches when the height is 6 inches and the length and width are each 24 inches. So the dimensions are $6 \times 24 \times 24$ inches.

(d)



The maximum point on the graph occurs at $x = 6$. This agrees with the maximum found in part (c).

93. $R = \dfrac{1}{100{,}000}\left(-x^3 + 600x^2\right)$

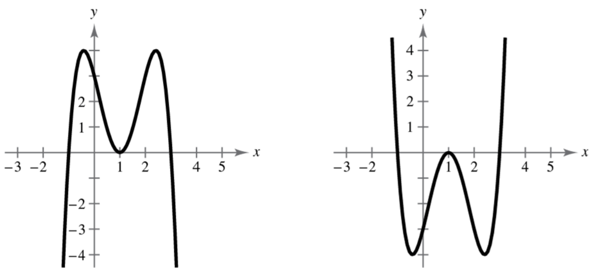
The point of diminishing returns (where the graph changes from curving upward to curving downward) occurs when $x = 200$. The point is $(200, 160)$ which corresponds to spending \$2,000,000 on advertising to obtain a revenue of \$160 million.

95. The graph will always cross the x-axis.

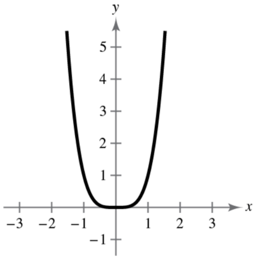
97. True. A polynomial function only falls to the right when the leading coefficient is negative.

99. False. The range of an even function cannot be $(-\infty, \infty)$. An even function's graph will fall to the left and right or rise to the left and right.

101. Answers will vary. *Sample answers:*


$a_4 < 0$ $\quad$ $a_4 > 0$

103. $f(x) = x^4$; $f(x)$ is even.



(a) $g(x) = f(x) + 2$

Vertical shift two units upward

$$\begin{aligned} g(-x) &= f(-x) + 2 \\ &= f(x) + 2 \\ &= g(x) \end{aligned}$$

Even

(b) $g(x) = f(x + 2)$

Horizontal shift two units to the left
Neither odd nor even

(c) $g(x) = f(-x) = (-x)^4 = x^4$

Reflection in the y-axis. The graph looks the same.
Even

(d) $g(x) = -f(x) = -x^4$

Reflection in the x-axis
Even

(e) $g(x) = f\left(\frac{1}{2}x\right) = \frac{1}{16}x^4$

Horizontal stretch
Even

(f) $g(x) = \frac{1}{2}f(x) = \frac{1}{2}x^4$

Vertical shrink

Even

(g) $g(x) = f\left(x^{3/4}\right) = \left(x^{3/4}\right)^4 = x^3, x \geq 0$

Neither odd nor even

(h) $g(x) = (f \circ f)(x) = f(f(x)) = f\left(x^4\right) = \left(x^4\right)^4 = x^{16}$

Even

105.
$$\begin{array}{r} 27 \\ 8\overline{)216} \\ \underline{16} \\ 56 \\ \underline{56} \\ 0 \end{array}$$

$216 \div 8 = 27$

107.
$$\begin{array}{r} 56 \\ 10\overline{)567} \\ \underline{50} \\ 67 \\ \underline{60} \\ 7 \end{array}$$

$567 \div 10 = 56\frac{7}{10}$

109.
$$\begin{array}{r} 28 \\ 151\overline{)4336} \\ \underline{302} \\ 1316 \\ \underline{1208} \\ 108 \end{array}$$

$4336 \div 151 = 28\frac{108}{151}$

111. $(12n - 1) + (3n^2 - 9n) = 3n^2 + 12n - 9n - 1 = 3n^2 + 3n - 1$

113. $(7x^2 + x + 4) - (2x^2 - 6x) = 7x^2 + x + 4 - 2x^2 + 6x = 5x^2 + 7x + 4$

115. $(3 + i) - (5 - 4i) = 3 + i - 5 + 4i = -2 + 5i$

117.
$$\begin{aligned}(1 - 6i)(9 + 2i) &= 9 - 54i + 2i - 12i^2\\ &= 9 - 52i + 12\\ &= 21 - 52i\end{aligned}$$

119.
$$\begin{aligned}(1 - 2i) \div (3 + i) &= \frac{1 - 2i}{3 + i} \cdot \frac{3 - i}{3 - i}\\ &= \frac{3 - 7i + 2i^2}{9 - i^2}\\ &= \frac{1 - 7i}{10}\\ &= \frac{1}{10} - \frac{7}{10}i\end{aligned}$$

121.
$$\begin{aligned}\frac{2}{1 + i} - \frac{3}{1 - i} &= \frac{2(1 - i) - 3(1 + i)}{(1 + i)(1 - i)}\\ &= \frac{2 - 2i - 3 - 3i}{1 + 1}\\ &= \frac{-1 - 5i}{2}\\ &= -\frac{1}{2} - \frac{5}{2}i\end{aligned}$$

Section 2.3 Polynomial and Synthetic Division

1. $f(x)$ is the dividend; $d(x)$ is the divisor; $q(x)$ is the quotient; $r(x)$ is the remainder

3. proper

5. synthetic division

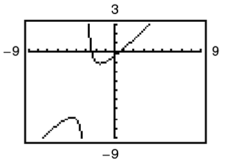
7. $y_1 = \dfrac{x^2}{x + 2}$ and $y_2 = x - 2 + \dfrac{4}{x + 2}$

$$\begin{array}{r} x - 2\phantom{{}+0} \\ x + 2\overline{\smash{)}x^2 + 0x + 0} \\ \underline{(x^2 + 2x)}\phantom{{}+0} \\ -2x + 0 \\ \underline{-2x - 4} \\ 4 \end{array}$$

So, $\dfrac{x^2}{x + 2} = x - 2 + \dfrac{4}{x + 2}$ and $y_1 = y_2$.

9. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, y_2 = x - 1 + \dfrac{2}{x + 3}$

(a) and (b)



(c)
$$\begin{array}{r} x - 1 \\ x + 3\overline{\smash{)}x^2 + 2x - 1} \\ \underline{(x^2 + 3x)}\phantom{{}-1} \\ -x - 1 \\ \underline{-(-x - 3)} \\ 2 \end{array}$$

So, $\dfrac{x^2 + 2x - 1}{x + 3} = x - 1 + \dfrac{2}{x + 3}$ and $y_1 = y_2$.

11.
$$\begin{array}{r} 6 \\ x + 1\overline{\smash{)}6x + 5} \\ \underline{6x + 6} \\ -1 \end{array}$$

$$\frac{6x + 5}{x + 1} = 6 - \frac{1}{x + 1}$$

13.
$$\begin{array}{r} 2x + 4\phantom{{}+12} \\ x + 3\overline{\smash{)}2x^2 + 10x + 12} \\ \underline{2x^2 + 6x}\phantom{{}+12} \\ 4x + 12 \\ \underline{4x + 12} \\ 0 \end{array}$$

$$\frac{2x^2 + 10x + 12}{x + 3} = 2x + 4, x \neq 3$$

15.
$$\begin{array}{r@{\,}rrrrr}
 & & x^3 & +\,3x^2 & & -1 \\ \cline{2-6}
x+2\big) & x^4 & +\,5x^3 & +\,6x^2 & -\,x & -\,2 \\
 & \underline{x^4} & \underline{+\,2x^3} & & & \\
 & & 3x^3 & +\,6x^2 & & \\
 & & \underline{3x^3} & \underline{+\,6x^2} & & \\
 & & & & -x & -\,2 \\
 & & & & \underline{-x} & \underline{-\,2} \\
 & & & & & 0
\end{array}$$

$$\frac{x^4 + 5x^3 + 6x^2 - x - 2}{x + 2} = x^3 + 3x^2 - 1,\ x \neq -2$$

17.
$$\begin{array}{r@{\,}rrrr}
 & & & x & \\ \cline{2-5}
x^2 + 0x + 1\big) & x^3 & +\,0x^2 & +\,0x & -\,9 \\
 & \underline{x^3} & \underline{+\,0x^2} & \underline{+\,x} & \\
 & & & -x & -\,9
\end{array}$$

$$\frac{x^3 - 9}{x^2 + 1} = x - \frac{x + 9}{x^2 + 1}$$

19.
$$\begin{array}{r@{\,}rrrr}
 & & & 2x & -\,8 \\ \cline{2-5}
x^2 + 0x + 1\big) & 2x^3 & -\,8x^2 & +\,3x & -\,9 \\
 & \underline{2x^3} & \underline{+\,0x^2} & \underline{+\,2x} & \\
 & & -8x^2 & +\,x & -\,9 \\
 & & \underline{-8x^2} & \underline{-\,0x} & \underline{-\,8} \\
 & & & x & -\,1
\end{array}$$

$$\frac{2x^3 - 8x^2 + 3x - 9}{x^2 + 1} = 2x - 8 + \frac{x - 1}{x^2 + 1}$$

21.
$$\begin{array}{r|rrrr}
4 & 2 & -10 & 14 & -24 \\
 & & 8 & -8 & 24 \\ \hline
 & 2 & -2 & 6 & 0
\end{array}$$

$$\frac{2x^3 - 10x^2 + 14x - 24}{x - 4} = 2x^2 - 2x + 6,\ x \neq 4$$

23.
$$\begin{array}{r|rrrr}
3 & 6 & 7 & -1 & 26 \\
 & & 18 & 75 & 222 \\ \hline
 & 6 & 25 & 74 & 248
\end{array}$$

$$\frac{6x^3 + 7x^2 - x + 26}{x - 3} = 6x^2 + 25x + 74 + \frac{248}{x - 3}$$

25.
$$\begin{array}{r|rrrr}
-2 & 4 & 8 & -9 & -18 \\
 & & -8 & 0 & 18 \\ \hline
 & 4 & 0 & -9 & 0
\end{array}$$

$$\frac{4x^3 + 8x^2 - 9x - 18}{x + 2} = 4x^2 - 9,\ x \neq -2$$

27.
$$\begin{array}{r|rrrr}
-10 & -1 & 0 & 75 & -250 \\
 & & 10 & -100 & 250 \\ \hline
 & -1 & 10 & -25 & 0
\end{array}$$

$$\frac{-x^3 + 75x - 250}{x + 10} = -x^2 + 10x - 25,\ x \neq -10$$

29.
$$\begin{array}{r|rrrr}
4 & 1 & -3 & 0 & 5 \\
 & & 4 & 4 & 16 \\ \hline
 & 1 & 1 & 4 & 21
\end{array}$$

$$\frac{x^3 - 3x^2 + 5}{x - 4} = x^2 + x + 4 + \frac{21}{x - 4}$$

31.
$$\begin{array}{r|rrrrr}
6 & 10 & -50 & 0 & 0 & -800 \\
 & & 60 & 60 & 360 & 2160 \\ \hline
 & 10 & 10 & 60 & 360 & 1360
\end{array}$$

$$\frac{10x^4 - 50x^3 - 800}{x - 6} = 10x^3 + 10x^2 + 60x + 360 + \frac{1360}{x - 6}$$

33.
$$\begin{array}{r|rrrr}
-8 & 1 & 0 & 0 & 512 \\
 & & -8 & 64 & -512 \\ \hline
 & 1 & -8 & 64 & 0
\end{array}$$

$$\frac{x^3 + 512}{x + 8} = x^2 - 8x + 64,\ x \neq -8$$

35.
$$\begin{array}{r|rrrr}
-\frac{1}{2} & 4 & 16 & -23 & -15 \\
 & & -2 & -7 & 15 \\ \hline
 & 4 & 14 & -30 & 0
\end{array}$$

$$\frac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}} = 4x^2 + 14x - 30,\ x \neq -\frac{1}{2}$$

37. $f(x) = x^3 - x^2 - 10x + 7, k = 3$

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -10 & 7 \\ & & 3 & 6 & -12 \\ \hline & 1 & 2 & -4 & -5 \end{array}$$

$$f(x) = (x - 3)(x^2 + 2x - 4) - 5$$

$$f(3) = 3^3 - 3^2 - 10(3) + 7 = -5$$

39. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14, k = -\frac{2}{3}$

$$\begin{array}{r|rrrrr} -\frac{2}{3} & 15 & 10 & -6 & 0 & 14 \\ & & -10 & 0 & 4 & -\frac{8}{3} \\ \hline & 15 & 0 & -6 & 4 & \frac{34}{3} \end{array}$$

$$f(x) = \left(x + \tfrac{2}{3}\right)\left(15x^3 - 6x + 4\right) + \tfrac{34}{3}$$

$$f\left(-\tfrac{2}{3}\right) = 15\left(-\tfrac{2}{3}\right)^4 + 10\left(-\tfrac{2}{3}\right)^3 - 6\left(-\tfrac{2}{3}\right)^2 + 14 = \tfrac{34}{3}$$

41. $f(x) = 2x^3 - 7x + 3$

(a) Using the Remainder Theorem:

$$f(1) = 2(1)^3 - 7(1) + 3 = -2$$

Using synthetic division:

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -7 & 3 \\ & & 2 & 2 & -5 \\ \hline & 2 & 2 & -5 & -2 \end{array}$$

Verify using long division:

$$\begin{array}{rrrrr}
 & & 2x^2 & +\,2x & -\,5 \\ \cline{2-5}
x - 1\big) & 2x^3 & +\,0x^2 & -\,7x & +\,3 \\
 & 2x^3 & -\,2x^2 & & \\ \cline{2-3}
 & & 2x^2 & -\,7x & \\
 & & 2x^2 & -\,2x & \\ \cline{3-4}
 & & & -5x & +\,3 \\
 & & & -5x & +\,5 \\ \cline{4-5}
 & & & & -2
\end{array}$$

(b) Using the Remainder Theorem:

$$f(-2) = 2(-2)^3 - 7(-2) + 3 = 1$$

Using synthetic division:

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -7 & 3 \\ & & -4 & 8 & -2 \\ \hline & 2 & -4 & 1 & 1 \end{array}$$

Verify using long division:

$$\begin{array}{rrrrr}
 & & 2x^2 & -\,4x & +\,1 \\ \cline{2-5}
x + 2\big) & 2x^3 & +\,0x^2 & -\,7x & +\,3 \\
 & 2x^3 & +\,4x^2 & & \\ \cline{2-3}
 & & -4x^2 & -\,7x & \\
 & & -4x^2 & -\,8x & \\ \cline{3-4}
 & & & x & +\,3 \\
 & & & x & +\,2 \\ \cline{4-5}
 & & & & 1
\end{array}$$

(c) Using the Remainder Theorem:

$$f(3) = 2(3)^3 - 7(3) + 3 = 36$$

Using synthetic division:

$$\begin{array}{r|rrrr} 3 & 2 & 0 & -7 & 3 \\ & & 6 & 18 & 33 \\ \hline & 2 & 6 & 11 & 36 \end{array}$$

Verify using long division:

$$\begin{array}{rrrrr}
 & & 2x^2 & +\,6x & +\,11 \\ \cline{2-5}
x - 3\big) & 2x^3 & +\,0x^2 & -\,7x & +\,3 \\
 & 2x^3 & -\,6x^2 & & \\ \cline{2-3}
 & & 6x^2 & -\,7x & +\,3 \\
 & & 6x^2 & -\,18x & \\ \cline{3-4}
 & & & 11x & +\,3 \\
 & & & 11x & -\,33 \\ \cline{4-5}
 & & & & 36
\end{array}$$

(d) Using the Remainder Theorem:

$$f(2) = 2(2)^3 - 7(2) + 3 = 5$$

Using synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & 0 & -7 & 3 \\ & & 4 & 8 & 2 \\ \hline & 2 & 4 & 1 & 5 \end{array}$$

Verify using long division:

$$\begin{array}{rrrrr}
 & & 2x^2 & +\,4x & +\,1 \\ \cline{2-5}
x - 2\big) & 2x^3 & +\,0x^2 & -\,7x & +\,3 \\
 & 2x^3 & -\,4x^2 & & \\ \cline{2-3}
 & & 4x^2 & -\,7x & \\
 & & 4x^2 & -\,8x & \\ \cline{3-4}
 & & & x & +\,3 \\
 & & & x & -\,2 \\ \cline{4-5}
 & & & & 5
\end{array}$$

43. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$

(a) Using the Remainder Theorem:

$f(1) = 4(1)^4 - 16(1)^3 + 7(1) + 20 = 15$

Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 4 & -16 & 7 & 0 & 20 \\ & & 4 & -12 & -5 & -5 \\ \hline & 4 & -12 & -5 & -5 & 15 \end{array}$$

Verify using long division:

$$\begin{array}{r} 4x^3 - 12x^2 - 5x - 5 \\ x-1\overline{)4x^4 - 16x^3 + 7x^2 + 0x + 20} \\ \underline{4x^4 - 4x^3} \qquad\qquad\qquad\quad \\ -12x^3 + 7x^2 \qquad\qquad \\ \underline{-12x^3 + 12x^2} \qquad\qquad \\ -5x^2 + 0x \qquad \\ \underline{-5x^2 + 5x} \qquad \\ -5x + 20 \\ \underline{-5x + 5} \\ 15 \end{array}$$

(b) Using the Remainder Theorem:

$f(-2) = 4(-2)^4 - 16(-2)^3 + 7(-2)^2 + 20 = 240$

Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 4 & -16 & 7 & 0 & 20 \\ & & -8 & 48 & -110 & 220 \\ \hline & 4 & -24 & 55 & -110 & 240 \end{array}$$

Verify using long division:

$$\begin{array}{r} 4x^3 - 24x^2 + 55x - 110 \\ x+2\overline{)4x^4 - 16x^3 + 7x^2 + 0x + 20} \\ \underline{4x^3 + 8x^3} \qquad\qquad\qquad\quad \\ -24x^3 + 7x^2 \qquad\qquad \\ \underline{-24x^3 - 48x^2} \qquad\qquad \\ 55x^2 + 0x \qquad \\ \underline{55x^2 + 110x} \qquad \\ -110x + 20 \\ \underline{-110x - 220} \\ 240 \end{array}$$

(c) Using the Remainder Theorem:

$f(5) = 4(5)^4 - 16(5)^3 + 7(5)^2 + 20 = 695$

Using synthetic division:

$$\begin{array}{r|rrrrr} 5 & 4 & -16 & 7 & 0 & 20 \\ & & 20 & 20 & 135 & 675 \\ \hline & 4 & 4 & 27 & 135 & 695 \end{array}$$

Verify using long division:

$$\begin{array}{r} 4x^3 + 4x^2 + 27x + 135 \\ x-5\overline{)4x^4 - 16x^3 + 7x^2 + 0x + 20} \\ \underline{4x^4 - 20x^3} \qquad\qquad\qquad\quad \\ 4x^3 + 7x^2 \qquad\qquad \\ \underline{4x^3 - 20x^2} \qquad\qquad \\ 27x^2 + 0x \qquad \\ \underline{27x^2 - 135x} \qquad \\ 135x + 20 \\ \underline{135x - 675} \\ 695 \end{array}$$

(d) Using the Remainder Theorem:

$f(-10) = 4(-10)^4 - 16(-10)^3 + 7(-10)^2 + 20 = 56{,}720$

Using synthetic division:

$$\begin{array}{r|rrrrr} -10 & 4 & -16 & 7 & 0 & 20 \\ & & -40 & 560 & -5670 & 56{,}700 \\ \hline & 4 & -56 & 567 & -5670 & 56{,}720 \end{array}$$

Verify using long division:

$$\begin{array}{r} 4x^3 - 56x^2 + 567x - 5670 \\ x+10\overline{)4x^4 - 16x^3 + 7x^2 + 0x + 20} \\ \underline{4x^4 + 40x^3} \qquad\qquad\qquad\quad \\ -56x^3 + 7x^2 \qquad\qquad \\ \underline{-56x^3 - 560x^2} \qquad\qquad \\ 567x^2 + 0x \qquad \\ \underline{567x^2 + 5670x} \qquad \\ -5670x + 20 \\ \underline{-5670x - 56{,}700} \\ 56{,}720 \end{array}$$

45.
$$\begin{array}{r|rrrr} -3 & 1 & 6 & 11 & 6 \\ & & -3 & -9 & -6 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$\begin{aligned} x^3 + 6x^2 + 11x + 6 &= (x + 3)(x^2 + 3x + 2) \\ &= (x + 3)(x + 2)(x + 1) \end{aligned}$$

Zeros: $-3, -2, -1$

47.
$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -15 & 27 & -10 \\ & & 1 & -7 & 10 \\ \hline & 2 & -14 & 20 & 0 \end{array}$$

$$\begin{aligned} 2x^3 - 15x^2 + 27x - 10 &= \left(x - \tfrac{1}{2}\right)(2x^2 - 14x + 20) \\ &= (2x - 1)(x - 2)(x - 5) \end{aligned}$$

Zeros: $\frac{1}{2}, 2, 5$

49.
$$\begin{array}{r|rrrr} \sqrt{3} & 1 & 2 & -3 & -6 \\ & & \sqrt{3} & 3 + 2\sqrt{3} & 6 \\ \hline & 1 & 2 + \sqrt{3} & 2\sqrt{3} & 0 \end{array}$$

$$\begin{array}{r|rrr} -\sqrt{3} & 1 & 2 + \sqrt{3} & 2\sqrt{3} \\ & & -\sqrt{3} & -2\sqrt{3} \\ \hline & 1 & 2 & 0 \end{array}$$

$$x^3 + 2x^2 - 3x - 6 = (x - \sqrt{3})(x + \sqrt{3})(x + 2)$$

Zeros: $-\sqrt{3}, \sqrt{3}, -2$

51. $f(x) = 2x^3 + x^2 - 5x + 2$; Factors: $(x + 2), (x - 1)$

(a)
$$\begin{array}{r|rrrr} -2 & 2 & 1 & -5 & 2 \\ & & -4 & 6 & -2 \\ \hline & 2 & -3 & 1 & 0 \end{array}$$

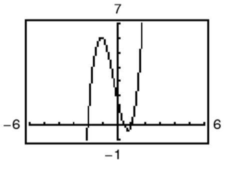
$$\begin{array}{r|rrr} 1 & 2 & -3 & 1 \\ & & 2 & -1 \\ \hline & 2 & -1 & 0 \end{array}$$

Both are factors of $f(x)$ because the remainders are zero.

(b) The remaining factor of $f(x)$ is $(2x - 1)$.

(c) $f(x) = (2x - 1)(x + 2)(x - 1)$

(d) Zeros: $\frac{1}{2}, -2, 1$

(e)


53. $f(x) = 6x^3 + 41x^2 - 9x - 14$;

Factors: $(2x + 1), (3x - 2)$

(a)
$$\begin{array}{r|rrrr} -\frac{1}{2} & 6 & 41 & -9 & -14 \\ & & -3 & -19 & 14 \\ \hline & 6 & 38 & -28 & 0 \end{array}$$

$$\begin{array}{r|rrr} \frac{2}{3} & 6 & 38 & -28 \\ & & 4 & 28 \\ \hline & 6 & 42 & 0 \end{array}$$

Both are factors of $f(x)$ because the remainders are zero.

(b) $6x + 42 = 6(x + 7)$

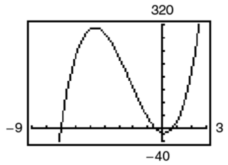
This shows that $\dfrac{f(x)}{\left(x + \dfrac{1}{2}\right)\left(x - \dfrac{2}{3}\right)} = 6(x + 7)$,

so $\dfrac{f(x)}{(2x + 1)(3x - 2)} = x + 7$.

The remaining factor is $(x + 7)$.

(c) $f(x) = (x + 7)(2x + 1)(3x - 2)$

(d) Zeros: $-7, -\dfrac{1}{2}, \dfrac{2}{3}$

(e)


55. $f(x) = x^3 - 2x^2 - 5x + 10$

(a) The zeros of f are $x = 2$ and $x \approx \pm 2.236$.

(b) An exact zero is $x = 2$.

(c)
$$\begin{array}{r|rrrr} 2 & 1 & -2 & -5 & 10 \\ & & 2 & 0 & -10 \\ \hline & 1 & 0 & -5 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x - 2)(x^2 - 5) \\ &= (x - 2)(x - \sqrt{5})(x + \sqrt{5}) \end{aligned}$$

57. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

(a) The zeros of h are $x = 0, x = 3, x = 4,$ $x \approx 1.414, x \approx -1.414.$

(b) An exact zero is $x = 4.$

(c)
$$\begin{array}{r|rrrrr} 4 & 1 & -7 & 10 & 14 & -24 \\ & & 4 & -12 & -8 & 24 \\ \hline & 1 & -3 & -2 & 6 & 0 \end{array}$$

$$\begin{aligned} h(x) &= (x-4)\left(x^4 - 3x^3 - 2x^2 + 6x\right) \\ &= x(x-4)(x-3)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) \end{aligned}$$

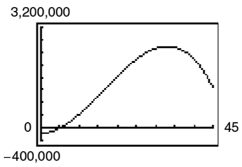
59. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$

$$\begin{array}{r|rrrr} \frac{3}{2} & 4 & -8 & 1 & 3 \\ & & 6 & -3 & -3 \\ \hline & 4 & -2 & -2 & 0 \end{array}$$

$$\frac{4x^3 - 8x^2 + x + 3}{x - \frac{3}{2}} = 4x^2 - 2x - 2 = 2\left(2x^2 - x - 1\right)$$

So, $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3} = 2x^2 - x - 1, x \neq \dfrac{3}{2}.$

61. (a)



(b) Using the trace and zoom features, when $x = 25$, an advertising expense of about \$250,000 would produce the same profit of \$2,174,375.

(c) $x = 25$

$$\begin{array}{r|rrrr} 25 & -152 & 7545 & 0 & -169{,}625 \\ & & -3800 & 93{,}625 & 2{,}340{,}625 \\ \hline & -152 & 3745 & 93{,}625 & 2{,}171{,}000 \end{array}$$

So, an advertising expense of \$250,000 yields a profit of \$2,171,000, which is close to \$2,174,375.

63. False. If $(7x + 4)$ is a factor of f, then $-\frac{4}{7}$ is a zero of f.

65. True. The degree of the numerator is greater than the degree of the denominator.

67. To divide $x^2 + 3x - 5$ by $x + 1$ using synthetic division, the value of k is $k = -1$ not $k = 1$ as shown.

$$\begin{array}{r|rrr} -1 & 1 & 3 & -5 \\ & & -1 & -2 \\ \hline & 1 & 2 & \boxed{-7} \end{array} \quad \leftarrow \text{Remainder: } -7$$

69.
$$\begin{array}{r} x^{2n} + 6x^n + 9 \\ x^n + 3\overline{\big)\,x^{3n} + 9x^{2n} + 27x^n + 27} \\ \underline{x^{3n} + 3x^{2n}} \\ 6x^{2n} + 27x^n \\ \underline{6x^{2n} + 18x^n} \\ 9x^n + 27 \\ \underline{9x^n + 27} \\ 0 \end{array}$$

$$\frac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3} = x^{2n} + 6x^n + 9, x^n \neq -3$$

71.
$$\begin{array}{r|rrrr} 5 & 1 & 4 & -3 & c \\ & & 5 & 45 & 210 \\ \hline & 1 & 9 & 42 & c + 210 \end{array}$$

To divide evenly, $c + 210$ must equal zero. So, c must equal -210.

73. If $x - 4$ is a factor of $f(x) = x^3 - kx^2 + 2kx - 8$, then $f(4) = 0$.

$$\begin{aligned} f(4) &= (4)^3 - k(4)^2 + 2k(4) - 8 \\ 0 &= 64 - 16k + 8k - 8 \\ -56 &= -8k \\ 7 &= k \end{aligned}$$

75. $f(x) = 2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

77. $f(x) = x^2 - 8x + 16 = 0 \Rightarrow (x-4)^2 = 0$
$\Rightarrow x = 4$

79. $f(x) = 6x^2 + x - 15 = 0 \Rightarrow (3x + 5)(2x - 3) = 0 \Rightarrow x = -\frac{5}{3}, \frac{3}{2}$

81. $f(x) = x^3 - x^2 - 25x + 25 = 0 \Rightarrow x^2(x - 1) - 25(x - 1) = 0 \Rightarrow \left(x^2 - 25\right)(x - 1) = 0 \Rightarrow x = \pm 5, 1$

83. $f(x) = 6x^4 - 3x^3 - 12x^2 + 6x = 0$

$$3x^3(2x-1) - 6x(2x-1) = 0$$
$$(3x^3 - 6x)(2x-1) = 0$$
$$3x(x^2-2)(2x-1) = 0$$
$$x = 0,\ \pm\sqrt{2},\ \tfrac{1}{2}$$

85. $2 + \sqrt{-25} = 2 + 5i$

87. $-6i + i^2 = -6i + (-1)$

$$= -1 - 6i$$

89. $x^2 + 4 = 0$

$$x^2 = -4$$
$$x = \pm 2i$$

91. $x^2 + 2x = -7$

$$x^2 + 2x + 7 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(7)}}{2}$$
$$= \frac{-2 \pm \sqrt{-24}}{2}$$
$$= \frac{-2 \pm 2\sqrt{6}i}{2}$$
$$= -1 \pm \sqrt{6}i$$

93. $4x^2 + 8x + 9 = 0$

$$x = \frac{-8 \pm \sqrt{64 - 4(4)(9)}}{2(4)}$$
$$= \frac{-8 \pm \sqrt{-80}}{8}$$
$$= \frac{-8 \pm 4\sqrt{5}i}{8}$$
$$= -1 \pm \frac{\sqrt{5}}{2}i$$

95. $x^{2n} - y^{2n} = (x^n)^2 - (y^n)^2 = (x^n + y^n)(x^n - y^n)$

This is not completely factored unless $n = 1$.

For $n = 2$: $(x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$

For $n = 3$: $(x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$

For $n = 4$: $(x^4 + y^4)(x^4 - y^4) = (x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$

97. Answers will vary. *Sample answer*: $x^2 - 3$

Section 2.4 Complex Numbers

1. $\sqrt{-1}$; -1

3. (a) ii (b) iii (c) i

5. The additive inverse of $2 - 4i$ is $-2 + 4i$.

7. $a + bi = 9 + 8i$

$$a = 9$$
$$b = 8$$

9. $(5 + i) + (2 + 3i) = 5 + i + 2 + 3i$

$$= 7 + 4i$$

11. $(9 - i) - (8 - i) = 1$

13. $(-2 + \sqrt{-8}) + (5 - \sqrt{-50}) = -2 + 2\sqrt{2}i$

$$+ 5 - 5\sqrt{2}i = 3 - 3\sqrt{2}i$$

15. $13i - (14 - 7i) + (2 - 11i) = -14 + 2 + (13 + 7 - 11)i = -12 + 9i$

17. $(1 + i)(3 - 2i) = 3 - 2i + 3i - 2i^2$

$$= 3 + i + 2 = 5 + i$$

19. $12i(1 - 9i) = 12i - 108i^2$

$$= 12i + 108$$
$$= 108 + 12i$$

21. $$\left(\sqrt{2} + 3i\right)\left(\sqrt{2} - 3i\right) = 2 - 9t^2$$
$$= 2 + 9 = 11$$

23. $$(6 + 7i)^2 = 36 + 84i + 49i^2$$
$$= 36 + 84i - 49$$
$$= -13 + 84i$$

25. The complex conjugate of $9 + 2i$ is $9 - 2i$.

$$(9 + 2i)(9 - 2i) = 81 - 4i^2$$
$$= 81 + 4$$
$$= 85$$

27. The complex conjugate of $-1 - \sqrt{5}i$ is $-1 + \sqrt{5}i$.

$$\left(-1 - \sqrt{5}i\right)\left(-1 + \sqrt{5}i\right) = 1 - 5i^2$$
$$= 1 + 5 = 6$$

29. The complex conjugate of $\sqrt{-20} = 2\sqrt{5}i$ is $-2\sqrt{5}i$.

$$\left(2\sqrt{5}i\right)\left(-2\sqrt{5}i\right) = -20i^2 = 20$$

31. The complex conjugate of $1 - \sqrt{-6}$ is $1 + \sqrt{-6}$.

$$\left(1 - \sqrt{-6}\right)\left(1 + \sqrt{-6}\right) = 1 - (-6) = 7$$

33. $$\frac{2}{4 - 5i} = \frac{2}{4 - 5i} \cdot \frac{4 + 5i}{4 + 5i}$$
$$= \frac{2(4 + 5i)}{16 + 25} = \frac{8 + 10i}{41} = \frac{8}{41} + \frac{10}{41}i$$

35. $$\frac{5 + i}{5 - i} \cdot \frac{(5 + i)}{(5 + i)} = \frac{25 + 10i + i^2}{25 - i^2}$$
$$= \frac{24 + 10i}{26} = \frac{12}{13} + \frac{5}{13}i$$

37. $$\frac{9 - 4i}{i} \cdot \frac{-i}{-i} = \frac{-9i + 4i^2}{-i^2} = -4 - 9i$$

39. $$\frac{3i}{(4 - 5i)^2} = \frac{3i}{16 - 40i + 25i^2} = \frac{3i}{-9 - 40i} \cdot \frac{-9 + 40i}{-9 + 40i}$$
$$= \frac{-27i + 120i^2}{81 + 1600} = \frac{-120 - 27i}{1681}$$
$$= -\frac{120}{1681} - \frac{27}{1681}i$$

41. $$\sqrt{-6} \cdot \sqrt{-2} = \left(\sqrt{6}i\right)\left(\sqrt{2}i\right) = \sqrt{12}i^2 = \left(2\sqrt{3}\right)(-1)$$
$$= -2\sqrt{3}$$

43. $$\left(\sqrt{-15}\right)^2 = \left(\sqrt{15}i\right)^2 = 15i^2 = -15$$

45. $$\sqrt{-8} + \sqrt{-50} = \sqrt{8}\,i + \sqrt{50}\,i$$
$$= 2\sqrt{2}\,i + 5\sqrt{2}\,i$$
$$= 7\sqrt{2}\,i$$

47. $$\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right) = \left(3 + \sqrt{5}i\right)\left(7 - \sqrt{10}i\right)$$
$$= 21 - 3\sqrt{10}i + 7\sqrt{5}i - \sqrt{50}i^2$$
$$= \left(21 + \sqrt{50}\right) + \left(7\sqrt{5} - 3\sqrt{10}\right)i$$
$$= \left(21 + 5\sqrt{2}\right) + \left(7\sqrt{5} - 3\sqrt{10}\right)i$$

49. $x^2 - 2x + 2 = 0;\ a = 1, b = -2, c = 2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2i}{2}$$
$$= 1 \pm i$$

51. $4x^2 + 16x + 17 = 0;\ a = 4, b = 16, c = 17$

$$x = \frac{-16 \pm \sqrt{(16)^2 - 4(4)(17)}}{2(4)}$$
$$= \frac{-16 \pm \sqrt{-16}}{8}$$
$$= \frac{-16 \pm 4i}{8}$$
$$= -2 \pm \frac{1}{2}i$$

53. $4x^2 + 16x + 21 = 0; a = 4, b = 16, c = 21$

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{(16)^2 - 4(4)(21)}}{2(4)} \\ &= \frac{-16 \pm \sqrt{-80}}{8} \\ &= \frac{-16 \pm \sqrt{80}\, i}{8} \\ &= \frac{-16 \pm 4\sqrt{5}\, i}{8} \\ &= -2 \pm \frac{\sqrt{5}}{2} i \end{aligned}$$

55. $\frac{3}{2}x^2 - 6x + 9 = 0$ Multiply both sides by 2.

$3x^2 - 12x + 18 = 0; a = 3, b = -12, c = 18$

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(18)}}{2(3)} \\ &= \frac{12 \pm \sqrt{-72}}{6} \\ &= \frac{12 \pm 6\sqrt{2}i}{6} \\ &= 2 \pm \sqrt{2}i \end{aligned}$$

57. $1.4x^2 - 2x + 10 = 0 \Rightarrow 14x^2 - 20x + 100 = 0;$

$a = 14, b = -20, c = 100$

$$\begin{aligned} x &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(14)(100)}}{2(14)} \\ &= \frac{20 \pm \sqrt{-5200}}{28} \\ &= \frac{20 \pm 20\sqrt{13}\, i}{28} \\ &= \frac{20}{28} \pm \frac{20\sqrt{13}\, i}{28} \\ &= \frac{5}{7} \pm \frac{5\sqrt{13}}{7} i \end{aligned}$$

59. $z_1 = 5 + 2i$

$z_2 = 3 - 4i$

$$\begin{aligned} \frac{1}{z} &= \frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{5 + 2i} + \frac{1}{3 - 4i} \\ &= \frac{(3 - 4i) + (5 + 2i)}{(5 + 2i)(3 - 4i)} \\ &= \frac{8 - 2i}{23 - 14i} \\ z &= \frac{23 - 14i}{8 - 2i}\left(\frac{8 + 2i}{8 + 2i}\right) \\ &= \frac{212 - 66i}{68} \approx 3.118 - 0.971i \end{aligned}$$

61. False.

Sample answer: $(1 + i) + (3 + i) = 4 + 2i$ which is not a real number.

63. True.

$$\begin{aligned} x^4 - x^2 + 14 &= 56 \\ \left(-i\sqrt{6}\right)^4 - \left(-i\sqrt{6}\right)^2 + 14 &\overset{?}{=} 56 \\ 36 + 6 + 14 &\overset{?}{=} 56 \\ 56 &= 56 \end{aligned}$$

65.

$$\begin{aligned} i &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i^4 i = i \\ i^6 &= i^4 i^2 = -1 \\ i^7 &= i^4 i^3 = -i \\ i^8 &= i^4 i^4 = 1 \\ i^9 &= i^4 i^4 i = i \\ i^{10} &= i^4 i^4 i^2 = -1 \\ i^{11} &= i^4 i^4 i^3 = -i \\ i^{12} &= i^4 i^4 i^4 = 1 \end{aligned}$$

The pattern $i, -1, -i, 1$ repeats. Divide the exponent by 4.

If the remainder is 1, the result is i.

If the remainder is 2, the result is -1.

If the remainder is 3, the result is $-i$.

If the remainder is 0, the result is 1.

67. $\sqrt{-6}\sqrt{-6} = \sqrt{6}i\sqrt{6}i = 6i^2 = -6$

69. $(a_1 + b_1i)(a_2 + b_2i) = a_1a_2 + a_1b_2i + a_2b_1i + b_1b_2i^2$
$= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$

The complex conjugate of this product is
$(a_1a_2 - b_1b_2) - (a_1b_2 + a_2b_1)i.$

The product of the complex conjugates is
$(a_1 - b_1i)(a_2 - b_2i) = a_1a_2 - a_1b_2i - a_2b_1i - b_1b_2i^2$
$= (a_1a_2 - b_1b_2) - (a_1b_2 + a_2b_1)i.$

So, the complex conjugate of the product of two complex numbers is the product of their complex conjugates.

71. $3x^4 - 48x^2 = 3x^2(x^2 - 16) = 3x^2(x - 4)(x + 4)$

73. $x^3 - 3x^2 + 3x - 9 = x^2(x - 3) + 3(x - 3)$
$= (x - 3)(x^2 + 3)$

75. $6x^3 - 27x^2 - 54x = 3x(2x^2 - 9x - 18)$
$= 3x(2x + 3)(x - 6)$

77. $x^4 - 3x^2 + 2 = (x^2 - 1)(x^2 - 2)$
$= (x - 1)(x + 1)(x^2 - 2)$

79. $9x^4 - 37x^2 + 4 = (9x^2 - 1)(x^2 - 4)$
$= (3x + 1)(3x - 1)(x + 2)(x - 2)$

81. (a) When $x = -3, \sqrt{2x + 7} - x$
$= \sqrt{2(-3) + 7} - (-3) = \sqrt{1} + 3 = 4.$

(b) When $x = 1, \sqrt{2x + 7} - x$
$= \sqrt{2(1) + 7} - 1 = \sqrt{9} - 1 = 2.$

83. (a) When $x = 3, \sqrt{2x - 5} - \sqrt{x - 3} - 1 = \sqrt{2(3) - 5} - \sqrt{3 - 3} - 1 = \sqrt{1} - \sqrt{0} - 1 = 1 - 1 = 0.$

(b) When $x = 7, \sqrt{2x - 5} - \sqrt{x - 3} - 1 = \sqrt{2(7) - 5} - \sqrt{7 - 3} - 1 = \sqrt{9} - \sqrt{4} - 1 = 3 - 2 - 1 = 0.$

85. (a) When $x = 129, (x - 4)^{2/3} = (129 - 4)^{2/3} = 125^{2/3} = 25.$

(b) When $x = -121, (x - 4)^{2/3} = (-121 - 4)^{2/3} = (-125)^{2/3} = 25.$

87. (a) When $x = 4, \frac{2}{x} - \frac{3}{x - 2} + 1 = \frac{2}{4} - \frac{3}{4 - 2} + 1 = \frac{1}{2} - \frac{3}{2} + 1 = -1 + 1 = 0.$

(b) When $x = -1, \frac{2}{x} - \frac{3}{x - 2} + 1 = \frac{2}{-1} - \frac{3}{-1 - 2} + 1 = -2 + 1 + 1 = 0.$

89. (a) When $x = -3, |x^2 - 3x| + 4x - 6 = |(-3)^2 - 3(-3)| + 4(-3) - 6 = |9 + 9| - 12 - 6 = 0.$

(b) When $x = 2, |x^2 - 3x| + 4x - 6 = |(2)^2 - 3(2)| + 4(2) - 6 = |4 - 6| + 8 - 6 = 4.$

Section 2.5 Zeros of Polynomial Functions

1. Fundamental Theorem of Algebra

3. Rational Zero

5. According to Descartes's Rule of Signs, given that $f(-x)$ has 5 variations in sign, there are either 5, 3, or 1 negative real zeros.

7. f is a 3rd degree polynomial function, so there are three zeros.

9. f is a 5th degree polynomial function, so there are five zeros.

11. f is a 2nd degree polynomial function, so there are two zeros.

13. $f(x) = x^3 + 2x^2 - x - 2$

Possible rational zeros: $\pm 1, \pm 2$

Zeros shown on graph: $-2, -1, 1, 2$

15. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

Possible rational zeros: $\pm1, \pm3, \pm5, \pm9, \pm15, \pm45,$
$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}$

Zeros shown on graph: $-1, \frac{3}{2}, 3, 5$

17. $f(x) = x^3 - 7x - 6$

Possible rational zeros: $\pm1, \pm2, \pm3, \pm6$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x - 3)(x^2 + 3x + 2) \\ &= (x - 3)(x + 2)(x + 1) \end{aligned}$$

So, the rational zeros are -2, -1, and 3.

19. $g(t) = t^3 - 4t^2 + 4$

Possible rational zeros: $\pm1, \pm2, \pm4$

After testing all six possible rational zeros by synthetic division, you can conclude there are no rational zeros.

21. $h(t) = t^3 + 8t^2 + 13t + 6$

Possible rational zeros: $\pm1, \pm2, \pm3, \pm6$

$$\begin{array}{r|rrrr} -6 & 1 & 8 & 13 & 6 \\ & & -6 & -12 & -6 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$\begin{aligned} t^3 + 8t^2 + 13t + 6 &= (t + 6)(t^2 + 2t + 1) \\ &= (t + 6)(t + 1)(t + 1) \end{aligned}$$

So, the rational zeros are -1 and -6.

23. $C(x) = 2x^3 + 3x^2 - 1$

Possible rational zeros: $\pm1, \pm\frac{1}{2}$

$$\begin{array}{r|rrrr} -1 & 2 & 3 & 0 & -1 \\ & & -2 & -1 & 1 \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

$$\begin{aligned} 2x^3 + 3x^2 - 1 &= (x + 1)(2x^2 + x - 1) \\ &= (x + 1)(x + 1)(2x - 1) \\ &= (x + 1)^2(2x - 1) \end{aligned}$$

So, the rational zeros are -1 and $\frac{1}{2}$.

25. $f(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$

Possible rational zeros:
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24,$
$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{9}, \pm\frac{2}{9}, \pm\frac{4}{9}, \pm\frac{8}{9}$

$$\begin{array}{r|rrrrr} -2 & 9 & -9 & -58 & 4 & 24 \\ & & -18 & 54 & 8 & -24 \\ \hline & 9 & -27 & -4 & 12 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 9 & -27 & -4 & 12 \\ & & 27 & 0 & -12 \\ \hline & 9 & 0 & -4 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x + 2)(x - 3)(9x^2 - 4) \\ &= (x + 2)(x - 3)(3x - 2)(3x + 2) \end{aligned}$$

So, the rational zeros are $-2, 3, \frac{2}{3},$ and $-\frac{2}{3}$.

27. $-5x^3 + 11x^2 - 4x - 2 = 0$

Possible rational zeros: $\dfrac{\pm1, \pm2}{\pm1, \pm5} = \pm\dfrac{1}{5}, \pm\dfrac{2}{5}, \pm1, \pm2$

$$\begin{array}{r|rrrr} 1 & -5 & 11 & -4 & -2 \\ & & -5 & 6 & 2 \\ \hline & -5 & 6 & 2 & 0 \end{array}$$

$$\begin{aligned} (x - 1)(-5x^2 + 6x + 2) &= 0 \\ -5x^2 + 6x + 2 &= 0 \\ 5x^2 - 6x - 2 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(-2)}}{2(5)} \\ x &= \frac{6 \pm \sqrt{76}}{10} \\ x &= \frac{2(3 \pm \sqrt{19})}{10} \\ x &= \frac{3 \pm \sqrt{19}}{5} \end{aligned}$$

So, the real zeros are $x = 1, x = \dfrac{3}{5} \pm \dfrac{\sqrt{19}}{5}$.

29. $x^4 + 6x^3 + 3x^2 - 16x + 6 = 0$

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrrr} 1 & 1 & 6 & 3 & -16 & 6 \\ & & 1 & 7 & 10 & -6 \\ \hline & 1 & 7 & 10 & -6 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 10 & -6 \\ & & -3 & -12 & 6 \\ \hline & 1 & 4 & -2 & 0 \end{array}$$

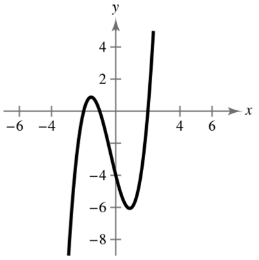
$$\begin{aligned} (x - 1)(x + 3)(x^2 + 4x - 2) &= 0 \\ x^2 + 4x - 2 &= 0 \\ x^2 + 4x + 4 &= 2 + 4 \\ (x + 2)^2 &= 6 \\ x + 2 &= \pm\sqrt{6} \\ x &= -2 \pm \sqrt{6} \end{aligned}$$

So the real zeros are $x = 1, -3, -2 \pm 2\sqrt{6}$.

31. $f(x) = x^3 + x^2 - 4x - 4$

(a) Possible rational zeros: $\pm 1, \pm 2, \pm 4$

(b)

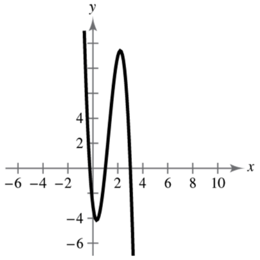


(c) Real zeros: $-2, -1, 2$

33. $f(x) = -4x^3 + 15x^2 - 8x - 3$

(a) Possible rational zeros: $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

(b)

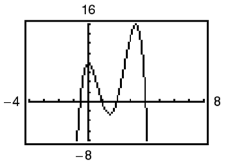


(c) Real zeros: $-\frac{1}{4}, 1, 3$

35. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

(a) Possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}$

(b)

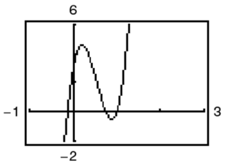


(c) Real zeros: $-\frac{1}{2}, 1, 2, 4$

37. $f(x) = 32x^3 - 52x^2 + 17x + 3$

(a) Possible rational zeros: $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{32}, \pm\frac{3}{32}$

(b)



(c) Real zeros: $-\frac{1}{8}, \frac{3}{4}, 1$

39.
$$\begin{aligned} f(x) &= (x - 1)(x - 5i)(x + 5i) \\ &= (x - 1)(x^2 + 25) \\ &= x^3 - x^2 + 25x - 25 \end{aligned}$$

Note: $f(x) = a(x^3 - x^2 + 25x - 25)$, where a is any nonzero real number, has the zeros 1 and $\pm 5i$.

41. If $1 + i$ is a zero, so is its conjugate, $1 - i$.

$$\begin{aligned} f(x) &= (x - 2)(x - 2)(x - (1 + i))(x - (1 - i)) \\ &= (x^2 - 4x + 4)(x^2 - 2x + 2) \\ &= x^4 - 6x^3 + 14x^2 - 16x + 8 \end{aligned}$$

Note: $f(x) = a(x^4 - 6x^3 + 14x^2 - 16x + 8)$, where a is any nonzero real number, has the zeros 2, 2 and $1 \pm i$.

43. If $3 + \sqrt{2}i$ is a zero, so is its conjugate, $3 - \sqrt{2}i$.

$$\begin{aligned} f(x) &= (3x - 2)(x + 1)\left[x - \left(3 + \sqrt{2}i\right)\right]\left[x - \left(3 - \sqrt{2}i\right)\right] \\ &= (3x - 2)(x + 1)\left[(x - 3) - \sqrt{2}i\right]\left[(x - 3) + \sqrt{2}i\right] \\ &= \left(3x^2 + x - 2\right)\left[(x - 3)^2 - \left(\sqrt{2}i\right)^2\right] \\ &= \left(3x^2 + x - 2\right)\left(x^2 - 6x + 9 + 2\right) \\ &= \left(3x^2 + x - 2\right)\left(x^2 - 6x + 11\right) \\ &= 3x^4 - 17x^3 + 25x^2 + 23x - 22 \end{aligned}$$

Note: $f(x) = a\left(3x^4 - 17x^3 + 25x^2 + 23x - 22\right)$, where a is any nonzero real number, has the zeros $\frac{2}{3}$, -1, and $3 \pm \sqrt{2}i$.

45.
$$\begin{aligned} f(x) &= a(x + 2)(x - 1)(x - i)(x + i) \\ &= a\left(x^2 + x - 2\right)\left(x^2 + 1\right) \\ &= a\left(x^4 + x^3 - x^2 + x - 2\right) \end{aligned}$$

Since $f(0) = -4$

$$\begin{aligned} -4 &= a\left((0)^4 + (0)^3 - (0)^2 + (0) - 2\right) \\ -4 &= -2a \\ a &= 2 \end{aligned}$$

So, $f(x) = 2\left(x^4 + x^3 - x^2 + x - 2\right)$

$= 2x^4 + 2x^3 - 2x^2 + 2x - 4.$

47.
$$\begin{aligned} f(x) &= a(x + 3)\left(x - \left(1 + \sqrt{3}i\right)\right)\left(x - \left(1 - \sqrt{3}i\right)\right) \\ &= a(x + 3)\left(x^2 - 2x + 4\right) \\ &= a\left(x^3 + x^2 - 2x + 12\right) \end{aligned}$$

Since $f(-2) = 12$

$$\begin{aligned} 12 &= a\left((-2)^3 + (-2)^2 - 2(-2) + 12\right) \\ 12 &= 12a \\ a &= 1 \end{aligned}$$

So, $f(x) = (1)\left(x^4 - x^3 - 2x - 4\right)$

$= x^3 + x^2 - 2x + 12.$

49. $f(x) = x^4 + 2x^2 - 8$

(a) $f(x) = \left(x^2 + 4\right)\left(x^2 - 2\right)$

(b) $f(x) = \left(x^2 + 4\right)\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

(c) $f(x) = (x + 2i)(x - 2i)\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

51. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

$$\begin{array}{r} x^2 - 2x + 3 \\ x^2 - 6\overline{\big)x^4 - 2x^3 - 3x^2 + 12x - 18} \\ \underline{x^4 \qquad - 6x^2 \qquad\qquad} \\ -2x^3 + 3x^2 + 12x \qquad \\ \underline{-2x^3 \qquad\quad + 12x} \qquad \\ 3x^2 \qquad -18 \\ \underline{3x^2 \qquad -18} \\ 0 \end{array}$$

(a) $f(x) = \left(x^2 - 6\right)\left(x^2 - 2x + 3\right)$

(b) $f(x) = \left(x + \sqrt{6}\right)\left(x - \sqrt{6}\right)\left(x^2 - 2x + 3\right)$

(c) $f(x) = \left(x + \sqrt{6}\right)\left(x - \sqrt{6}\right)\left(x - 1 - \sqrt{2}i\right)$
$\left(x - 1 + \sqrt{2}i\right)$

Note: Use the Quadratic Formula for (c).

53. $f(x) = x^3 - x^2 + 4x - 4$

Because $2i$ is a zero, so is $-2i$

$$\begin{array}{r|rrrr} 2i & 1 & -1 & 4 & -4 \\ & & 2i & -4-2i & 4 \\ \hline & 1 & 2i-1 & -2i & 0 \end{array}$$

$$\begin{array}{r|rrr} -2i & 1 & 2i-1 & -2i \\ & & -2i & 2i \\ \hline & 1 & -1 & 0 \end{array}$$

$f(x) = (x - 2i)(x + 2i)(x - 1)$

The zeros of $f(x)$ are $x = 1, \pm 2i$.

Alternate Solution:

Because $x = \pm 2i$ are zeros of $f(x)$,

$(x + 2i)(x - 2i) = x^2 + 4$ is a factor of $f(x)$.

By long division, you have:

$$\begin{array}{r} x - 1 \\ x^2 + 0x + 4 \overline{\smash{\big)}\, x^3 - x^2 + 4x - 4} \\ \underline{x^3 + 0x^2 + 4x} \phantom{{}-4} \\ -x^2 + 0x - 4 \\ \underline{-x^2 + 0x - 4} \\ 0 \end{array}$$

$f(x) = (x^2 + 4)(x - 1)$

The zeros of $f(x)$ are $x = 1, \pm 2i$.

55. $f(x) = x^3 - 8x^2 + 25x - 26$

Because $3 + 2i$ is a zero, so is $3 - 2i$.

$$\begin{array}{r|rrrr} 3+2i & 1 & -8 & 25 & -26 \\ & & 3+2i & -19-4i & 26 \\ \hline & 1 & -5+2i & 6-4i & 0 \end{array}$$

$$\begin{array}{r|rrr} 3-2i & 1 & -5+2i & 6-4i \\ & & 3-2i & -6+4i \\ \hline & 1 & -2 & 0 \end{array}$$

$f(x) = (x - (3 + 2i))(x - (3 - 2i))(x - 2)$

The zeros of $f(x)$ are $x = 3 \pm 2i, 2$.

Alternate Solution:

Because $x = 3 \pm 2i$ are zeros of $f(x), (x - (3 + 2i))(x - (3 - 2i)) = x^2 - 6x + 13$ is a factor of $f(x.)$

By long division, you have:

$$\begin{array}{r} x - 2 \\ x^2 - 6x + 13 \overline{\smash{\big)}\, x^3 - 8x^2 + 25x - 26} \\ \underline{x^3 - 6x^2 + 13x} \phantom{{}-26} \\ -2x^2 + 12x - 26 \\ \underline{-2x^2 + 12x^2 - 26} \\ 0 \end{array}$$

$f(x) = (x^2 - 6x + 13)(x - 2)$

The zeros of $f(x)$ are $x = 3 \pm 2i, 2$.

57. $f(x) = x^4 - 6x^3 + 14x^2 - 18x + 9$

Because $1 - \sqrt{2}i$ is a zero, so is $1 + \sqrt{2}i$, and

$$\begin{aligned} \left[x - \left(1 - \sqrt{2}i\right)\right]\left[x - \left(1 + \sqrt{2}i\right)\right] &= \left[(x - 1) - \sqrt{2}i\right]\left[(x - 1) - \sqrt{2}i\right]\left[(x - 1) + \sqrt{2}i\right] \\ &= (x - 1)^2 - \left(\sqrt{2}i\right)^2 \\ &= x^2 - 2x + 1 - 2i^2 \\ &= x^2 - 2x + 3 \end{aligned}$$

is a factor of $f(x)$. By long division, you have:

$$\begin{array}{r} x^2 - 4x + 3 \\ x^2 - 2x + 3 \overline{\smash{\big)}\, x^4 - 6x^3 + 14x^2 - 18x + 9} \\ \underline{x^4 - 2x^3 + 3x^2} \phantom{{}- 18x + 9} \\ -4x^3 + 11x^2 - 18x + 9 \\ \underline{-4x^3 + 8x^2 - 12x} \phantom{{}+ 9} \\ 3x^2 - 6x + 9 \\ \underline{3x^2 - 6x + 9} \\ 0 \end{array}$$

$$\begin{aligned} f(x) &= (x^2 - 2x + 3)(x^2 - 4x + 3) \\ &= (x^2 - 2x + 3)(x - 1)(x - 3) \end{aligned}$$

The zeros of $f(x)$ are $x = 1 \pm \sqrt{2}i, 1, 3$.

59. $f(x) = x^2 + 36$

$$= (x + 6i)(x - 6i)$$

The zeros of $f(x)$ are $x = \pm 6i$.

61. $h(x) = x^2 - 2x + 17$

By the Quadratic Formula, the zeros of $f(x)$ are

$$x = \frac{2 \pm \sqrt{4 - 68}}{2} = \frac{2 \pm \sqrt{-64}}{2} = 1 \pm 4i.$$

$$f(x) = (x - (1 + 4i))(x - (1 - 4i))$$
$$= (x - 1 - 4i)(x - 1 + 4i)$$

63. $f(x) = x^4 - 16$

$$= (x^2 - 4)(x^2 + 4)$$
$$= (x - 2)(x + 2)(x - 2i)(x + 2i)$$

Zeros: $\pm 2, \pm 2i$

65. $f(z) = z^2 - 2z + 2$

By the Quadratic Formula, the zeros of $f(z)$ are

$$z = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i.$$

$$f(z) = [z - (1 + i)][z - (1 - i)]$$
$$= (z - 1 - i)(z - 1 + i)$$

67. $g(x) = x^3 - 3x^2 + x + 5$

Possible rational zeros: $\pm 1, \pm 5$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 1 & 5 \\ & & -1 & 4 & -5 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

By the Quadratic Formula, the zeros of $x^2 - 4x + 5$ are:

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

Zeros: $-1, 2 \pm i$

$$g(x) = (x + 1)(x - 2 - i)(x - 2 + i)$$

69. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

Possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

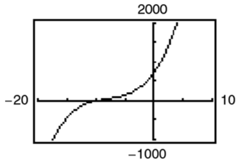
$$\begin{array}{r|rrrr} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$$g(x) = (x - 2)(x - 2)(x^2 + 4)$$
$$= (x - 2)^2(x + 2i)(x - 2i)$$

Zeros: $2, \pm 2i$

71. $f(x) = x^3 + 24x^2 + 214x + 740$

Possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm 37,$
$\pm 74, \pm 148, \pm 185, \pm 370, \pm 740$



Based on the graph, try $x = -10$.

$$\begin{array}{r|rrrr} -10 & 1 & 24 & 214 & 740 \\ & & -10 & -140 & -740 \\ \hline & 1 & 14 & 74 & 0 \end{array}$$

By the Quadratic Formula, the zeros of $x^2 + 14x + 74$

are $x = \dfrac{-14 \pm \sqrt{196 - 296}}{2} = -7 \pm 5i.$

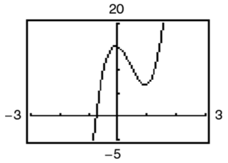
The zeros of $f(x)$ are $x = -10$ and $x = -7 \pm 5i$.

73. $f(x) = 16x^3 - 20x^2 - 4x + 15$

Possible rational zeros:

$\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{4}, \pm\frac{3}{4},$

$\pm\frac{5}{4}, \pm\frac{15}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{5}{8}, \pm\frac{15}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{5}{16}, \pm\frac{15}{16}$



Based on the graph, try $x = -\frac{3}{4}$.

$$\begin{array}{r|rrrr} -\frac{3}{4} & 16 & -20 & -4 & 15 \\ & & -12 & 24 & -15 \\ \hline & 16 & -32 & 20 & 0 \end{array}$$

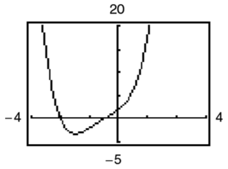
By the Quadratic Formula, the zeros of $16x^2 - 32x + 20 = 4(4x^2 - 8x + 5)$ are

$$x = \frac{8 \pm \sqrt{64 - 80}}{8} = 1 \pm \frac{1}{2}i.$$

The zeros of $f(x)$ are $x = -\frac{3}{4}$ and $x = 1 \pm \frac{1}{2}i$.

75. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

Possible rational zeros: $\pm 1, \pm 2, \pm\frac{1}{2}$



Based on the graph, try $x = -2$ and $x = -\frac{1}{2}$.

$$\begin{array}{r|rrrrr} -2 & 2 & 5 & 4 & 5 & 2 \\ & & -4 & -2 & -4 & -2 \\ \hline & 2 & 1 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & 2 & 1 \\ & & -1 & 0 & -1 \\ \hline & 2 & 0 & 2 & 0 \end{array}$$

The zeros of $2x^2 + 2 = 2(x^2 + 1)$ are $x = \pm i$.

The zeros of $f(x)$ are $x = -2$, $x = -\frac{1}{2}$, and $x = \pm i$.

77. $g(x) = 2x^3 - 3x^2 - 3$

Sign variations: 1, positive zeros: 1

$g(-x) = -2x^3 - 3x^2 - 3$

Sign variations: 0, negative zeros: 0

79. $h(x) = 2x^3 + 3x^2 + 1$

Sign variations: 0, positive zeros: 0

$h(-x) = -2x^3 + 3x^2 + 1$

Sign variations: 1, negative zeros: 1

81. $g(x) = 6x^4 + 2x^3 - 3x^2 + 2$

Sign variations: 2, positive zeros: 2 or 0

$g(-x) = 6x^4 - 2x^3 - 3x^2 + 2$

Sign variations: 2, negative zeros: 2 or 0

83. $f(x) = 5x^3 + x^2 - x + 5$

Sign variations: 2, positive zeros: 2 or 0

$f(-x) = -5x^3 + x^2 + x + 5$

Sign variations: 1, negative zeros: 1.

85. $f(x) = x^3 + 3x^2 - 2x + 1$

(a)
$$\begin{array}{r|rrrr} 1 & 1 & 3 & -2 & 1 \\ & & 1 & 4 & 2 \\ \hline & 1 & 4 & 2 & 3 \end{array}$$

1 is an upper bound.

(b)
$$\begin{array}{r|rrrr} -4 & 1 & 3 & -2 & 1 \\ & & -4 & 4 & -8 \\ \hline & 1 & -1 & 2 & -7 \end{array}$$

−4 is a lower bound.

87. $f(x) = x^4 - 4x^3 + 16x - 16$

(a)
$$\begin{array}{r|rrrrr} 5 & 1 & -4 & 0 & 16 & -16 \\ & & 5 & 5 & 25 & 205 \\ \hline & 1 & 1 & 5 & 41 & 189 \end{array}$$

5 is an upper bound.

(b)
$$\begin{array}{r|rrrrr} -3 & 1 & -4 & 0 & 16 & -16 \\ & & -3 & 21 & -63 & 141 \\ \hline & 1 & -7 & 21 & -47 & 125 \end{array}$$

−3 is a lower bound.

89. $f(x) = 16x^3 - 12x^2 - 4x + 3$

Possible rational zeros:

$$\frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16} = \pm\tfrac{1}{16}, \pm\tfrac{1}{8}, \pm\tfrac{3}{16}, \pm\tfrac{1}{4}, \pm\tfrac{3}{8}, \pm\tfrac{3}{4},$$

$$\pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \pm 3$$

However, the function factors by grouping.

$$\begin{aligned} 16x^2 - 12x^2 - 4x + 3 &= 0 \\ 4x^2(4x - 3) - (4x - 3) &= 0 \\ (4x - 3)(4x^2 - 1) &= 0 \\ (4x - 3)(2x - 1)(2x + 1) &= 0 \end{aligned}$$

$$4x - 3 = 0 \rightarrow x = \tfrac{3}{4}$$

$$2x - 1 = 0 \rightarrow x = \tfrac{1}{2}$$

$$2x + 1 = 0 \rightarrow x = -\tfrac{1}{2}$$

So, the zeros are $x = \frac{3}{4}, \pm\frac{1}{2}$.

91. $f(y) = 4y^3 + 3y^2 + 8y + 6$

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 4 & 3 & 8 & 6 \\ & & -3 & 0 & -6 \\ \hline & 4 & 0 & 8 & 0 \end{array}$$

$$\begin{aligned} 4y^3 + 3y^2 + 8y + 6 &= \left(y + \tfrac{3}{4}\right)(4y^2 + 8) \\ &= \left(y + \tfrac{3}{4}\right)4(y^2 + 2) \\ &= (4y + 3)(y^2 + 2) \end{aligned}$$

So, the only real zero is $-\frac{3}{4}$.

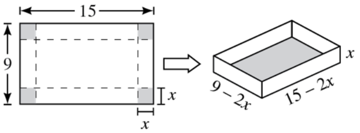
93.
$$\begin{aligned} P(x) &= x^4 - \tfrac{25}{4}x^2 + 9 \\ &= \tfrac{1}{4}(4x^4 - 25x^2 + 36) \\ &= \tfrac{1}{4}(4x^2 - 9)(x^2 - 4) \\ &= \tfrac{1}{4}(2x + 3)(2x - 3)(x + 2)(x - 2) \end{aligned}$$

The rational zeros are $\pm\frac{3}{2}$ and ± 2.

95.
$$\begin{aligned} f(x) &= x^3 - \tfrac{1}{4}x^2 - x + \tfrac{1}{4} \\ &= \tfrac{1}{4}(4x^3 - x^2 - 4x + 1) \\ &= \tfrac{1}{4}\left[x^2(4x - 1) - 1(4x - 1)\right] \\ &= \tfrac{1}{4}(4x - 1)(x^2 - 1) \\ &= \tfrac{1}{4}(4x - 1)(x + 1)(x - 1) \end{aligned}$$

The rational zeros are $\frac{1}{4}$ and ± 1.

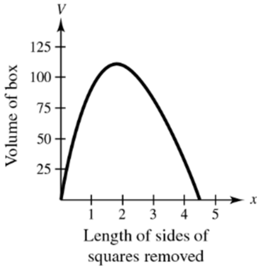
97. (a)



(b)
$$\begin{aligned} V &= l \cdot w \cdot h = (15 - 2x)(9 - 2x)x \\ &= x(9 - 2x)(15 - 2x) \end{aligned}$$

Because length, width, and height must be positive, you have $0 < x < \frac{9}{2}$ for the domain.

(c)



The volume is maximum when $x \approx 1.82$.

The dimensions are:

$r = ak^3 + bk^2 + ck + d, f(k) = r.$

1.82 cm × 5.36 cm × 11.36 cm

(d)
$$\begin{aligned} 56 &= x(9 - 2x)(15 - 2x) \\ 56 &= 135x - 48x^2 + 4x^3 \\ 0 &= 4x^3 - 48x^2 + 135x - 56 \end{aligned}$$

The zeros of this polynomial are $\frac{1}{2}, \frac{7}{2}$, and 8.

x cannot equal 8 because it is not in the domain of V. [The length cannot equal -1 and the width cannot equal -7. The product of $(8)(-1)(-7) = 56$ so it showed up as an extraneous solution.]

So, the volume is 56 cubic centimeters when $x = \frac{1}{2}$ centimeter or $x = \frac{7}{2}$ centimeters

99. (a) Current bin: $V = 2 \times 3 \times 4 = 24$ cubic feet

New bin: $V = 5(24) = 120$ cubic feet

$$V(x) = (2 + x)(3 + x)(4 + x) = 120$$

(b)
$$\begin{aligned} x^3 + 9x^2 + 26x + 24 &= 120 \\ x^3 + 9x^2 + 26x - 96 &= 0 \end{aligned}$$

The only real zero of this polynomial is $x = 2$. All the dimensions should be increased by 2 feet, so the new bin will have dimensions of 4 feet by 5 feet by 6 feet.

101. False. The most complex zeros it can have is two, and the Linear Factorization Theorem guarantees that there are three linear factors, so one zero must be real.

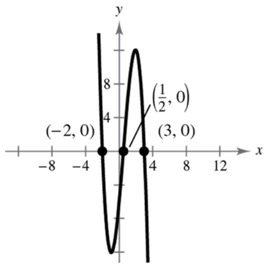
103. $g(x) = -f(x)$. This function would have the same zeros as $f(x)$, so r_1, r_2, and r_3 are also zeros of $g(x)$.

105. $g(x) = f(x - 5)$. The graph of $g(x)$ is a horizontal shift of the graph of $f(x)$ five units of the right, so the zeros of $g(x)$ are $5 + r_1, 5 + r_2,$ and $5 + r_3$.

107. $g(x) = 3 + f(x)$. Because $g(x)$ is a vertical shift of the graph of $f(x)$, the zeros of $g(x)$ cannot be determined.

109. Zeros: $-2, \frac{1}{2}, 3$

$$\begin{aligned} f(x) &= -(x + 2)(2x - 1)(x - 3) \\ &= -2x^3 + 3x^2 + 11x - 6 \end{aligned}$$

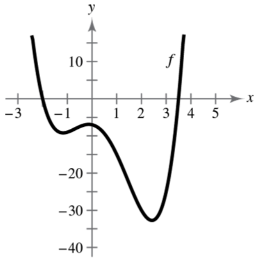


Any nonzero scalar multiple of *f* would have the same three zeros. Let $g(x) = af(x), a > 0$. There are infinitely many possible functions for *f*.

111. Because $1 + i$ is a zero of *f*, so is $1 - i$. From the graph, 1 is also a zero.

$$\begin{aligned} f(x) &= (x - (1 + i))(x - (1 - i))(x - 1) \\ &= (x^2 - 2x + 2)(x - 1) \\ &= x^3 - 3x^2 + 4x - 2 \end{aligned}$$

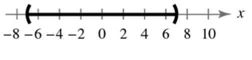
113.



If $x = i$ is a zero, then $x = -i$ is also a zero. So the function is $f(x) = (x - 2)(x - 3.5)(x - i)(x + i)$.

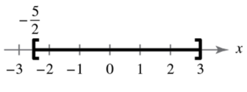
115. $|x| < 7$

$-7 < x < 7$



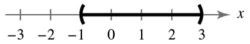
117. $|4x - 1| \le 11$

$$\begin{aligned} -11 &\le 4x - 1 \le 11 \\ -10 &\le 4x \le 12 \\ -\tfrac{5}{2} &\le x \le 3 \end{aligned}$$



119. $(x - 1)^2 < 4$

$$\begin{aligned} -2 &< x - 1 < 2 \\ -1 &< \quad x < 3 \end{aligned}$$



Alternate solution:

$$\begin{aligned} x^2 - 2x + 1 &< 4 \\ x^2 - 2x - 3 &< 0 \\ (x - 3)(x + 1) &< 0 \\ -1 < x &< 3 \end{aligned}$$

121. $(1, 1), (2, 5)$

$$\text{Slope} = \frac{5 - 1}{2 - 1} = \frac{4}{1} = 4$$

123. $(1, 1), (6, 0.5)$

$$\text{Slope} = \frac{1 - 0.5}{1 - 6} = \frac{0.5}{-5} = -0.1$$

125. $(1, 1), \left(\frac{-2}{5}, \frac{-3}{5}\right)$

$$\text{Slope} = \frac{1 - \left(\frac{-3}{5}\right)}{1 - \left(\frac{-2}{5}\right)} = \frac{\frac{8}{5}}{\frac{7}{5}} = \frac{8}{7}$$

127.
$$\begin{aligned} y &= \frac{2k - 5}{x} \\ -2 &= \frac{2k - 5}{4} \\ -8 &= 2k - 5 \\ -3 &= 2k \\ k &= \tfrac{-3}{2} \end{aligned}$$

129. $z = \dfrac{kx}{y}$

$$k = \frac{zy}{x} = \frac{(-3)(-2)}{4} = \frac{6}{4} = \frac{3}{2}$$

131.
$$\begin{aligned} kz &= \frac{x}{-5y - 7} \\ -3k &= \frac{4}{-5(-2) - 7} \\ -3k &= \frac{4}{3} \\ k &= -\frac{4}{9} \end{aligned}$$

133.
$$\begin{aligned} z &= k \times y \\ -3 &= k(4)(-2) \\ k &= \tfrac{3}{8} \end{aligned}$$

Section 2.6 Rational Functions

1. rational functions

3. To determine the vertical asymptote(s) of the graph of $f(x) = \frac{9}{x-3}$, find the real zeros of the denominator of the equation. (Assuming no common factors in the numerator and denominator)

5. Because the denominator is zero when $x - 1 = 0$, the domain of *f* is all real numbers except $x = 1$.

x	0	0.5	0.9	0.99	$\to 1$
$f(x)$	-1	-2	-10	-100	$\to -\infty$

x	$1 \leftarrow$	1.01	1.1	1.5	2
$f(x)$	$\infty \leftarrow$	100	10	2	1

As *x* approaches 1 from the left, $f(x)$ decreases without bound towards $-\infty$. As *x* approaches 1 from the right, $f(x)$ increases without bound towards $+\infty$.

7. Because the denominator is zero when $x^2 - 1 = 0$, the domain of *f* is all real numbers except $x = -1$ and $x = 1$.

x	-2	-1.5	-1.1	-1.01	$\to -1$
$f(x)$	4	5.4	17.3	152.3	$\to \infty$

x	$-1 \leftarrow$	-0.99	-0.9	-0.5	0
$f(x)$	$-\infty \leftarrow$	-147.8	-12.8	-1	0

As *x* approaches -1 from the left, $f(x)$ increases without bound (∞). As *x* approaches -1 from the right, $f(x)$ decreases without bound $(-\infty)$.

x	0	0.5	0.9	0.99	$\to 1$
$f(x)$	0	-1	-12.8	-147.8	$\to -\infty$

x	$1 \leftarrow$	1.01	1.1	1.5	2
$f(x)$	$\infty \leftarrow$	152.3	17.3	5.4	4

As *x* approaches 1 from the left, $f(x)$ decreases without bound $(-\infty)$. As *x* approaches 1 from the right, $f(x)$ increases without bound (∞).

9. $f(x) = \frac{4}{x^2}$

Domain: all real numbers except $x = 0$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

$[\text{Degree of } N(x) < \text{degree of } D(x)]$

11. $f(x) = \frac{5+x}{5-x} = \frac{x+5}{-x+5}$

Domain: all real numbers except $x = 5$

Vertical asymptote: $x = 5$

Horizontal asymptote: $y = -1$

$[\text{Degree of } N(x) = \text{degree of } D(x)]$

13. $f(x) = \frac{x^3}{x^2 - 1}$

Domain: all real numbers except $x = \pm 1$

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: None

$[\text{Degree of } N(x) > \text{degree of } D(x)]$

15.
$$\begin{aligned} f(x) &= \frac{x^2 - 3x - 4}{2x^2 + x - 1} \\ &= \frac{(x+1)(x-4)}{(2x-1)(x+1)} \\ &= \frac{x-4}{2x-1}, x \neq -1 \end{aligned}$$

Horizontal asymptote: $y = \frac{1}{2}$

$(\text{Degree of } N(x) = \text{degree of } D(x))$

Vertical asymptote: $x = \frac{1}{2}$

(Because $x + 1$ is a common factor of $N(x)$ and $D(x)$, $x = -1$ is not a vertical asymptote of $f(x)$.)

17. $f(x) = \frac{1}{x + 1}$

(a) Domain: all real numbers x except $x = -1$

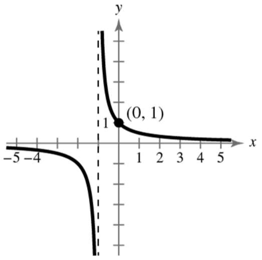
(b) y-intercept: $(0, 1)$

(c) Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 0$

(d)

x	-4	-3	0	1	2	3
$f(x)$	$-\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



19. $h(x) = \frac{-1}{x + 4}$

(a) Domain: all real numbers x except $x = -4$

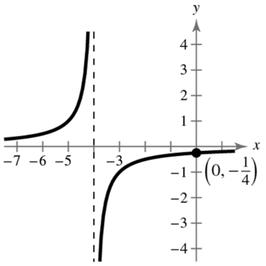
(b) y-intercept: $\left(0, -\frac{1}{4}\right)$

(c) Vertical asymptote: $x = -4$

Horizontal asymptote: $y = 0$

(d)

x	-6	-5	-3	-2	-1	0
$h(x)$	$\frac{1}{2}$	1	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$



21. $f(x) = \frac{x^2}{x^2 + 9}$

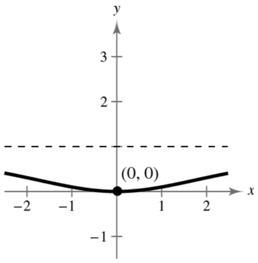
(a) Domain: all real numbers x

(b) Intercept: $(0, 0)$

(c) Horizontal asymptote: $y = 1$

(d)

x	± 1	± 2	± 3
$f(x)$	$\frac{1}{10}$	$\frac{4}{13}$	$\frac{1}{2}$



23. $g(s) = \dfrac{4s}{s^2 + 4}$

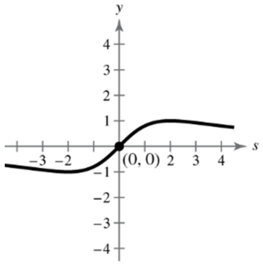
(a) Domain: all real numbers s

(b) Intercept: $(0, 0)$

(c) Horizontal asymptote: $y = 0$

(d)

s	-2	-1	0	1	2
$g(s)$	-1	$-\frac{4}{5}$	0	$\frac{4}{5}$	1



25. $f(x) = \dfrac{2x}{x^2 - 3x - 4} = \dfrac{2x}{(x - 4)(x + 1)}$

(a) Domain: all real numbers x except $x = 4$ and $x = -1$

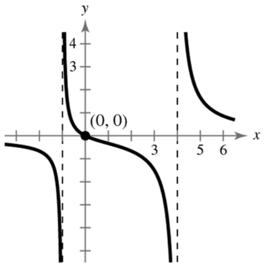
(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = 4, x = -1$

Horizontal asymptote: $y = 0$

(d)

x	-3	-2	0	1	2	3	5
$f(x)$	$-\frac{3}{7}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{3}{2}$	$\frac{5}{3}$



27. $f(x) = \dfrac{x - 4}{x^2 - 16} = \dfrac{x - 4}{(x + 4)(x - 4)} = \dfrac{1}{x + 4}, x \neq 4$

(a) Domain: all real numbers except $x = \pm 4$

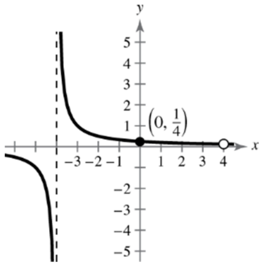
(b) y-intercept: $\left(0, \dfrac{1}{4}\right)$

(c) Vertical asymptote: $x = -4$

Horizontal asymptote: $y = 0$

(d)

x	-5	-4	-3	-2	-1
$f(x)$	-1	Undef.	1	$\frac{1}{2}$	$\frac{1}{3}$



29. $f(x) = \dfrac{x^2 - 25}{x^2 - 4x - 5} = \dfrac{(x + 5)(x - 5)}{(x - 5)(x + 1)} = \dfrac{x + 5}{x + 1}, x \neq 5$

(a) Domain: all real numbers x except $x = 5$ and $x = -1$

(b) x-intercept: $(-5, 0)$

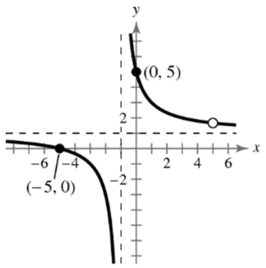
y-intercept: $(0, 5)$

(c) Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 1$

(d)

x	-5	-3	0	3	5
$f(x)$	0	-1	5	2	Undef.



31. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6} = \dfrac{x(x + 3)}{(x + 3)(x - 2)} = \dfrac{x}{x - 2}, \quad x \neq -3$

(a) Domain: all real numbers x except $x = -3$ and $x = 2$

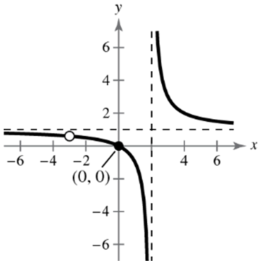
(b) Intercept: $(0, 0)$

(c) Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 1$

(d)

x	-1	0	1	3	4
$f(x)$	$\frac{1}{3}$	0	-1	3	2



33. $f(x) = \dfrac{2x^2 - 5x - 3}{x^3 - 2x^2 - x + 2} = \dfrac{(2x + 1)(x - 3)}{(x - 2)(x + 1)(x - 1)}$

(a) Domain: all real numbers x except $x = 2, x = \pm 1$

(b) x-intercepts: $\left(-\frac{1}{2}, 0\right), (3, 0)$

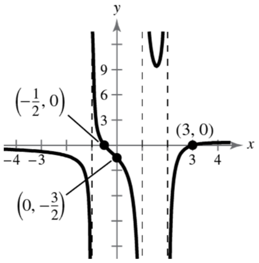
y-intercept: $\left(0, -\frac{3}{2}\right)$

(c) Vertical asymptotes: $x = 2, x = -1,$ and $x = 1$

Horizontal asymptote: $y = 0$

(d)

x	-3	-2	0	$\frac{3}{2}$	3	4
$f(x)$	$-\frac{3}{4}$	$-\frac{5}{4}$	$-\frac{3}{2}$	$\frac{48}{5}$	0	$\frac{3}{10}$



35. Because the function has a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 0$,

$f(x) = \dfrac{4}{x + 2}$ matches graph (d).

36. Because the function has a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 0$,

$f(x) = \dfrac{5}{x - 2}$ matches graph (a).

37. Because the function has vertical asymptotes at $x = \pm 2$ and a horizontal asymptote at $y = 2$,

$f(x) = \dfrac{2x^2}{x^2 - 4}$ matches graph (c).

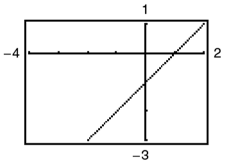
38. Because the function has a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 0$,

$f(x) = \dfrac{3x}{(x + 2)^2}$ matches graph (b).

39. (a) Domain of f: all real numbers x except $x = -1$

Domain of g: all real numbers x

(b)

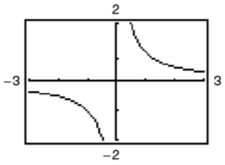


(c) Because there are only a finite number of pixels, the graphing utility may not attempt to evaluate the function where it does not exist.

41. (a) Domain of f: all real numbers x except $x = 0, 2$

Domain of g: all real numbers $x = 0$

(b)



(c) Because there are only a finite number of pixels, the graphing utility may not attempt to evaluate the function where it does not exist.

43. $h(x) = \dfrac{x^2 - 4}{x} = x - \dfrac{4}{x}$

(a) Domain: all real numbers x except $x = 0$

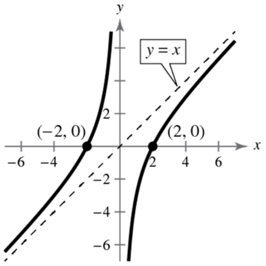
(b) x-intercepts: $(-2, 0), (2, 0)$

(c) Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

(d)

x	-4	-3	-1	1	3	4
$h(x)$	-3	$\frac{-5}{3}$	3	-3	$\frac{5}{3}$	3



45. $f(x) = \dfrac{2x^2 + 1}{x} = 2x + \dfrac{1}{x}$

(a) Domain: all real numbers x except $x = 0$

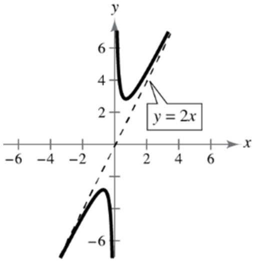
(b) No intercepts

(c) Vertical asymptote: $x = 0$

Slant asymptote: $y = 2x$

(d)

x	-4	-2	2	4	6
$f(x)$	$-\frac{33}{4}$	$-\frac{9}{2}$	$\frac{9}{2}$	$\frac{33}{4}$	$\frac{73}{6}$



47. $g(x) = \dfrac{x^2 + 1}{x} = x + \dfrac{1}{x}$

(a) Domain: all real numbers x except $x = 0$

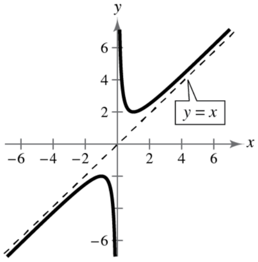
(b) No intercepts

(c) Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

(d)

x	-4	-2	2	4	6
$g(x)$	$-\frac{17}{4}$	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{17}{4}$	$\frac{37}{6}$



49. $f(t) = \dfrac{t^2 + 1}{t + 5} = -t + 5 - \dfrac{26}{t + 5}$

(a) Domain: all real numbers t except $t = -5$

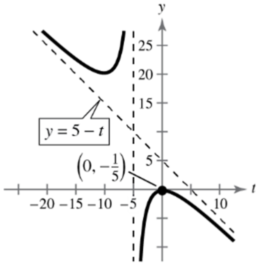
(b) Intercept: $\left(0, -\dfrac{1}{5}\right)$

(c) Vertical asymptote: $t = -5$

Slant asymptote: $y = -t + 5$

(d)

t	-7	-6	-4	-3	0
$f(t)$	25	37	-17	-5	$-\frac{1}{5}$



51. $f(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{4x}{x^2 - 4}$

(a) Domain: all real numbers x except $x = \pm 2$

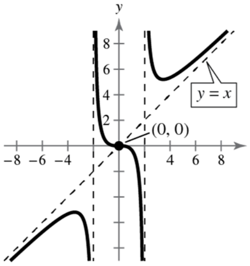
(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$

(d)

x	-6	-4	-1	0	1	4	6
$f(x)$	$-\frac{27}{4}$	$-\frac{16}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{16}{3}$	$\frac{27}{4}$



53. $f(x) = \dfrac{x^2 - x + 1}{x - 1} = x + \dfrac{1}{x - 1}$

(a) Domain: all real numbers x except $x = 1$

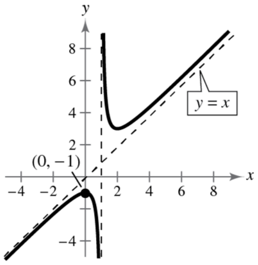
(b) y-intercept: $(0, -1)$

(c) Vertical asymptote: $x = 1$

Slant asymptote: $y = x$

(d)

x	-4	-2	0	2	4
$f(x)$	$-\frac{21}{5}$	$-\frac{7}{3}$	-1	3	$\frac{13}{3}$



55. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2} = \dfrac{(2x - 1)(x + 1)(x - 1)}{(x + 1)(x + 2)} = \dfrac{(2x - 1)(x - 1)}{x + 2}, \quad x \neq -1$

$= \dfrac{2x^2 - 3x + 1}{x + 2} = 2x - 7 + \dfrac{15}{x + 2}, \quad x \neq -1$

(a) Domain: all real numbers x except $x = -1$ and $x = -2$

(b) y-intercept: $\left(0, \frac{1}{2}\right)$

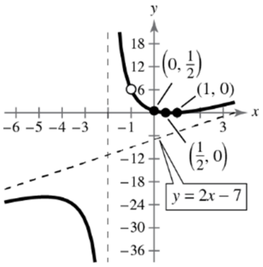
x-intercepts: $\left(\frac{1}{2}, 0\right), (1, 0)$

(c) Vertical asymptote: $x = -2$

Slant asymptote: $y = 2x - 7$

(d)

x	-4	-3	$-\frac{3}{2}$	0	1
$f(x)$	$-\frac{45}{2}$	-28	20	$\frac{1}{2}$	0



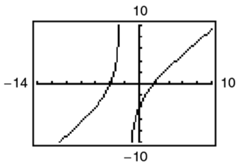
57. $f(x) = \dfrac{x^2 + 2x - 8}{x + 2} = x - \dfrac{8}{x + 2}$

Domain: all real numbers x except $x = -2$

Vertical asymptote: $x = -2$

Slant asymptote: $y = x$

Line: $y = x$



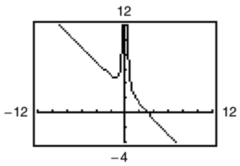
59. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2} = \dfrac{1}{x^2} + 3 - x = -x + 3 + \dfrac{1}{x^2}$

Domain: all real numbers x except $x = 0$

Vertical asymptote: $x = 0$

Slant asymptote: $y = -x + 3$

Line: $y = -x + 3$



61. $y = \dfrac{x+1}{x-3}$

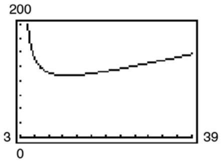
(a) x-intercept: $(-1, 0)$

(b) $$\begin{aligned} 0 &= \frac{x+1}{x-3} \\ 0 &= x+1 \\ -1 &= x \end{aligned}$$

63. $A = xy$ and

$$\begin{aligned} (x-3)(y-2) &= 64 \\ y - 2 &= \frac{64}{x-3} \\ y &= 2 + \frac{64}{x-3} = \frac{2x+58}{x-3} \end{aligned}$$

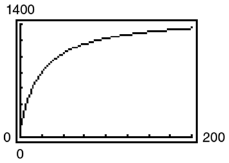
Thus, $A = xy = x\left(\dfrac{2x+58}{x-3}\right) = \dfrac{2x(x+29)}{x-3}$, $x > 3$.



By graphing the area function, we see that A is minimum when $x \approx 12.8$ inches and $y \approx 8.5$ inches.

65. $N = \dfrac{20(5+3t)}{1+0.04t}$, $t \geq 0$

(a)



(b) $N(5) \approx 333$ deer

$N(10) = 500$ deer

$N(25) = 800$ deer

(c) The herd is limited by the horizontal asymptote:

$N = \dfrac{60}{0.04} = 1500$ deer

67. (a) Let t_1 = time from Akron to Columbus and t_2 = time from Columbus back to Akron.

$xt_1 = 100 \Rightarrow t_1 = \dfrac{100}{x}$

$yt_2 = 100 \Rightarrow t_2 = \dfrac{100}{y}$

$$\begin{aligned} 50(t_1 + t_2) &= 200 \\ t_1 + t_2 &= 4 \\ \frac{100}{x} + \frac{100}{y} &= 4 \\ 100y + 100x &= 4xy \\ 25y + 25x &= xy \\ 25x &= xy - 25y \\ 25x &= y(x - 25) \end{aligned}$$

Thus, $y = \dfrac{25x}{x-25}$.

69. False. Polynomial functions do not have vertical asymptotes.

71. Yes. No. Every rational function is the ratio of two polynomial functions of the form $f(x) = \dfrac{N(x)}{D(x)}$.

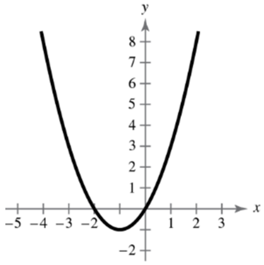
73. $f(x) = x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$

Vertex: $(-1, -1)$

Axis of symmetry: $x = -1$

$f(x) = x(x + 2) = 0 \Rightarrow x = 0, -2$

x-intercepts: $(0, 0), (-2, 0)$



75. $f(x) = -x^2 - 16x - 13$

$= -(x^2 + 16x + 64) - 13 + 64$

$= -(x + 8)^2 + 51$

Vertex: $(-8, 51)$

Axis of symmetry: $x = -8$

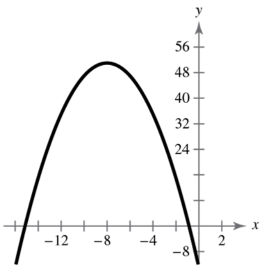
$f(x) = -x^2 - 16x - 13 = 0$

$\Rightarrow x^2 + 16x + 13 = 0$

$\Rightarrow x = \dfrac{-16 \pm \sqrt{256 - 4(13)}}{2}$

$= -8 \pm \sqrt{51}$

x-intercepts: $(-8 \pm \sqrt{51}, 0)$



77. $y = x^2 + 12x + 20$

$y = 0 \Rightarrow (x + 2)(x + 10) = 0 \Rightarrow x = -2, -10$

x-intercepts: $(-2, 0), (-10, 0)$

$x = 0 \Rightarrow y = 0^2 + 12(0) + 20 \Rightarrow y = 20$

y-intercept: $(0, 20)$

79. $y = \dfrac{3 + 2x}{1 + x}$

$y = 0 \Rightarrow 3 + 2x = 0 \Rightarrow x = -\frac{3}{2}$

x-intercept: $\left(-\frac{3}{2}, 0\right)$

$x = 0 \Rightarrow y = \dfrac{3 + 2(0)}{1 + 0} \Rightarrow y = 3$

y-intercept: $(0, 3)$

81. $f(x) = \dfrac{x}{x^2 - x - 2}$

$f(x) = 0 \Rightarrow x = 0$

$x = 0 \Rightarrow f(0) = \dfrac{0}{0^2 - 0 - 2} \Rightarrow y = 0$

Intercept: $(0, 0)$

83. $f(x) = \dfrac{x^2 - 9}{x^2 - x - 6} = \dfrac{(x - 3)(x + 3)}{(x - 3)(x + 2)}$

$= \dfrac{x + 3}{x + 2}, x \neq 3$

$y = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow (x + 3)(x - 3) = 0$

$\Rightarrow x = -3, 3$

$x = 0 \Rightarrow y = \dfrac{0^2 - 9}{0^2 - 0 - 6} = \dfrac{-9}{-6} = \dfrac{3}{2}$

x-intercept: $(-3, 0)$ ($x = 3$ is a hole.)

y-intercept: $\left(0, \frac{3}{2}\right)$

85. $6x^2 - 3x - 9 = 6$

$2x^2 - x - 5 = 0$

$x = \dfrac{-(-1) \pm \sqrt{1 - 4(2)(-5)}}{4}$

$= \dfrac{1 \pm \sqrt{41}}{4}$

87. $\dfrac{2x + 4}{8x + 7} = 3$

$2x + 4 = 3(8x + 7)$

$2x + 4 = 24x + 21$

$-17 = 22x$

$-\dfrac{17}{22} = x$

Section 2.7 Nonlinear Inequalities

1. positive; negative

3. The key numbers of the inequality are −2 and 5.

5. $x^2 - 3 < 0$

(a) $x = 3$

$$(3)^2 - 3 \overset{?}{<} 0$$
$$6 \not< 0$$

No, $x = 3$ *is not* a solution.

(b) $x = 0$

$$(0)^2 - 3 \overset{?}{<} 0$$
$$-3 < 0$$

Yes, $x = 0$ *is* a solution.

(c) $x = \frac{3}{2}$

$$\left(\frac{3}{2}\right)^2 - 3 \overset{?}{<} 0$$
$$-\frac{3}{4} < 0$$

Yes, $x = \frac{3}{2}$ *is* a solution.

(d) $x = -5$

$$(-5)^2 - 3 \overset{?}{<} 0$$
$$22 \not< 0$$

No, $x = -5$ *is not* a solution

7. $\dfrac{x + 2}{x - 4} \geq 3$

(a) $x = 5$

$$\frac{5 + 2}{5 - 4} \overset{?}{\geq} 3$$
$$7 \geq 3$$

Yes, $x = 5$ *is* a solution.

(b) $x = 4$

$$\frac{4 + 2}{4 - 4} \overset{?}{\geq} 3$$

$\dfrac{6}{0}$ is undefined.

No, $x = 4$ *is not* a solution.

(c) $x = -\dfrac{9}{2}$

$$\frac{-\frac{9}{2} + 2}{-\frac{9}{2} - 4} \overset{?}{\geq} 3$$
$$\frac{5}{17} \not\geq 3$$

No, $x = -\dfrac{9}{2}$ *is not* a solution.

(d) $x = \dfrac{9}{2}$

$$\frac{\frac{9}{2} + 2}{\frac{9}{2} - 4} \overset{?}{\geq} 3$$
$$13 \geq 3$$

Yes, $x = \dfrac{9}{2}$ *is* a solution.

9. $x^2 - 3x - 18 = (x + 3)(x - 6)$

$x + 3 = 0 \Rightarrow x = -3$

$x - 6 = 0 \Rightarrow x = 6$

The key numbers are −3 and 6.

11. $\dfrac{1}{x - 5} + 1 = \dfrac{1 + 1(x - 5)}{x - 5}$

$= \dfrac{x - 4}{x - 5}$

$x - 4 = 0 \Rightarrow x = 4$

$x - 5 = 0 \Rightarrow x = 5$

The key numbers are 4 and 5.

13. $2x^2 + 4x < 0$

$2x(x + 2) < 0$

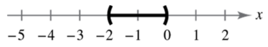
Key numbers: $x = 0, -2$

Test intervals: $(-\infty, -2), (-2, 0), (0, \infty)$

Test: Is $2x(x + 2) < 0$?

Interval	x-Value	Value of $2x(x + 2)$	Conclusion
$(-\infty, -2)$	−3	6	Positive
$(-2, 0)$	−1	−2	Negative
$(0, \infty)$	1	3	Positive

Solution set: $(-2, 0)$



15.

$$\begin{aligned} x^2 &< 9 \\ x^2 - 9 &< 0 \\ (x + 3)(x - 3) &< 0 \end{aligned}$$

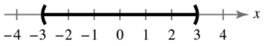
Key numbers: $x = \pm 3$

Test intervals: $(-\infty, -3), (-3, 3), (3, \infty)$

Test: Is $(x + 3)(x - 3) < 0$?

Interval	x-Value	Value of $x^2 - 9$	Conclusion
$(-\infty, -3)$	-4	7	Positive
$(-3, 3)$	0	-9	Negative
$(3, \infty)$	4	7	Positive

Solution set: $(-3, 3)$



17.

$$\begin{aligned} (x + 2)^2 &\le 25 \\ x^2 + 4x + 4 &\le 25 \\ x^2 + 4x - 21 &\le 0 \\ (x + 7)(x - 3) &\le 0 \end{aligned}$$

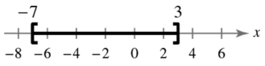
Key numbers: $x = -7, x = 3$

Test intervals: $(-\infty, -7), (-7, 3), (3, \infty)$

Test: Is $(x + 7)(x - 3) \le 0$?

Interval	x-Value	Value of $(x + 7)(x - 3)$	Conclusion
$(-\infty, -7)$	-8	$(-1)(-11) = 11$	Positive
$(-7, 3)$	0	$(7)(-3) = -21$	Negative
$(3, \infty)$	4	$(11)(1) = 11$	Positive

Solution set: $[-7, 3]$



19.

$$\begin{aligned} x^2 + 6x + 1 &\ge -7 \\ x^2 + 6x + 8 &\ge 0 \\ (x + 2)(x + 4) &\ge 0 \end{aligned}$$

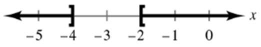
Key numbers: $x = -2, x = -4$

Test Intervals: $(-\infty, -4), (-4, -2), (-2, \infty)$

Test: Is $(x + 2)(x + 4) > 0$?

Interval	x-Value	Value of $(x + 2)(x + 4)$	Conclusion
$(-\infty, -4)$	-6	8	Positive
$(-4, -2)$	-3	-1	Negative
$(-2, \infty)$	0	8	Positive

Solution set: $(-\infty, -4] \cup [-2, \infty)$



21.

$$\begin{aligned} x^2 + x &< 6 \\ x^2 + x - 6 &< 0 \\ (x + 3)(x - 2) &< 0 \end{aligned}$$

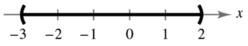
Key numbers: $x = -3, x = 2$

Test intervals: $(-\infty, -3), (-3, 2), (2, \infty)$

Test: Is $(x + 3)(x - 2) < 0$?

Interval	x-Value	Value of $(x + 3)(x - 2)$	Conclusion
$(-\infty, -3)$	-4	$(-1)(-6) = 6$	Positive
$(-3, 2)$	0	$(3)(-2) = -6$	Negative
$(2, \infty)$	3	$(6)(1) = 6$	Positive

Solution set: $(-3, 2)$



23.

$$x^2 < 3 - 2x$$
$$x^2 + 2x - 3 < 0$$
$$(x + 3)(x - 1) < 0$$

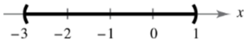
Key numbers: $x = -3, x = 1$

Test intervals: $(-\infty, -3), (-3, 1), (1, \infty)$

Test: Is $(x + 3)(x - 1) < 0$?

Interval	x-Value	Value of $(x + 3)(x - 1)$	Conclusion
$(-\infty, -3)$	-4	$(-1)(-5) = 5$	Positive
$(-3, 1)$	0	$(3)(-1) = -3$	Negative
$(1, \infty)$	2	$(5)(1) = 5$	Positive

Solution set: $(-3, 1)$



25.

$$3x^2 - 11x > 20$$
$$3x^2 - 11x - 20 > 0$$
$$(3x + 4)(x - 5) > 0$$

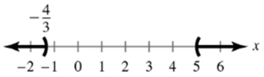
Key numbers: $x = 5, x = -\frac{4}{3}$

Test intervals: $\left(-\infty, -\frac{4}{5}\right), \left(-\frac{4}{3}, 5\right), (5, \infty)$

Test: Is $(3x + 4)(x - 5) > 0$?

Interval	x-Value	Value of $(3x + 4)(x - 5)$	Conclusion
$\left(-\infty, -\frac{4}{3}\right)$	-3	$(-5)(-8) = 40$	Positive
$\left(-\frac{4}{3}, 5\right)$	0	$(4)(-5) = -20$	Negative
$(5, \infty)$	6	$(22)(1) = 22$	Positive

Solution set: $\left(-\infty, -\frac{4}{3}\right) \cup (5, \infty)$



27.

$$x^3 - 3x^2 - x + 3 > 0$$
$$x^2(x - 3) - (x - 3) > 0$$
$$(x - 3)(x^2 - 1) > 0$$
$$(x - 3)(x + 1)(x - 1) > 0$$

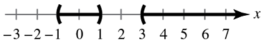
Key numbers: $x = -1, x = 1, x = 3$

Test intervals: $(-\infty, -1), (-1, 1), (1, 3), (3, \infty)$

Test: Is $(x - 3)(x + 1)(x - 1) > 0$?

Interval	x-Value	Value of $(x - 3)(x + 1)(x - 1)$	Conclusion
$(-\infty, -1)$	-2	$(-5)(-1)(-3) = -15$	Negative
$(-1, 1)$	0	$(-3)(1)(-1) = 3$	Positive
$(1, 3)$	2	$(-1)(3)(1) = -3$	Negative
$(3, \infty)$	4	$(1)(5)(3) = 15$	Positive

Solution set: $(-1, 1) \cup (3, \infty)$



29.

$$\begin{aligned} -x^3 + 7x^2 + 9x &> 63 \\ x^3 - 7x^2 - 9x &< -63 \\ x^3 - 7x^2 - 9x + 63 &< 0 \\ x^2(x - 7) - 9(x - 7) &< 0 \\ (x - 7)(x^2 - 9) &< 0 \\ (x - 7)(x + 3)(x - 3) &< 0 \end{aligned}$$

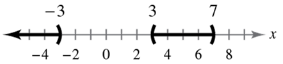
Key numbers: $x = -3, x = 3, x = 7$

Test intervals: $(-\infty, -3), (-3, 3), (3, 7), (7, \infty)$

Test: Is $(x - 7)(x + 3)(x - 3) < 0$?

Interval	**x-Value**	Value of $(x - 7)(x + 3)(x - 3)$	Conclusion
$(-\infty, -3)$	-4	$(-11)(-1)(-7) = -77$	Negative
$(-3, 3)$	0	$(-7)(3)(-3) = 63$	Positive
$(3, 7)$	4	$(-3)(7)(1) = -21$	Negative
$(7, \infty)$	8	$(1)(11)(5) = 55$	Positive

Solution set: $(-\infty, -3) \cup (3, 7)$



31. $4x^3 - 6x^2 < 0$

$2x^2(2x - 3) < 0$

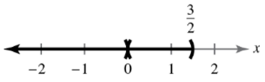
Key numbers: $x = 0, x = \frac{3}{2}$

Test intervals: $(-\infty, 0) \Rightarrow 2x^2(2x - 3) < 0$

$\left(0, \frac{3}{2}\right) \Rightarrow 2 \Rightarrow 2x^2(2x - 3) < 0$

$\left(\frac{3}{2}, \infty\right) \Rightarrow 2x^2(2x - 3) > 0$

Solution set: $(-\infty, 0) \cup \left(0, \frac{3}{2}\right)$



33. $(x - 1)^2(x + 2)^3 \geq 0$

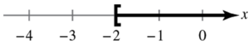
Key numbers: $x = 1, x = -2$

Test intervals: $(-\infty, -2) \Rightarrow (x - 1)^2(x + 2)^3 < 0$

$(-2, 1) \Rightarrow (x - 1)^2(x + 2)^3 > 0$

$(1, \infty) \Rightarrow (x - 1)^2(x + 2)^3 > 0$

Solution set: $[-2, \infty)$



35. $4x^2 - 4x + 1 \leq 0$

$(2x - 1)^2 \leq 0$

Key number: $x = \frac{1}{2}$

Test Interval	x-Value	Polynomial Value	Conclusion
$\left(-\infty, \frac{1}{2}\right)$	$x = 0$	$[2(0) -1]^2 = 1$	Positive
$\left(\frac{1}{2}, \infty\right)$	$x = 1$	$[2(1) - 1]^2 = 1$	Positive

The solution set consists of the single real number $\frac{1}{2}$.

37. $x^2 - 6x + 12 \le 0$

Using the Quadratic Formula, you can determine that the key numbers are $x = 3 \pm \sqrt{3}i$.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, \infty)$	$x = 0$	$(0)^2 - 6(0) + 12 = 12$	Positive

The solution set is empty, that is there are no real solutions.

39. $\dfrac{4x - 1}{x} > 0$

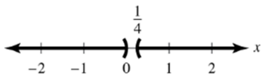
Key numbers: $x = 0, x = \dfrac{1}{4}$

Test intervals: $(-\infty, 0), \left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \infty\right)$

Test: Is $\dfrac{4x - 1}{x} > 0$?

Interval	x-Value	Value of $\dfrac{4x - 1}{x}$	Conclusion
$(-\infty, 0)$	-1	$\dfrac{-5}{-1} = 5$	Positive
$\left(0, \dfrac{1}{4}\right)$	$\dfrac{1}{8}$	$\dfrac{-\frac{1}{2}}{\frac{1}{8}} = -4$	Negative
$\left(\dfrac{1}{4}, \infty\right)$	1	$\dfrac{3}{1} = 3$	Positive

Solution set: $(-\infty, 0) \cup \left(\dfrac{1}{4}, \infty\right)$



41.

$$\frac{3x + 5}{x - 1} < 2$$

$$\frac{3x + 5}{x - 1} - 2 < 0$$

$$\frac{3x + 5 - 2(x - 1)}{x - 1} < 0$$

$$\frac{x + 7}{x - 1} < 0$$

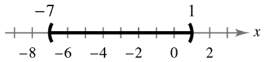
Key numbers: $x = -7, x = 1$

Test intervals: $(-\infty, -7), (-7, 1), (1, \infty)$

Test: Is $\dfrac{x + 7}{x - 1} < 0$?

Interval	x-Value	Value of $\dfrac{x + 7}{x - 1}$	Conclusion
$(-\infty, -7)$	-8	$\dfrac{-1}{-9} = \dfrac{1}{9}$	Positive
$(-7, 1)$	0	$\dfrac{0 + 7}{0 - 1} = -7$	Negative
$(1, \infty)$	2	$\dfrac{2 + 9}{2 - 1} = 11$	Positive

Solution set: $(-7, 1)$



43. $$\frac{2}{x+5} > \frac{1}{x-3}$$

$$\frac{2}{x+5} - \frac{1}{x-3} > 0$$

$$\frac{2(x-3) - 1(x+5)}{(x+5)(x-3)} > 0$$

$$\frac{x-11}{(x+5)(x-3)} > 0$$

Key numbers: $x = -5, x = 3, x = 11$

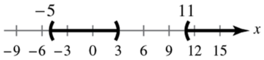
Test intervals: $(-\infty, -5) \Rightarrow \frac{x-11}{(x+5)(x-3)} < 0$

$(-5, 3) \Rightarrow \frac{x-11}{(x+5)(x-3)} > 0$

$(3, 11) \Rightarrow \frac{x-11}{(x+5)(x-3)} < 0$

$(11, \infty) \Rightarrow \frac{x-11}{(x+5)(x-3)} > 0$

Solution set: $(-5, 3) \cup (11, \infty)$



45. $$\frac{x^2+2x}{x^2-9} \le 0$$

$$\frac{x(x+2)}{(x+3)(x-3)} \le 0$$

Key numbers: $x = 0, x = -2, x = \pm 3$

Test intervals: $(-\infty, -3) \Rightarrow \frac{x(x+2)}{(x+3)(x-3)} > 0$

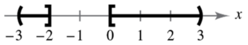
$(-3, -2) \Rightarrow \frac{x(x+2)}{(x+3)(x-3)} < 0$

$(-2, 0) \Rightarrow \frac{x(x+2)}{(x+3)(x-3)} > 0$

$(0, 3) \Rightarrow \frac{x(x+2)}{(x+3)(x-3)} < 0$

$(3, \infty) \Rightarrow \frac{x(x+2)}{(x+3)(x-3)} > 0$

Solution set: $(-3, -2] \cup [0, 3)$



47. $$\frac{3}{x-1} + \frac{2x}{x+1} > -1$$

$$\frac{3(x+1) + 2x(x-1) + 1(x+1)(x-1)}{(x-1)(x+1)} > 0$$

$$\frac{3x^2+x+2}{(x-1)(x+1)} > 0$$

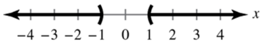
Key numbers: $x = -1, x = 1$

Test intervals: $(-\infty, -1) \Rightarrow \frac{3x^2+x+2}{(x-1)(x+1)} > 0$

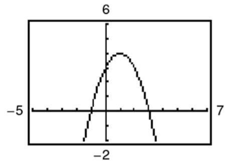
$(-1, 1) \Rightarrow \frac{3x^2+x+2}{(x-1)(x+1)} < 0$

$(1, \infty) \Rightarrow \frac{3x^2+x+2}{(x-1)(x+1)} > 0$

Solution set: $(-\infty, -1) \cup (1, \infty)$



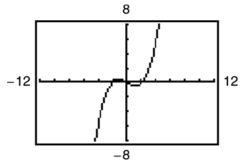
49. $y = -x^2 + 2x + 3$



(a) $y \le 0$ when $x \le -1$ or $x \ge 3$.

(b) $y \ge 3$ when $0 \le x \le 2$.

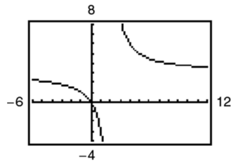
51. $y = \frac{1}{8}x^3 - \frac{1}{2}x$



(a) $y \ge 0$ when $-2 \le x \le 0$ or $2 \le x < \infty$.

(b) $y \le 6$ when $x \le 4$.

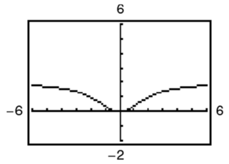
53. $y = \frac{3x}{x-2}$



(a) $y \le 0$ when $0 \le x < 2$.

(b) $y \ge 6$ when $2 < x \le 4$.

55. $y = \dfrac{2x^2}{x^2 + 4}$



(a) $y \geq 1$ when $x \leq -2$ or $x \geq 2$.

This can also be expressed as $|x| \geq 2$.

(b) $y \leq 2$ for all real numbers x.

This can also be expressed as $-\infty < x < \infty$.

57. $0.3x^2 + 6.26 < 10.8$

$0.3x^2 + 4.54 < 0$

Key numbers: $x \approx \pm 3.89$

Test intervals: $(-\infty, -3.89), (-3.89, 3.89), (3.89, \infty)$

Solution set: $(-3.89, 3.89)$

59. $12.5x + 1.6 > 0.5x^2$

$-0.5x^2 + 12.5x + 1.6 > 0$

Key numbers: $x \approx -0.13, x \approx 25.13$

Test intervals: $(-\infty, -0.13), (-0.13, 25.13), (25.13, \infty)$

Solution set: $(-0.13, 25.13)$

61.
$$\frac{1}{2.3x - 5.2} > 3.4$$
$$\frac{1}{2.3x - 5.2} - 3.4 > 0$$
$$\frac{1 - 3.4(2.3x - 5.2)}{2.3x - 5.2} > 0$$
$$\frac{-7.82x + 18.68}{2.3x - 5.2} > 0$$

Key numbers: $x \approx 2.39, x \approx 2.26$

Test intervals: $(-\infty, 2.26), (2.26, 2.39), (2.39, \infty)$

Solution set: $(2.26, 2.39)$

63. $s = -16t^2 + v_0t + s_0 = -16t^2 + 160t$

(a)
$$\begin{aligned} -16t^2 + 160t &= 0 \\ -16t(t - 10) &= 0 \\ t = 0, t &= 10 \end{aligned}$$

It will be back on the ground in 10 seconds.

(b)
$$\begin{aligned} -16t^2 + 160t &> 384 \\ -16t^2 + 160t - 384 &> 0 \\ -16(t^2 - 10t + 24) &> 0 \\ t^2 - 10t + 24 &< 0 \\ (t - 4)(t - 6) &< 0 \end{aligned}$$

Key numbers: $t = 4, t = 6$

Test intervals: $(-\infty, 4), (4, 6), (6, \infty)$

Solution set: 4 seconds $< t <$ 6 seconds

65. $R = x(75 - 0.0005x)$ and $C = 30x + 250{,}000$

$$\begin{aligned} P &= R - C \\ &= (75x - 0.0005x^2) - (30x + 250{,}000) \\ &= -0.0005x^2 + 45x - 250{,}000 \end{aligned}$$

$$\begin{aligned} P &\geq 750{,}000 \\ -0.0005x^2 + 45x - 250{,}000 &\geq 750{,}000 \\ -0.0005x^2 + 45x - 1{,}000{,}000 &\geq 0 \end{aligned}$$

Key numbers: $x = 40{,}000, x = 50{,}000$

(These were obtained by using the Quadratic Formula.)

Test intervals:

$(0, 40{,}000), (40{,}000, 50{,}000), (50{,}000, \infty)$

The solution set is $[40{,}000, 50{,}000]$ or $40{,}000 \leq x \leq 50{,}000$. The price per unit is

$$p = \frac{R}{x} = 75 - 0.0005x.$$

For $x = 40{,}000$, $p = \$55$. For $x = 50{,}000$, $p = \$50$. So, for $40{,}000 \leq x \leq 50{,}000$, $\$50.00 \leq p \leq \$55.00$.

67.
$$\begin{aligned} 4 - x^2 &\geq 0 \\ (2 + x)(2 - x) &\geq 0 \end{aligned}$$

Key numbers: $x = \pm 2$

Test intervals: $(-\infty, -2) \Rightarrow 4 - x^2 < 0$

$(-2, 2) \Rightarrow 4 - x^2 > 0$

$(2, \infty) \Rightarrow 4 - x^2 < 0$

Domain: $[-2, 2]$

69. $x^2 - 9x + 20 \geq 0$

$(x - 4)(x - 5) \geq 0$

Key numbers: $x = 4, x = 5$

Test intervals: $(-\infty, 4), (4, 5), (5, \infty)$

Interval	x-Value	Value of $(x - 4)(x - 5)$	Conclusion
$(-\infty, 4)$	0	$(-4)(-5) = 20$	Positive
$(4, 5)$	$\frac{9}{2}$	$\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{4}$	Negative
$(5, \infty)$	6	$(2)(1) = 2$	Positive

Domain: $(-\infty, 4] \cup [5, \infty)$

71. $\dfrac{x}{x^2 - 2x - 35} \geq 0$

$\dfrac{x}{(x + 5)(x - 7)} \geq 0$

Key numbers: $x = 0, x = -5, x = 7$

Test intervals: $(-\infty, -5) \Rightarrow \dfrac{x}{(x + 5)(x - 7)} < 0$

$(-5, 0) \Rightarrow \dfrac{x}{(x + 5)(x - 7)} > 0$

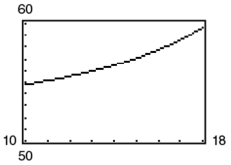
$(0, 7) \Rightarrow \dfrac{x}{(x + 5)(x - 7)} < 0$

$(7, \infty) \Rightarrow \dfrac{x}{(x + 5)(x - 7)} > 0$

Domain: $(-5, 0] \cup (7, \infty)$

73. $S = \dfrac{52.88 - 1.89t}{1 - 0.038t}, 10 \leq t \leq 18$

(a)



(b) $S = \dfrac{52.88 - 1.89t}{1 - 0.038t}$

$$57 < \frac{52.88 - 1.89t}{1 - 0.038t}$$

$$0 < \frac{52.88 - 1.89t}{1 - 0.038t} - 57$$

$$0 < \frac{-4.12 + 0.276t}{1 - 0.038t}$$

Key numbers: 14.9, 26.3

Use the domain of the model to create test intervals.

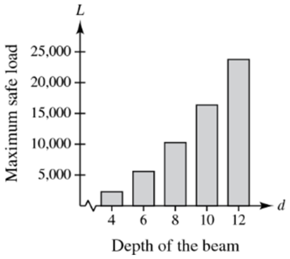
Test Interval	t-Value	Value of expression	Conclusion
$(10, 14.9)$	11	-0.76	Negative
$(14.9, 18)$	15	0.01	Positive

The mean salary was less than $57,000 for $t < 14.9$, or during the year 2014.

(c) *Sample answer:* No. For $t < 22$, the model rapidly increases then decreases.

75. (a)

d	4	6	8	10	12
Load	2223.9	5593.9	10,312	16,378	23,792



(b)

$$2000 \leq 168.5d^2 - 472.1$$

$$2472.1 \leq 168.5d^2$$

$$14.67 \leq d^2$$

$$3.83 \leq d$$

The minimum depth is 3.83 inches.

77. $2L + 2W = 100 \Rightarrow W = 50 - L$

$$LW \ge 500$$
$$L(50 - L) \ge 500$$
$$-L^2 + 50L - 500 \ge 0$$

By the Quadratic Formula you have:

Key numbers: $L = 25 \pm 5\sqrt{5}$

Test: Is $-L^2 + 50L - 500 \ge 0$?

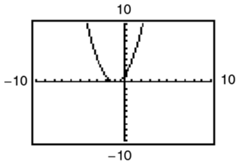
Solution set: $25 - 5\sqrt{5} \le L \le 25 + 5\sqrt{5}$

$13.8 \text{ meters} \le L \le 36.2 \text{ meters}$

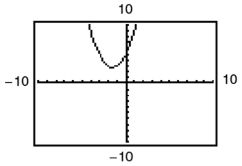
79. False.

There are four test intervals. The test intervals are $(-\infty, -3), (-3, 1), (1, 4),$ and $(4, \infty)$.

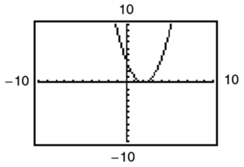
81.



For part (b), the y-values that are less than or equal to 0 occur only at $x = -1$.



For part (c), there are no y-values that are less than 0.



For part (d), the y-values that are greater than 0 occur for all values of x except 2.

83. $\frac{1}{x}$ is undefined when $x = 0$. The correct solution set is $(0, \infty)$.

85. $x^2 + bx + 9 = 0$

(a) To have at least one real solution, $b^2 - 4ac \ge 0$.

$$b^2 - 4(1)(9) \ge 0$$
$$b^2 - 36 \ge 0$$

Key numbers: $b = -6, b = 6$

Test intervals: $(-\infty, -6) \Rightarrow b^2 - 36 > 0$

$(-6, 6) \Rightarrow b^2 - 36 < 0$

$(6, \infty) \Rightarrow b^2 - 36 > 0$

Solution set: $(-\infty, -6] \cup [6, \infty)$

(b) $b^2 - 4ac \ge 0$

Key numbers: $b = -2\sqrt{ac}, b = 2\sqrt{ac}$

Similar to part (a), if $a > 0$ and $c > 0$, $b \le -2\sqrt{ac}$ or $b \ge 2\sqrt{ac}$.

87. $3x^2 + bx + 10 = 0$

(a) To have at least one real solution, $b^2 - 4ac \ge 0$.

$$b^2 - 4(3)(10) \ge 0$$
$$b^2 - 120 \ge 0$$

Key numbers: $b = -2\sqrt{30}, b = 2\sqrt{30}$

Test intervals: $(-\infty, -2\sqrt{30}) \Rightarrow b^2 - 120 > 0$

$(-2\sqrt{30}, 2\sqrt{30}) \Rightarrow b^2 - 120 < 0$

$(2\sqrt{30}, \infty) \Rightarrow b^2 - 120 > 0$

Solution set: $(-\infty, -2\sqrt{30}] \cup [2\sqrt{30}, \infty)$

(b) $b^2 - 4ac \ge 0$

Similar to part (a), if $a > 0$ and $c > 0$, $b \le -2\sqrt{ac}$ or $b \ge 2\sqrt{ac}$.

89. $\frac{5 - 7}{12 - 18} = \frac{-2}{-6} = \frac{1}{3}$

91. $\frac{3 - 3}{4 - 0} = \frac{0}{4} = 0$

93. x-intercept: $(-1, 0)$

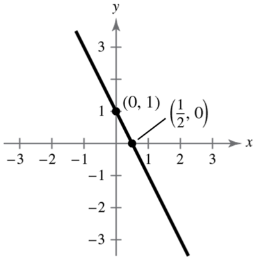
y-intercept: $(0, 1)$

95. $2x + y = 1$

$2(-x) + y = 1 \Rightarrow -2x + y = 1 \Rightarrow$ No y-axis symmetry

$2x + (-y) = 1 \Rightarrow 2x - y = 1 \Rightarrow$ No x-axis symmetry

$2(-x) + (-y) = 1 \Rightarrow -2x - y = 1 \Rightarrow$ No origin symmetry



x-intercept: $\left(\frac{1}{2}, 0\right)$

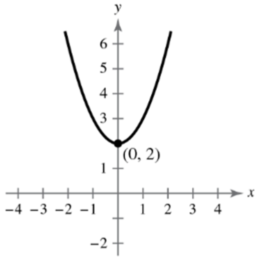
y-intercept: $(0, 1)$

97. $y = x^2 + 2$

$y = (-x)^2 + 2 \Rightarrow y = x^2 + 2 \Rightarrow y$-axis symmetry

$-y = x^2 + 2 \Rightarrow y = -x^2 - 2 \Rightarrow$ No x-axis symmetry

$-y = (-x)^2 + 2 \Rightarrow y = -x^2 - 2 \Rightarrow$ No origin symmetry



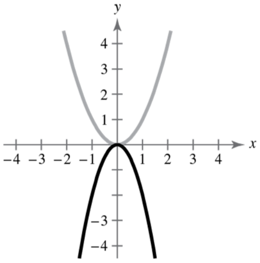
No x-intercepts

y-intercept: $(0, 2)$

Review Exercises for Chapter 2

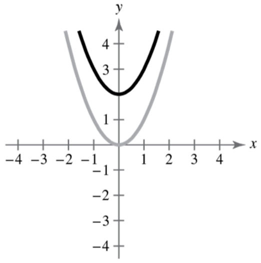
1. (a) $y = -2x^2$

Vertical stretch and a reflection in the x-axis



(b) $y = x^2 + 2$

Upward shift of two units



3. $g(x) = x^2 - 2x$

$= x^2 - 2x + 1 - 1$

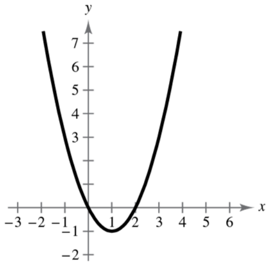
$= (x - 1)^2 - 1$

Vertex: $(1, -1)$

Axis of symmetry: $x = 1$

$0 = x^2 - 2x = x(x - 2)$

x-intercepts: $(0, 0), (2, 0)$



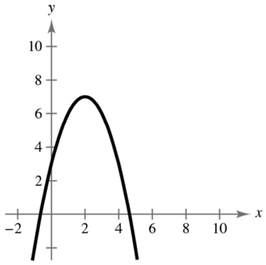
5. $h(x) = 3 + 4x - x^2$

$$= -\left(x^2 - 4x - 3\right)$$

$$= -\left(x^2 - 4x + 4 - 4 - 3\right)$$

$$= -\left[(x - 2)^2 - 7\right]$$

$$= -(x - 2)^2 + 7$$



Vertex: $(2, 7)$

Axis of symmetry: $x = 2$

$0 = 3 + 4x - x^2$

$0 = x^2 - 4x - 3$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{28}}{2} = 2 \pm \sqrt{7}$$

x-intercepts: $\left(2 \pm \sqrt{7}, 0\right)$

7. $h(x) = 4x^2 + 4x + 13$

$$= 4\left(x^2 + x\right) + 13$$

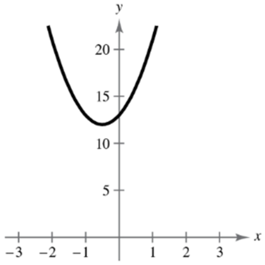
$$= 4\left(x^2 + x + \tfrac{1}{4} - \tfrac{1}{4}\right) + 13$$

$$= 4\left(x^2 + x + \tfrac{1}{4}\right) - 1 + 13$$

$$= 4\left(x + \tfrac{1}{2}\right)^2 + 12$$

Vertex: $\left(-\tfrac{1}{2}, 12\right)$

Axis of symmetry: $x = -\tfrac{1}{2}$



$$0 = 4\left(x + \tfrac{1}{2}\right)^2 + 12$$

$$\left(x + \tfrac{1}{2}\right)^2 = -3$$

No real zeros

x-intercepts: none

9. (a) $x + x + y + y = P$

$$2x + 2y = 1000$$

$$y = 500 - x$$

$A = xy$

$$= x(500 - x)$$

$$= 500x - x^2$$

(b) $A = 500x - x^2$

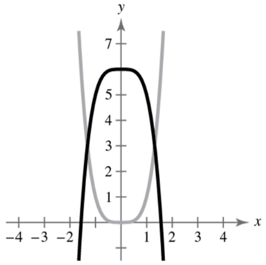
$$= -\left(x^2 - 500x + 62{,}500\right) + 62{,}500$$

$$= -(x - 250)^2 + 62{,}500$$

The maximum area occurs at the vertex when $x = 250$ and $y = 500 - 250 = 250$.

The dimensions with the maximum area are $x = 250$ meters and $y = 250$ meters.

11. $y = x^4$, $f(x) = 6 - x^4$



Transformation: Reflection in the x-axis and a vertical shift six units upward

13. $f(x) = -2x^2 - 5x + 12$

The degree is even and the leading coefficient is negative. The graph falls to the left and falls to the right.

15. $g(x) = -3x^3 - 8x^4 + x^5$

The degree is odd and the leading coefficient is positive. The graph falls to the left and rises to the right.

17. $g(x) = 2x^3 + 4x^2$

(a) The degree is odd and the leading coefficient is positive. The graph falls to the left and rises to the right.

(b) $g(x) = 2x^3 + 4x^2$

$$0 = 2x^3 + 4x^2$$

$$0 = 2x^2(x + 2)$$

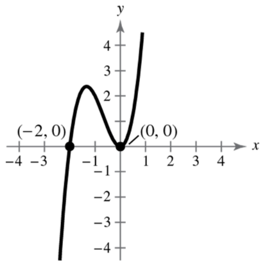
$$0 = x^2(x + 2)$$

Zeros: $x = -2, 0$

(c)

x	−3	−2	−1	0	1
$g(x)$	−18	0	2	0	6

(d)



19. $f(x) = -x^3 + x^2 - 2$

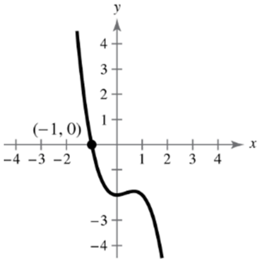
(a) The degree is odd and the leading coefficient is negative. The graph rises to the left and falls to the right.

(b) Zero: $x = -1$

(c)

x	-3	-2	-1	0	1	2
$f(x)$	34	10	0	-2	-2	-6

(d)



21. (a) $f(x) = 3x^3 - x^2 + 3$

x	-3	-2	-1	0	1	2	3
$f(x)$	-87	-25	-1	3	5	23	75

The zero is in the interval $[-1, 0]$.

(b) Zero: $x \approx -0.900$

23.

$$\begin{array}{r}
6x + 3 \\
5x - 3\,\overline{\smash{)}\,30x^2 - 3x + 8} \\
\underline{30x^2 - 18x} \\
15x + 8 \\
\underline{15x - 9} \\
17
\end{array}$$

$$\frac{30x^2 - 3x + 8}{5x - 3} = 6x + 3 + \frac{17}{5x - 3}$$

25.

$$\begin{array}{r|rrrr}
8 & 2 & -25 & 66 & 48 \\
 & & 16 & -72 & -48 \\
\hline
 & 2 & -9 & -6 & 0
\end{array}$$

$$\frac{2x^3 - 25x^2 + 66x + 48}{x - 8} = 2x^2 - 9x - 6, \quad x \neq 8$$

27. $f(x) = 2x^3 + 11x^2 - 21x - 90$; Factor: $(x + 6)$

(a)
$$\begin{array}{r|rrrr}
-6 & 2 & 11 & -21 & -90 \\
 & & -12 & 6 & 90 \\
\hline
 & 2 & -1 & -15 & 0
\end{array}$$

Yes, $(x + 6)$ is a factor of $f(x)$.

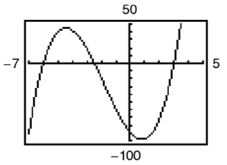
(b) $2x^2 - x - 15 = (2x + 5)(x - 3)$

The remaining factors are $(2x + 5)$ and $(x - 3)$.

(c) $f(x) = (2x + 5)(x - 3)(x + 6)$

(d) Zeros: $x = -\frac{5}{2}, 3, -6$

(e)



29. $4 + \sqrt{-9} = 4 + 3i$

31.
$$\begin{aligned}(6 - 4i) + (-9 + i) &= (6 + (-9)) + (-4i + i) \\ &= -3 - 3i\end{aligned}$$

33.
$$\begin{aligned}-3i(-2 + 5i) &= 6i - 15i^2 \\ &= 6i - 15(-1) \\ &= 15 + 6i\end{aligned}$$

35.
$$\begin{aligned}\frac{4}{1 - 2i} &= \frac{4}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} \\ &= \frac{4 + 8i}{1 - 4i^2} \\ &= \frac{4 + 8i}{5} \\ &= \frac{4}{5} + \frac{8}{5}i\end{aligned}$$

37.
$$\begin{aligned}\frac{4}{2 - 3i} + \frac{2}{1 + i} &= \frac{4}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} + \frac{2}{1 + i} \cdot \frac{1 - i}{1 - i} \\ &= \frac{8 + 12i}{4 + 9} + \frac{2 - 2i}{1 + 1} \\ &= \frac{8}{13} + \frac{12}{13}i + 1 - i \\ &= \left(\frac{8}{13} + 1\right) + \left(\frac{12}{13}i - i\right) \\ &= \frac{21}{13} - \frac{1}{13}i\end{aligned}$$

39. $x^2 - 2x + 10 = 0$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-36}}{2} \\ &= \frac{2 \pm 6i}{2} \\ &= 1 \pm 3i \end{aligned}$$

41. Since $g(x) = x^2 - 2x - 8$ is a 2nd degree polynomial function, it has two zeros.

43. $f(x) = 4x^3 - 27x^2 + 11x + 42$

Possible rational zeros: $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm1, \pm\frac{3}{2}, \pm\frac{7}{4}, \pm2, \pm3, \pm\frac{7}{2}, \pm\frac{21}{4}, \pm6, \pm7, \pm\frac{21}{2}, \pm14, \pm21, \pm42$

$$\begin{array}{r|rrrr} -1 & 4 & -27 & 11 & 42 \\ & & -4 & 31 & -42 \\ \hline & 4 & -31 & 42 & 0 \end{array}$$

$$\begin{aligned} 4x^3 - 27x^2 + 11x + 42 &= (x + 1)(4x^2 - 31x + 42) \\ &= (x + 1)(x - 6)(4x - 7) \end{aligned}$$

The zeros of $f(x)$ are $x = -1, x = \frac{7}{4},$ and $x = 6$.

45. $h(x) = -x^3 + 2x^2 - 16x + 32$

Because $-4i$ is a zero, so is $4i$.

$$\begin{array}{r|rrrr} -4i & -1 & 2 & -16 & 32 \\ & & 4i & 16 - 8i & -32 \\ \hline & -1 & 2 + 4i & -8i & 0 \end{array}$$

$$\begin{array}{r|rrr} 4i & -1 & 2 + 4i & -8i \\ & & -4i & 8i \\ \hline & -1 & 2 & 0 \end{array}$$

$h(x) = (x + 4i)(x - 4i)(-x + 2)$

Zeros: $x = \pm 4i, 2$

47. $h(x) = -2x^5 + 4x^3 - 2x^2 + 5$

$h(x)$ has three variations in sign, so h has either three or one positive real zeros.

$$\begin{aligned} h(-x) &= -2(-x)^5 + 4(-x)^3 - 2(-x)^2 + 5 \\ &= 2x^5 - 4x^3 - 2x^2 + 5 \end{aligned}$$

$h(-x)$ has two variations in sign, so h has either two or no negative real zeros.

49. Because the denominator is zero when $x + 10 = 0$, the domain of f is all real numbers except $x = -10$.

x	-11	-10.5	-10.1	-10.01	-10.001	$\to -10$
$f(x)$	33	63	303	3003	30,003	$\to \infty$

x	$-10 \leftarrow$	-9.999	-9.99	-9.9	-9.5	-9
$f(x)$	$-\infty \leftarrow$	$-29,997$	-2997	-297	-57	-27

As x approaches -10 from the left, $f(x)$ increases without bound.

As x approaches -10 from the right, $f(x)$ decreases without bound.

Vertical asymptote: $x = -10$

Horizontal asymptote: $y = 3$

51. $f(x) = \dfrac{4}{x}$

(a) Domain: all real numbers x except $x = 0$

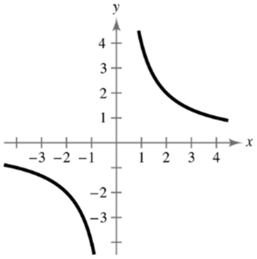
(b) No intercepts

(c) Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

(d)

x	-3	-2	-1	1	2	3
$f(x)$	$-\dfrac{4}{3}$	-2	-4	4	2	$\dfrac{4}{3}$



53. $f(x) = \dfrac{x}{x^2 - 16}$

(a) Domain: all real numbers x except $x \neq \pm 4$

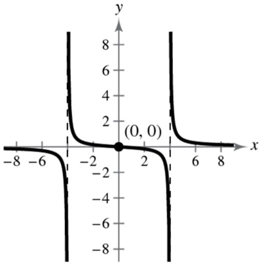
(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = \pm 4$

Horizontal asymptote: $y = 0$

(d)

x	-5	-3	-2	-1	0	1	2	3	5
$f(x)$	$-\dfrac{5}{9}$	$\dfrac{3}{7}$	$\dfrac{1}{6}$	$\dfrac{1}{15}$	0	$-\dfrac{1}{15}$	$-\dfrac{1}{6}$	$-\dfrac{3}{7}$	$\dfrac{5}{9}$



55. $f(x) = \dfrac{6x^2 - 11x + 3}{3x^2 - x} = \dfrac{(3x - 1)(2x - 3)}{x(3x - 1)} = \dfrac{2x - 3}{x}, \quad x \neq \dfrac{1}{3}$

(a) Domain: all real numbers x except $x = 0$ and $x = \dfrac{1}{3}$

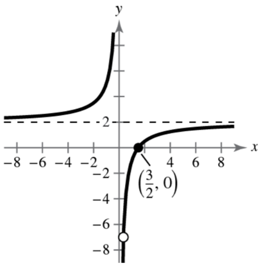
(b) x-intercept: $\left(\dfrac{3}{2}, 0\right)$

(c) Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 2$

(d)

x	-2	-1	1	2	3	4
$f(x)$	$\dfrac{7}{2}$	5	-1	$\dfrac{1}{2}$	1	$\dfrac{5}{4}$



57. $f(x) = \dfrac{2x^3}{x^2 + 1} = 2x - \dfrac{2x}{x^2 + 1}$

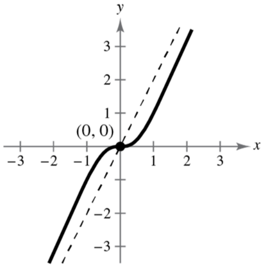
(a) Domain: all real numbers x

(b) Intercept: $(0, 0)$

(c) Slant asymptote: $y = 2x$

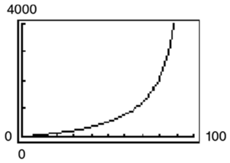
(d)

x	-2	-1	0	1	2
$f(x)$	$-\frac{16}{5}$	-1	0	1	$\frac{16}{5}$



59. $C = \dfrac{528p}{100 - p}, \; 0 \le p < 100$

(a)



(b) When $p = 25, C = \dfrac{528(25)}{100 - 25} = \$176$ million.

When $p = 50, C = \dfrac{528(50)}{100 - 50} = \$528$ million.

When $p = 75, C = \dfrac{528(75)}{100 - 75} = \$1584$ million.

(c) As $p \to 100, C \to \infty$. No, the function is undefined when $p = 100$.

61.

$$
\begin{aligned}
12x^2 + 5x &< 2\\
12x^2 + 5x - 2 &< 0\\
(4x - 1)(3x + 2) &< 0
\end{aligned}
$$

Key numbers: $x = -\frac{2}{3}, x = \frac{1}{4}$

Test intervals: $\left(-\infty, -\frac{2}{3}\right), \left(-\frac{2}{3}, \frac{1}{4}\right), \left(\frac{1}{4}, \infty\right)$

Test: Is $(4x - 1)(3x + 2) < 0$?

By testing an x-value in each test interval in the inequality, you see that the solution set is $\left(-\frac{2}{3}, \frac{1}{4}\right)$.

63.

$$
\begin{aligned}
\frac{2}{x + 1} &\le \frac{3}{x - 1}\\
\frac{2(x - 1) - 3(x + 1)}{(x + 1)(x - 1)} &\le 0\\
\frac{2x - 2 - 3x - 3}{(x + 1)(x - 1)} &\le 0\\
\frac{-(x + 5)}{(x + 1)(x - 1)} &\le 0
\end{aligned}
$$

Key numbers: $x = -5, x = \pm 1$

Test intervals: $(-\infty, -5), (-5, -1), (-1, 1), (1, \infty)$

Test: Is $\dfrac{-(x + 5)}{(x + 1)(x - 1)} \le 0$?

By testing an x-value in each test interval in the inequality, you see that the solution set is $[-5, -1) \cup (1, \infty)$.

65.

$$
\begin{aligned}
P &= \frac{1000(1 + 3t)}{5 + t}\\
2000 &\le \frac{1000(1 + 3t)}{5 + t}\\
2000(5 + t) &\le 1000(1 + 3t)\\
10{,}000 + 2000t &\le 1000 + 3000t\\
-1000t &\le -9000\\
t &\ge 9 \text{ days}
\end{aligned}
$$

67. False. The domain of $f(x) = \dfrac{1}{x^2 + 1}$ is the set of all real numbers x.

Problem Solving for Chapter 2

1. $f(x) = ax^3 + bx^2 + cx + d$

$$
\begin{array}{rl}
 & ax^2 + (ak + b)x + \left(ak^2 + bk + c\right) \\
x - k \,\big) & \overline{ax^3 + bx^2 \qquad + cx \qquad + d} \\
 & \underline{ax^3 - akx^2} \\
 & \qquad (ak + b)x^2 + cx \\
 & \qquad \underline{(ak + b)x^2 - \left(ak^2 + bk\right)x} \\
 & \qquad\qquad \left(ak^2 + bk + c\right)x + d \\
 & \qquad\qquad \underline{\left(ak^2 + bk + c\right)x - \left(ak^3 + bk^2 + ck\right)} \\
 & \qquad\qquad\qquad\qquad \left(ak^3 + bk^2 + ck + d\right)
\end{array}
$$

So, $f(x) = ax^3 + bx^2 + cx + d = (x - k)\left[ax^2 + (ak + b)x + \left(ak^2 + bx + c\right)\right] + ak^3 + bk^2 + ck + d$ and $f(x) = ak^3 + bk^2 + ck + d$. Because the remainder is $r = ak^3 + bk^2 + ck + d$, $f(k) = r$.

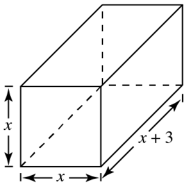
3. $V = l \cdot w \cdot h = x^2(x + 3)$

$x^2(x + 3) = 20$

$x^3 + 3x^2 - 20 = 0$

Possible rational zeros: $\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$

$$
\begin{array}{r|rrrr}
2 & 1 & 3 & 0 & -20 \\
 & & 2 & 10 & 20 \\
\hline
 & 1 & 5 & 10 & 0
\end{array}
$$



$(x - 2)\left(x^2 + 5x + 10\right) = 0$

$x = 2$ or $x = \dfrac{-5 \pm \sqrt{15}i}{2}$

Choosing the real positive value for x we have: $x = 2$ and $x + 3 = 5$.

The dimensions of the mold are 2 inches × 2 inches × 5 inches.

5. (a) $y = ax^2 + bx + c$

$(0, -4)$: $-4 = a(0)^2 + b(0) + c$

$-4 = c$

$(4, 0)$: $0 = a(4)^2 + b(4) - 4$

$0 = 16a + 4b - 4 = 4(4a + b - 1)$

$0 = 4a + b - 1$ or $b = 1 - 4a$

$(1, 0)$: $0 = a(1)^2 + b(1) - 4$

$4 = a + b$

$4 = a + (1 - 4a)$

$4 = 1 - 3a$

$3 = -3a$

$a = -1$

$b = 1 - 4(-1) = 5$

$y = -x^2 + 5x - 4$

(b) Enter the data points $(0, -4)$, $(1, 0)$, $(2, 2)$, $(4, 0)$, $(6, -10)$ and use the regression feature to obtain $y = -x^2 + 5x - 4$.

7. $f(x) = (x - k)q(x) + r$

(a) Cubic, passes through $(2, 5)$, rises to the right

One possibility:

$f(x) = (x - 2)x^2 + 5$

$= x^3 - 2x^2 + 5$

(b) Cubic, passes through $(-3, 1)$, falls to the right

One possibility:

$f(x) = -(x + 3)x^2 + 1$

$= -x^3 - 3x^2 + 1$

9. $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$

Since a and b are real numbers, $a^2 + b^2$ is also a real number.

11. $f(x) = \dfrac{ax}{(x - b)^2}$

(a) $b \neq 0 \Rightarrow x = b$ is a vertical asymptote.

a causes a vertical stretch if $|a| > 1$ and a vertical shrink if $0 < |a| < 1$. For $|a| > 1$, the graph becomes wider as $|a|$ increases. When a is negative, the graph is reflected about the x-axis.

(b) $a \neq 0$. Varying the value of b varies the vertical asymptote of the graph of f. For $b > 0$, the graph is translated to the right. For $b < 0$, the graph is reflected in the x-axis and is translated to the left.

13. Because complex zeros always occur in conjugate pairs, and a cubic function has three zeros and not four, a cubic function with real coefficients cannot have two real zeros and one complex zero.

Practice Test for Chapter 2

1. Sketch the graph of $f(x) = x^2 - 6x + 5$ and identify the vertex and the intercepts.

2. Find the number of units x that produce a minimum cost C if
   $C = 0.01x^2 - 90x + 15{,}000$.

3. Find the quadratic function that has a maximum at $(1, 7)$ and passes through the point $(2, 5)$.

4. Find two quadratic functions that have x-intercepts $(2, 0)$ and $\left(\frac{4}{3}, 0\right)$.

5. Use the leading coefficient test to determine the right and left end behavior of the graph of the polynomial function $f(x) = -3x^5 + 2x^3 - 17$.

6. Find all the real zeros of $f(x) = x^5 - 5x^3 + 4x$.

7. Find a polynomial function with 0, 3, and -2 as zeros.

8. Sketch $f(x) = x^3 - 12x$.

9. Divide $3x^4 - 7x^2 + 2x - 10$ by $x - 3$ using long division.

10. Divide $x^3 - 11$ by $x^2 + 2x - 1$.

11. Use synthetic division to divide $3x^5 + 13x^4 + 12x - 1$ by $x + 5$.

12. Use synthetic division to find $f(-6)$ given $f(x) = 7x^3 + 40x^2 - 12x + 15$.

13. Find the real zeros of $f(x) = x^3 - 19x - 30$.

14. Find the real zeros of $f(x) = x^4 + x^3 - 8x^2 - 9x - 9$.

15. List all possible rational zeros of the function $f(x) = 6x^3 - 5x^2 + 4x - 15$.

16. Find the rational zeros of the polynomial $f(x) = x^3 - \frac{20}{3}x^2 + 9x - \frac{10}{3}$.

17. Write $f(x) = x^4 + x^3 + 5x - 10$ as a product of linear factors.

18. Find a polynomial with real coefficients that has $2, 3 + i,$ and $3 - 2i$ as zeros.

19. Use synthetic division to show that $3i$ is a zero of $f(x) = x^3 + 4x^2 + 9x + 36$.

20. Sketch the graph of $f(x) = \dfrac{x - 1}{2x}$ and label all intercepts and asymptotes.

21. Find all the asymptotes of $f(x) = \dfrac{8x^2 - 9}{x^2 + 1}$.

22. Find all the asymptotes of $f(x) = \dfrac{4x^2 - 2x + 7}{x - 1}$.

23. Given $z_1 = 4 - 3i$ and $z_2 = -2 + i$, find the following:

(a) $z_1 - z_2$

(b) $z_1 z_2$

(c) z_1/z_2

24. Solve the inequality: $x^2 - 49 \leq 0$

25. Solve the inequality: $\dfrac{x + 3}{x - 7} \geq 0$

CHAPTER 3
Exponential and Logarithmic Functions

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CHAPTER 3
Exponential and Logarithmic Functions

Section 3.1 Exponential Functions and Their Graphs

1. transcendental

3. The graph of $f(x+1)$ is a horizontal shift one unit to the left of $f(x) = 5^x$.

5. $f(1.4) = (0.9)^{1.4} \approx 0.863$

7. $f(-1.5) = 5000(2^{-1.5})$
≈ 1767.767

9. $f(x) = 2^{1-x}$

Decreasing
Asymptote: $y = 0$
Intercept: $(0, 2)$
Matches graph (d).

10. $f(x) = 2^x + 1$

Increasing
Asymptote: $y = 1$
Intercept: $(0, 2)$
Matches graph (c).

11. $f(x) = -2^x$

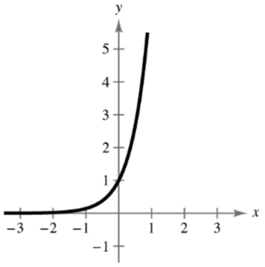
Decreasing
Asymptote: $y = 0$
Intercept: $(0, -1)$
Matches graph (a).

12. $f(x) = 2^{x-2}$

Increasing
Asymptote: $y = 0$
Intercept: $(0, \frac{1}{4})$
Matches graph (b).

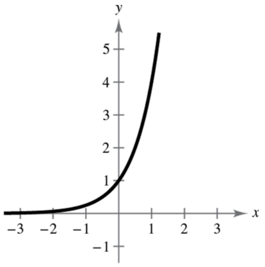
13. $f(x) = 7^x$

x	-2	-1	0	1	2
$f(x)$	0.020	0.143	1	7	49



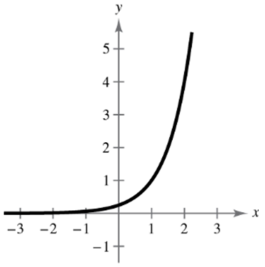
15. $f(x) = \left(\frac{1}{4}\right)^{-x}$

x	-2	-1	0	1	2
$f(x)$	0.063	0.25	1	4	16



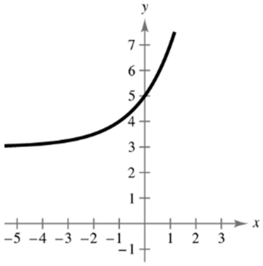
17. $f(x) = 4^{x-1}$

x	-2	-1	0	1	2
$f(x)$	0.016	0.063	0.25	1	4



19. $f(x) = 2^{x+1} + 3$

x	-3	-2	-1	0	1
$f(x)$	3.25	3.5	4	5	7



21.

$$\begin{aligned} 3^{x+1} &= 27 \\ 3^{x+1} &= 3^3 \\ x + 1 &= 3 \\ x &= 2 \end{aligned}$$

23.

$$\begin{aligned} \left(\tfrac{1}{2}\right)^x &= 32 \\ \left(\tfrac{1}{2}\right)^x &= \left(\tfrac{1}{2}\right)^{-5} \\ x &= -5 \end{aligned}$$

25. $f(x) = 3^x$, $g(x) = 3^x + 1$

Because $g(x) = f(x) + 1$, the graph of g can be obtained by shifting the graph of f one unit upward.

27. $f(x) = 10^x$, $g(x) = 10^{-x+3}$

Because $g(x) = f(-x + 3)$, the graph of g can be obtained by reflecting the graph of f in the y-axis and shifting f three units to the right. (**Note:** This is equivalent to shifting f three units to the left and then reflecting the graph in the y-axis.)

29.

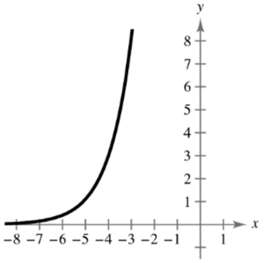
$$\begin{aligned} f(x) &= e^x \\ f(1.9) &= e^{1.9} \approx 6.686 \end{aligned}$$

31. $f(6) = 5000e^{0.06(6)} \approx 7166.647$

33. $f(x) = 3e^{x+4}$

x	-8	-7	-6	-5	-4
$f(x)$	0.055	0.149	0.406	1.104	3

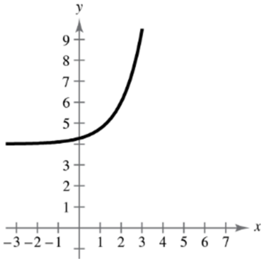
Asymptote: $y = 0$



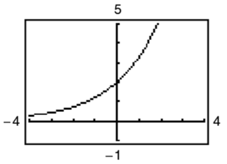
35. $f(x) = 2e^{x-2} + 4$

x	-2	-1	0	1	2
$f(x)$	4.037	4.100	4.271	4.736	6

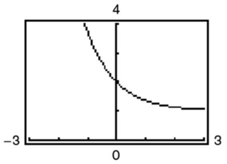
Asymptote: $y = 4$



37. $s(t) = 2e^{0.5t}$



39. $g(x) = 1 + e^{-x}$



41.

$$\begin{aligned} e^{3x+2} &= e^3 \\ 3x + 2 &= 3 \\ 3x &= 1 \\ x &= \tfrac{1}{3} \end{aligned}$$

43.
$$\begin{aligned} e^{x^2-3} &= e^{2x} \\ x^2 - 3 &= 2x \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x = 3 \quad \text{or} \quad x &= -1 \end{aligned}$$

45. $P = \$1500, r = 2\%, t = 10$ years

Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt} = 1500\left(1 + \frac{0.02}{n}\right)^{10n}$

Compounded continuously: $A = Pe^{rt} = 1500e^{0.02(10)}$

n	1	2	4	12	365	Continuous
A	\$1828.49	\$1830.29	\$1831.19	\$1831.80	\$1832.09	\$1832.10

47. $P = \$2500, r = 4\%, t = 20$ years

Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.04}{n}\right)^{20n}$

Compounded continuously: $A = Pe^{rt} = 2500e^{0.04(20)}$

n	1	2	4	12	365	Continuous
A	\$5477.81	\$5520.10	\$5541.79	\$5556.46	\$5563.61	\$5563.85

49. $A = Pe^{rt} = 12{,}000e^{0.04t}$

t	10	20	30	40	50
A	\$17,901.90	\$26,706.49	\$39,841.40	\$59,436.39	\$88,668.67

51. $A = Pe^{rt} = 12{,}000e^{0.065t}$

t	10	20	30	40	50
A	\$22,986.49	\$44,031.56	\$84,344.25	\$161,564.86	\$309,484.08

53. $A = 30{,}000e^{(0.05)(25)} \approx \$104{,}710.29$

55. $C(t) = 29.88(1.04)^t$

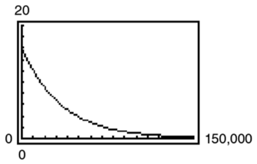
Ten years from today, $t = 10$: $C(10) = 29.88(1.04)^{10} \approx \$44.23$

57. $Q = 16\left(\frac{1}{2}\right)^{t/24{,}100}$

(a) $Q(0) = 16$ grams

(b) $Q(75{,}000) \approx 1.85$ grams

(c)



59. (a) $V(t) = 52{,}490\left(\frac{7}{8}\right)^t$

(b) $V(4) = 52{,}490\left(\frac{7}{8}\right)^4 \approx 30{,}769$ infections

61. True. The line $y = -2$ is a horizontal asymptote for the graph of $f(x) = 10^x - 2$. As $x \to -\infty$, $f(x) \to -2$ but never reaches -2.

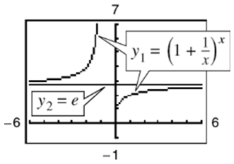
63. $$\begin{aligned} f(x) &= 3^{x-2} \\ &= 3^x 3^{-2} \\ &= 3^x\left(\frac{1}{3^2}\right) \\ &= \frac{1}{9}\left(3^x\right) \\ &= h(x) \end{aligned}$$

So, $f(x) \neq g(x)$, but $f(x) = h(x)$.

65. $$\begin{array}{ll} f(x) = 16\left(4^{-x}\right) \quad \text{and} & f(x) = 16\left(4^{-x}\right) \\ \quad\;\; = 4^2\left(4^{-x}\right) & \quad\;\; = 16\left(2^2\right)^{-x} \\ \quad\;\; = 4^{2-x} & \quad\;\; = 16\left(2^{-2x}\right) \\ \quad\;\; = \left(\frac{1}{4}\right)^{-(2-x)} & \quad\;\; = h(x) \\ \quad\;\; = \left(\frac{1}{4}\right)^{x-2} & \\ \quad\;\; = g(x) & \end{array}$$

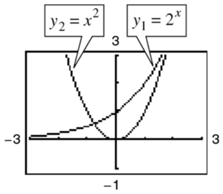
So, $f(x) = g(x) = h(x)$.

67.



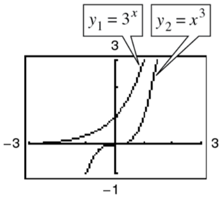
As x increases, the graph of y_1 approaches e, which is y_2.

69. (a)



At $x = 2$, both functions have a value of 4. The function y_1 increases for all values of x. The function y_2 is symmetric with respect to the y-axis.

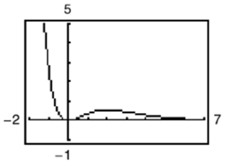
(b)



Both functions are increasing for all values of x. For $x > 0$, both functions have a similar shape. The function y_2 is symmetric with respect to the origin.

In both viewing windows, the constant raised to a variable power increases more rapidly than the variable raised to a constant power.

71. (a) $f(x) = x^2e^{-x}$



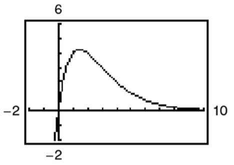
Decreasing: $(-\infty, 0), (2, \infty)$

Increasing: $(0, 2)$

Relative maximum: $\left(2, 4e^{-2}\right)$

Relative minimum: $(0, 0)$

(b) $g(x) = x2^{3-x}$



Decreasing: $(1.44, \infty)$

Increasing: $(-\infty, 1.44)$

Relative maximum: $(1.44, 4.25)$

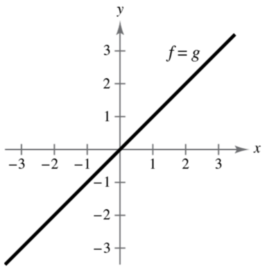
73. The functions (c) $h(x) = 3^x$ and (d) $k(x) = 2^{-x}$ are exponential.

75. $f(x) = x$, $g(x) = x$

(a) $f(g(x)) = f(x) = x$

$g(f(x)) = g(x) = x$

(b)

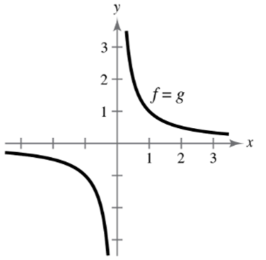


77. $f(x) = \frac{1}{x},\ g(x) = \frac{1}{x}$

(a) $f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = 1 \div \frac{1}{x} = 1 \cdot \frac{x}{1} = x$

$g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{1/x} = 1 \div \frac{1}{x} = 1 \cdot \frac{x}{1} = x$

(b)



79. $f(x) = \frac{2x+1}{x+1},\ g(x) = \frac{1-x}{x-2}$

(a) $$f(g(x)) = f\left(\frac{1-x}{x-2}\right) = \frac{2\left(\frac{1-x}{x-2}\right)+1}{\left(\frac{1-x}{x-2}\right)+1}$$

$$= \frac{2(1-x)+(x-2)}{(1-x)+(x-2)}$$

$$= \frac{2-2x+x-2}{1-x+x-2}$$

$$= \frac{-x}{-1} = x$$

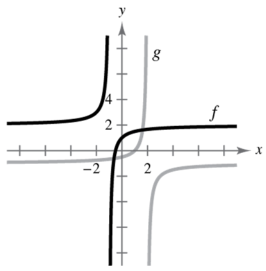
$$g(f(x)) = g\left(\frac{2x+1}{x+1}\right) = \frac{1-\left(\frac{2x+1}{x+1}\right)}{\left(\frac{2x+1}{x+1}-2\right)}$$

$$= \frac{(x+1)-(2x+1)}{(2x+1)-2(x+1)}$$

$$= \frac{x+1-2x-1}{2x+1-2x-2}$$

$$= \frac{-x}{-1} = x$$

(b)



81. $f(x) = |x+2|$

domain of $f: x \ge -2$, range of $f: y \ge 0$

$$f(x) = |x+2|$$
$$y = |x+2|$$
$$x = y+2$$
$$x-2 = y$$

So, $f^{-1}(x) = x - 2$.

domain of $f^{-1}: x \ge 0$, range of $f^{-1}: y \ge -2$

83. $f(x) = (x+6)^2$

domain of $f: x \ge -6$, range of $f: y \ge 0$

$$f(x) = (x+6)^2$$
$$y = (x+6)^2$$
$$x = (y+6)^2$$
$$\sqrt{x} = y+6$$
$$\sqrt{x} - 6 = y$$

So, $f^{-1}(x) = \sqrt{x} - 6$.

domain of $f^{-1}: x \ge 0$, range of $f^{-1}: y \ge -6$

85. $f(x) = -2x^2 + 5$

domain of $f: x \ge 0$, range of $f: y \le 5$

$$f(x) = -2x^2 + 5$$
$$y = -2x^2 + 5$$
$$x = -2y^2 + 5$$
$$x - 5 = -2y^2$$
$$5 - x = 2y^2$$
$$\sqrt{\frac{5-x}{2}} = y$$
$$\frac{\sqrt{5-x}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = y$$
$$\frac{\sqrt{2(5-x)}}{2} = y$$

So, $f^{-1}(x) = \frac{\sqrt{-2(x-5)}}{2}$.

domain of $f^{-1}(x): x \le 5$, range of $f^{-1}(x): y \ge 0$

87. $f(x) = |x - 4| + 1$

domain of $f: x \geq 4$, range of $f: y \geq 1$

$$\begin{aligned} f(x) &= |x - 4| + 1 \\ y &= x - 3 \\ x &= y - 3 \\ x + 3 &= y \end{aligned}$$

So, $f^{-1}(x) = x + 3$.

domain of $f^{-1}: x \geq 1$, range of $f^{-1}: y \geq 4$

Section 3.2 Logarithmic Functions and Their Graphs

1. logarithmic

3. natural; e

5. (a) $\log_a a^x = x \log_a a = x$

(b) $b \log_b y^2 = y^2$

7. $\log_4 16 = 2 \Rightarrow 4^2 = 16$

9. $\log_9 \frac{1}{81} = -2 \Rightarrow 9^{-2} = \frac{1}{81}$

11. $5^3 = 125 \Rightarrow \log_5 125 = 3$

13. $4^{-3} = \frac{1}{64} \Rightarrow \log_4 \frac{1}{64} = -3$

15. $f(x) = \log_2 x$

$f(64) = \log_2 64 = 6$ because $2^6 = 64$

17. $f(x) = \log x$

$f(10) = \log 10 = 1$ because $10^1 = 10$

19. $g(x) = \log_b x$

$g(b^{-3}) = \log_b \sqrt{b} = \log_b b^{1/2} = \frac{1}{2}$

21. $f(x) = \log x$

$f(12.5) = \log 12.5 \approx 1.097$

23. $f(x) = \log x$

$f\left(\frac{7}{8}\right) = \log\left(\frac{7}{8}\right) \approx -0.058$

25. $\log_8 8 = 1$ because $8^1 = 8$

27. $\log_\pi \pi^2 = 2$ because $\pi^2 = \pi^2$

29. $\log_5(x + 1) = \log_5 6$

$$\begin{aligned} x + 1 &= 6 \\ x &= 5 \end{aligned}$$

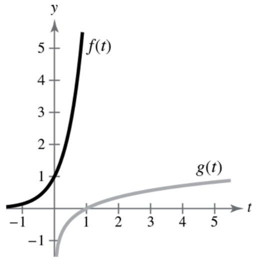
31. $\log 11 = \log(x^2 + 7)$

$$\begin{aligned} 11 &= x^2 + 7 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

33.

t	-2	-1	0	1	2
$f(t) = 7^t$	$\frac{1}{49}$	$\frac{1}{7}$	1	7	49

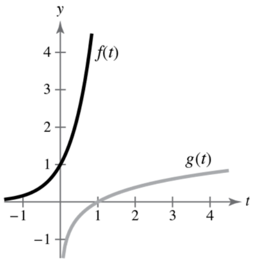
t	$\frac{1}{49}$	$\frac{1}{7}$	1	7	49
$g(t) = \log_7 t$	-2	-1	0	1	2



35.

t	-2	-1	0	1	2
$f(t) = 6^t$	$\frac{1}{36}$	$\frac{1}{6}$	1	6	36

t	$\frac{1}{36}$	$\frac{1}{6}$	1	6	36
$g(t) = \log_6 t$	-2	-1	0	1	2



37. $f(x) = \log_3 x + 2$

Asymptote: $x = 0$

Point on graph: $(1, 2)$

Matches graph (a).

The graph of $f(x)$ is obtained from $g(x)$ by shifting the graph two units upward.

38. $f(x) = \log_3(x - 1)$

Asymptote: $x = 1$

Point on graph: $(2, 0)$

Matches graph (d).

$f(x)$ shifts $g(x)$ one unit to the right.

39. $f(x) = \log_3(1 - x) = \log_3\left[-(x - 1)\right]$

Asymptote: $x = 1$

Point on graph: $(0, 0)$

Matches graph (b).

The graph of $f(x)$ is obtained by reflecting the graph of $g(x)$ in the y-axis and shifting the graph one unit to the right.

40. $f(x) = -\log_3 x$

Asymptote: $x = 0$

Point on graph: $(1, 0)$

Matches graph (c).

$f(x)$ reflects $g(x)$ in the x-axis.

41. $f(x) = \log_4 x$

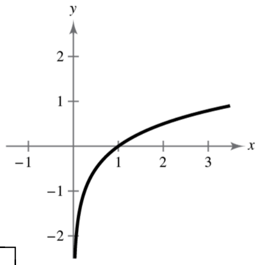
Domain: $(0, \infty)$

x-intercept: $(1, 0)$

Vertical asymptote: $x = 0$

$y = \log_4 x \Rightarrow 4^y = x$

x	$\frac{1}{4}$	1	4	2
$f(x)$	-1	0	1	$\frac{1}{2}$



43. $y = \log_3 x + 1$

Domain: $(0, \infty)$

x-intercept:

$$\begin{aligned} \log_3 x + 1 &= 0 \\ \log_3 x &= -1 \\ 3^{-1} &= x \\ \tfrac{1}{3} &= x \end{aligned}$$

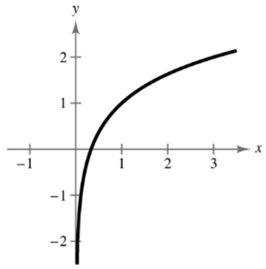
The x-intercept is $\left(\frac{1}{3}, 0\right)$.

Vertical asymptote: $x = 0$

$y = \log_3 x + 1$

$\log_3 x = y - 1 \Rightarrow 3^{y-1} = x$

x	$\frac{1}{9}$	$\frac{1}{3}$	0	3	9
y	-1	0	1	2	3



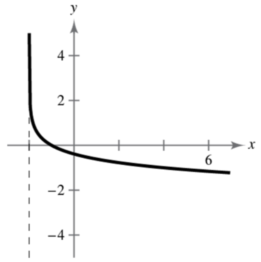
45. $f(x) = -\log_6(x + 2)$

Domain: $x + 2 > 0 \Rightarrow x > -2$

The domain is $(-2, \infty)$.

x-intercept:

$$\begin{aligned} 0 &= -\log_6(x + 2) \\ 0 &= \log_6(x + 2) \\ 6^0 &= x + 2 \\ 1 &= x + 2 \\ -1 &= x \end{aligned}$$



The x-intercept is $(-1, 0)$.

Vertical asymptote: $x + 2 = 0 \Rightarrow x = -2$

$$\begin{aligned} y &= -\log_6(x + 2) \\ -y &= \log_6(x + 2) \\ 6^{-y} - 2 &= x \end{aligned}$$

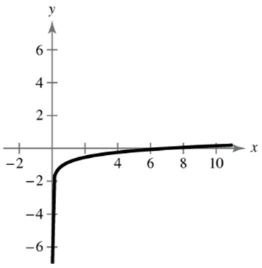
x	4	-1	$-1\frac{5}{6}$	$-1\frac{35}{36}$
$f(x)$	-1	0	1	2

47. $y = \log\left(\frac{x}{7}\right)$

Domain: $\frac{x}{7} > 0 \Rightarrow x > 0$

The domain is $(0, \infty)$.

x-intercept:
$$\begin{aligned}\log\left(\frac{x}{7}\right) &= 0\\ \frac{x}{7} &= 10^0\\ \frac{x}{7} &= 1\\ x &= 7\end{aligned}$$



The x-intercept is $(7, 0)$.

Vertical asymptote: $\frac{x}{7} = 0 \Rightarrow x = 0$

The vertical asymptote is the y-axis.

x	1	2	3	4	5
y	-0.85	-0.54	-0.37	-0.24	-0.15

x	6	7	8
y	-0.069	0	0.06

49. $\ln \frac{1}{2} = -0.693\ldots \Rightarrow e^{-0.693\ldots} = \frac{1}{2}$

51. $\ln 250 = 5.521\ldots \Rightarrow e^{5.521\ldots} = 250$

53. $e^2 = 7.3890\ldots \Rightarrow \ln 7.3890\ldots = 2$

55. $e^{-4x} = \frac{1}{2} \Rightarrow \ln \frac{1}{2} = -4x$

57. $f(x) = 8 \ln x$

$f(18) = 8 \ln(18) \approx 23.123$

59. $g(x) = 8 \ln x$

$g\left(\sqrt{5}\right) = 8 \ln \sqrt{5} \approx 6.438$

61. $e^{\ln 4} = 4$

63. $2.5 \ln 1 = 2.5(0) = 0$

65. $\ln e^{\ln e} = \ln e^1 = 1$

67. $f(x) = \ln(x - 4)$

Domain: $x - 4 > 0 \Rightarrow x > 4$

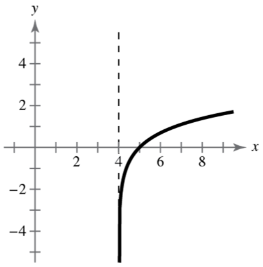
The domain is $(4, \infty)$.

x-intercept:
$$\begin{aligned}0 &= \ln(x - 4)\\ e^0 &= x - 4\\ 5 &= x\end{aligned}$$

The x-intercept is $(5, 0)$.

Vertical asymptote: $x - 4 = 0 \Rightarrow x = 4$

x	4.5	5	6	7
$f(x)$	-0.69	0	0.69	1.10



69. $g(x) = \ln(-x)$

Domain: $-x > 0 \Rightarrow x < 0$

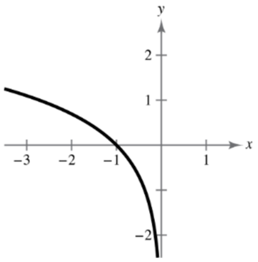
The domain is $(-\infty, 0)$.

x-intercept:
$$\begin{aligned}0 &= \ln(-x)\\ e^0 &= -x\\ -1 &= x\end{aligned}$$

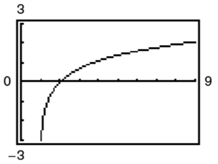
The x-intercept is $(-1, 0)$.

Vertical asymptote: $-x = 0 \Rightarrow x = 0$

x	-0.5	-1	-2	-3
$g(x)$	-0.69	0	0.69	1.10



71. $f(x) = \ln(x - 1)$



73. $\ln(x + 4) = \ln 12$

$$x + 4 = 12$$
$$x = 8$$

75.
$$\ln(x^2 - x) = \ln 6$$
$$x^2 - x = 6$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = -2 \text{ or } x = 3$$

77. $t = 16.708 \ln\left(\dfrac{x}{x - 750}\right)$

(a) When $x = 897.72$: $t = 16.708 \ln\left(\dfrac{897.72}{897.72 - 750}\right)$

≈ 30 years

When $x = 1659.24$:

$$t = 16.708 \ln\left(\frac{1659.24}{1659.24 - 750}\right)$$

≈ 10 years

(b) If $x = 897.72$ for 30 years. Then the total amount paid is $(897.72)(30)(12) \approx \$323{,}179$ and the total interest is $323{,}179 - 150{,}000 = \$173{,}179$.

If $x = 1659.24$ for 10 years, then the total amount paid is $(1659.24)(10)(12) \approx \$199{,}109$ and the total interest is $199{,}109 - 150{,}000 = \$49{,}109$.

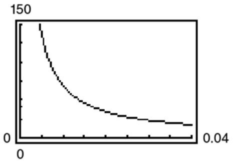
(c) The vertical asymptote is $x = 750$. The closer the payment is to \$750 per month, the longer the length of the mortgage will be. Also, the monthly payment must be greater than \$750.00.

79. $t = \dfrac{\ln 2}{r}$

(a)

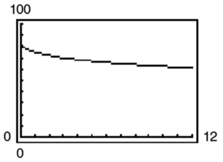
r	0.005	0.010	0.015	0.020	0.025	0.030
t	138.6	69.3	46.2	34.7	27.7	23.1

(b)



81. $f(t) = 80 - 17\log(t + 1), 0 \le t \le 12$

(a)

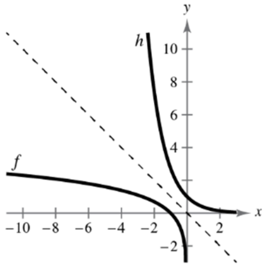


(b) $f(0) = 80 - 17\log 1 = 80.0$

(c) $f(4) = 80 - 17\log 5 \approx 68.1$

(d) $f(10) = 80 - 17\log 11 \approx 62.3$

83.



True. The graph of $f(x) = \ln(-x)$ is a reflection of $h(x) = e^{-x}$ in the line $y = -x$.

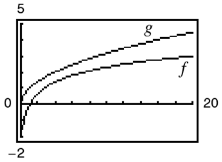
85. $f(x) = \log_b x$ contains the point $\left(\dfrac{1}{81}, 2\right)$. Substitute these values in the equation for f and solve for b.

$$2 = \log_b\left(\frac{1}{81}\right)$$
$$b^2 = \frac{1}{81}$$
$$b = \frac{1}{9}$$

87. The ordered pairs (1, 0), (2, 1), and (8, 3) model the logarithmic function $y = \log_2 x$.

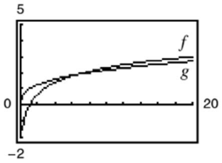
So, y is a logarithmic function of x.

89. (a)

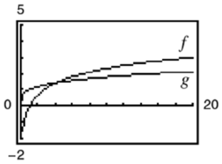


(b) Using the graph from part (a), $g(x) = x^{1/2}$ is increasing at a greater rate than $f(x) = \ln x$ as x approaches infinity.

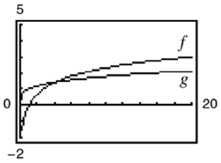
(c)



When $n = 3$, $f(x) = \ln x$ is increasing at a greater rate than $g(x) = x^{1/3}$.



When $n = 4$, $f(x) = \ln x$ is increasing at a greater rate than $g(x) = x^{1/4}$.



When $n = 5$, $f(x) = \ln x$ is increasing at a greater rate than $g(x) = x^{1/5}$.

As n increases, $n = 2, 3, 4, \ldots$, the rate at which $y = x^{1/n}$ increases decrease.

91. $(x^{-1}y^2)(x^4y^3) = x^{-1+4}y^{2+3} = x^3y^5, x \neq 0$

93. $\dfrac{m^0n^2}{m^{-3}n^3} = m^3n^{2-3} = \dfrac{m^3}{n},\ m \neq 0$

95.

$$\begin{array}{r}
5x + 3 \\
x - 4\overline{)5x^2 - 17x - 12} \\
\underline{5x^2 - 20x} \\
3x - 12 \\
\underline{3x - 12} \\
0
\end{array}$$

$$\frac{5x^2 - 17x - 12}{x - 4} = 5x + 3, x \neq 4$$

97.

$$\begin{array}{r}
-x^3 - 6x^2 - 36x - 36 \\
x - 6\overline{)\ -x^4 + 180x} \\
\underline{-x^4 + 6x^3} \\
-6x^3 + 180x \\
\underline{-6x^3 + 36x^2} \\
-36x^2 + 180x \\
\underline{-36x^2\quad 216x} \\
-36x \\
\underline{-36x + 216} \\
-216
\end{array}$$

$$\frac{-x^4 + 180x}{x - 6} = -x^3 - 6x^2 - 36x - 36 - \frac{216}{x - 6}$$

99. $f(x) = x - 3,\ g(x) = 4x + 1$

(a) $f(g(x)) = f(4x + 1) = (4x + 1) - 3 = 4x - 2$

(b) $g(f(x)) = g(x - 3) = 4(x - 3) + 1 = 4x - 12 + 1 = 4x - 11$

(c) $g(g(x)) = g(4x + 1) = 4(4x + 1) + 1 = 16x + 4 + 1 = 16x + 5$

101. $f(x) = \sqrt{x^2 - 9},\ g(x) = \frac{-1}{x}$

(a) $f(g(x)) = f\left(-\frac{1}{x}\right) = \sqrt{\left(-\frac{1}{x}\right)^2 - 9} = \sqrt{\frac{1}{x^2} - 9}$

(b) $g(f(x)) = g\left(\sqrt{x^2 - 9}\right) = -\frac{1}{\sqrt{x^2 - 9}}$

(c) $g(g(x)) = g\left(\frac{-1}{x}\right) = \frac{-1}{\left(\frac{-1}{x}\right)} = x,\ x \neq 0$

Section 3.3 Properties of Logarithms

1. change-of-base

3. Using the change-of-base formula:

$$\log_3 24 = \frac{\ln 24}{\ln 3}$$

5. (a) $\log_5 16 = \frac{\log 16}{\log 5}$

(b) $\log_5 16 = \frac{\ln 16}{\ln 5}$

7. (a) $\log_x \frac{3}{10} = \frac{\log(3/10)}{\log x}$

(b) $\log_x \frac{3}{10} = \frac{\ln(3/10)}{\ln x}$

9. $\log_3 17 = \frac{\log 17}{\log 3} = \frac{\ln 17}{\ln 3} \approx 2.579$

10. $\log_{0.4} 12 = \frac{\log 12}{\log 0.4} = \frac{\ln 12}{\ln 0.4} \approx -2.712$

11. $\log_\pi 0.5 = \frac{\log 0.5}{\log \pi} = \frac{\ln 0.5}{\ln \pi} \approx -0.606$

13.
$$\begin{aligned} \log_3 35 &= \log_3 (5 \cdot 7) \\ &= \log_3 5 + \log_3 7 \end{aligned}$$

15.
$$\begin{aligned} \log_3\left(\tfrac{7}{25}\right) &= \log_3 7 - \log_3 25 \\ &= \log_3 7 - \log_3 5^2 \\ &= \log_3 7 - 2\log_3 5 \end{aligned}$$

17.
$$\begin{aligned} \log_3\left(\tfrac{21}{5}\right) &= \log_3 21 - \log_3 5 \\ &= \log_3 (3 \cdot 7) - \log_3 5 \\ &= \log_3 3 + \log_3 7 - \log_3 5 \\ &= 1 + \log_3 7 - \log_3 5 \end{aligned}$$

19. $\log_3 9 = 2\log_3 3 = 2$

21.
$$\begin{aligned} \log_6 \sqrt[3]{\tfrac{1}{6}} &= \log_6 \left(\tfrac{1}{6}\right)^{1/3} \\ &= \tfrac{1}{3}\log_6\left(\tfrac{1}{6}\right) \\ &= \tfrac{1}{3}\log_6 6^{-1} \\ &= \tfrac{1}{3}(-1) \\ &= -\tfrac{1}{3} \end{aligned}$$

23. $\log_2(-2)$ is undefined. -2 is not in the domain of $\log_2 x$.

25.
$$\begin{aligned} \ln \sqrt[4]{e^3} &= \ln e^{3/4} \\ &= \tfrac{3}{4}\ln e \\ &= \tfrac{3}{4}(1) \\ &= \tfrac{3}{4} \end{aligned}$$

27. $\ln e^2 + \ln e^5 = 2 + 5 = 7$

29.
$$\begin{aligned} \log_5 75 - \log_5 3 &= \log_5 \tfrac{75}{3} \\ &= \log_5 25 \\ &= \log_5 5^2 \\ &= 2\log_5 5 \\ &= 2 \end{aligned}$$

31.
$$\begin{aligned} \log_4 8 &= \log_4 (4 \cdot 2) \\ &= \log_4 4 + \log_4 2 \\ &= \log_4 4 + \log_4 4^{1/2} \\ &= 1 + \tfrac{1}{2} \\ &= \tfrac{3}{2} \end{aligned}$$

33. $$\begin{aligned}\log_b 10 &= \log_b 2\cdot 5\\ &= \log_b 2 + \log_b 5\\ &\approx 0.3562 + 0.8271\\ &= 1.1833\end{aligned}$$

35. $$\begin{aligned}\log_b 0.04 &= \log_b \tfrac{4}{100} = \log_b \tfrac{1}{25}\\ &= \log_b 1 - \log_b 25\\ &= \log_b 1 - \log_b 5^2\\ &= 0 - 2\log_b 5\\ &\approx -2(0.8271)\\ &= -1.6542\end{aligned}$$

37. $$\begin{aligned}\log_b 45 &= \log_b 9\cdot 5\\ &= \log_b 9 + \log_b 5\\ &= \log_b 3^2 + \log_b 5\\ &= 2\log_b 3 + \log_b 5\\ &\approx 2(0.5646) + 0.8271\\ &= 1.9563\end{aligned}$$

39. $$\begin{aligned}\log_b (2b)^{-2} &= -2\log_b 2b\\ &= -2(\log_b 2 + \log_b b)\\ &\approx -2(0.3562 + 1)\\ &= -2.7124\end{aligned}$$

41. $\ln 7x = \ln 7 + \ln x$

43. $\log_8 x^4 = 4\log_8 x$

45. $$\begin{aligned}\log_5 \frac{5}{x} &= \log_5 5 - \log_5 x\\ &= 1 - \log_5 x\end{aligned}$$

47. $\ln\sqrt{z} = \ln z^{1/2} = \frac{1}{2}\ln z$

49. $$\begin{aligned}\ln xyz^2 &= \ln x + \ln y + \ln z^2\\ &= \ln x + \ln y + 2\ln z\end{aligned}$$

51. $$\begin{aligned}\ln z(z-1)^2 &= \ln z + \ln(z-1)^2\\ &= \ln z + 2\ln(z-1),\ z > 1\end{aligned}$$

53. $$\begin{aligned}\log_2\left(\frac{\sqrt{a^2-4}}{7}\right) &= \log_2\sqrt{a^2-4} - \log_2 7\\ &= \log_2 (a^2-4)^{1/2} - \log_2 7\\ &= \tfrac{1}{2}\log_2 (a^2-4) - \log_2 7\\ &= \tfrac{1}{2}\log_2 \left[(a-2)(a+2)\right] - \log_2 7\\ &= \tfrac{1}{2}\left[\log_2 (a-2) + \log_2 (a+2)\right] - \log_2 7\\ &= \tfrac{1}{2}\log_2 (a-2) + \tfrac{1}{2}\log_2 (a+2) - \log_2 7\end{aligned}$$

55. $$\begin{aligned}\log_5\left(\frac{x^2}{y^2z^3}\right) &= \log_5 x^2 - \log_5 y^2z^3\\ &= \log_5 x^2 - \left(\log_5 y^2 + \log_5 z^3\right)\\ &= 2\log_5 x - 2\log_5 y - 3\log_5 z\end{aligned}$$

57. $$\begin{aligned}\ln\sqrt{\frac{x^2}{y^3}} &= \ln\left(\frac{x^2}{y^3}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{x^2}{y^3}\right)\\ &= \frac{1}{2}\left(\ln x^2 - \ln y^3\right)\\ &= \frac{1}{2}(2\ln x - 3\ln y)\\ &= \ln x - \frac{3}{2}\ln y\end{aligned}$$

59. $$\begin{aligned}\ln x^2\sqrt{\frac{y}{z}} &= \ln x^2 + \ln\sqrt{\frac{y}{z}}\\ &= \ln x^2 + \frac{1}{2}\ln\frac{y}{z}\\ &= \ln x^2 + \frac{1}{2}[\ln y - \ln z]\\ &= 2\ln x + \frac{1}{2}\ln y - \frac{1}{2}\ln z\end{aligned}$$

61. $$\begin{aligned}\ln\sqrt[4]{x^3(x^2+3)} &= \tfrac{1}{4}\ln x^3(x^2+3)\\ &= \tfrac{1}{4}\left[\ln x^3 + \ln(x^2+3)\right]\\ &= \tfrac{1}{4}\left[3\ln x + \ln(x^2+3)\right]\\ &= \tfrac{3}{4}\ln x + \tfrac{1}{4}\ln(x^2+3)\end{aligned}$$

63. $\ln 3 + \ln x = \ln(3x)$

65. $\frac{2}{3}\log_7(z-2) = \log_7(z-2)^{2/3}$

67.
$$\begin{aligned}\log_3 5x - 4\log_3 x &= \log_3 5x - \log_3 x^4\\ &= \log_3\left(\frac{5x}{x^4}\right)\\ &= \log_3\left(\frac{5}{x^3}\right)\end{aligned}$$

69.
$$\begin{aligned}\log x + 2\log(x+1) &= \log x + \log(x+1)^2\\ &= \log\left[x(x+1)^2\right]\end{aligned}$$

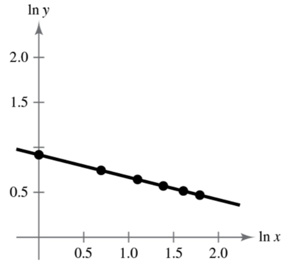
71.
$$\begin{aligned}\log x - 2\log y + 3\log z &= \log x - \log y^2 + \log z^3\\ &= \log\frac{x}{y^2} + \log z^3\\ &= \log\frac{xz^3}{y^2}\end{aligned}$$

73. $\ln x - \left[\ln(x+1) + \ln(x-1)\right] = \ln x - \ln(x+1)(x-1) = \ln\dfrac{x}{(x+1)(x-1)}$

75.
$$\begin{aligned}\frac{1}{2}\left[2\ln(x+3) + \ln x - \ln\left(x^2-1\right)\right] &= \frac{1}{2}\left[\ln(x+3)^2 + \ln x - \ln\left(x^2-1\right)\right]\\ &= \frac{1}{2}\left[\ln\left[x(x+3)^2\right] - \ln\left(x^2-1\right)\right]\\ &= \frac{1}{2}\left[\ln\left(\frac{x(x+3)^2}{x^2-1}\right)\right]\\ &= \frac{1}{2}\ln\left[\frac{x(x+3)^2}{x^2-1}\right]\\ &= \ln\sqrt{\frac{x(x+3)^2}{x^2-1}}\end{aligned}$$

77.

x	1	2	3	4	5	6
y	2.500	2.102	1.900	1.768	1.672	1.597
$\ln x$	0	0.693	1.099	1.386	1.609	1.792
$\ln y$	0.916	0.743	0.642	0.570	0.514	0.468



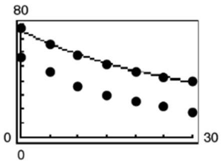
The slope of the line is $-\frac{1}{4}$ So, $\ln y = -\frac{1}{4}\ln x + \ln\frac{5}{2}$.

79.

Weight, x	25	35	50	75	500	1000
Galloping Speed, y	191.5	182.7	173.8	164.2	125.9	114.2
$\ln x$	3.219	3.555	3.912	4.317	6.215	6.908
$\ln y$	5.255	5.208	5.158	5.101	4.835	4.738

$\ln y = -0.14\ln x + 5.7$

81. (a)



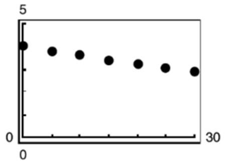
(b) $T - 21 = 54.4(0.964)^t$

$T = 54.4(0.964)^t + 21$

See graph in (a).

(c)

t (in minutes)	T (°C)	$T - 21$ (°C)	$\ln(T - 21)$	$1/(T - 21)$
0	78	57	4.043	0.0175
5	66	45	3.807	0.0222
10	57.5	36.5	3.597	0.0274
15	51.2	30.2	3.408	0.0331
20	46.3	25.3	3.231	0.0395
25	42.5	21.5	3.068	0.0465
30	39.6	18.6	2.923	0.0538



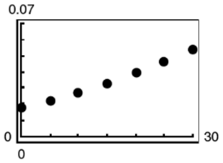
$\ln(T - 21) = -0.037t + 4$

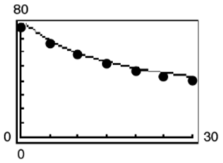
$T = e^{-0.037t+3.997} + 21$

This graph is identical to T in (b).

(d) $\dfrac{1}{T - 21} = 0.0012t + 0.016$

$T = \dfrac{1}{0.001t + 0.016} + 21$





83. $f(x) = \ln x$

False, $f(0) \neq 0$ because 0 is not in the domain of $f(x)$.

$f(1) = \ln 1 = 0$

85. False.

$f(x) - f(2) = \ln x - \ln 2 = \ln \dfrac{x}{2} \neq \ln(x - 2)$

87. False.

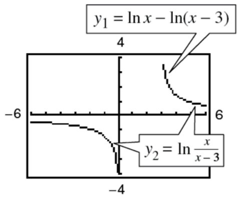
$f(u) = 2f(v) \Rightarrow \ln u = 2$

$\ln v \Rightarrow \ln u = \ln v^2 \Rightarrow u = v^2$

89. The power property cannot be used because $\ln e$ is raised to the second power, not just e.

A correct statement is $(\ln e)^2 = (1)^2 = 1$.

91.

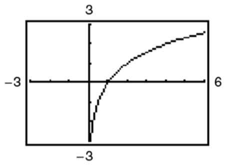


The graphing utility does not show the functions with the same domain. The domain of $y_1 = \ln x - \ln(x - 3)$ is $(3, \infty)$ and the domain of $y_2 = \ln \dfrac{x}{x - 3}$ is $(-\infty, 0) \cup (3, \infty)$.

93. $\log_2 \dfrac{32}{4} = \log_2 32 - \log_2 4 \neq \dfrac{\log_2 32}{\log_2 4}$

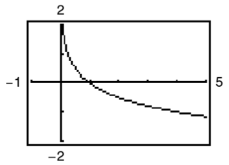
The second and third expressions are equal by Property 2.

95. $f(x) = \log_2 x = \dfrac{\log x}{\log 2} = \dfrac{\ln x}{\ln 2}$



97. $f(x) = \log_{1/4} x$

$$= \frac{\log x}{\log(1/4)} = \frac{\ln x}{\ln(1/4)}$$



99. $\ln 2 \approx 0.6931, \ln 3 \approx 1.0986, \ln 5 \approx 1.6094$

$\ln 1 = 0$

$\ln 2 \approx 0.6931$

$\ln 3 \approx 1.0986$

$\ln 4 = \ln(2 \cdot 2) = \ln 2 + \ln 2 \approx 0.6931 + 0.6931 = 1.3862$

$\ln 5 \approx 1.6094$

$\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3 \approx 0.6931 + 1.0986 = 1.7917$

$\ln 8 = \ln 2^3 = 3 \ln 2 \approx 3(0.6931) = 2.0793$

$\ln 9 = \ln 3^2 = 2 \ln 3 \approx 2(1.0986) = 2.1972$

$\ln 10 = \ln(5 \cdot 2) = \ln 5 + \ln 2 \approx 1.6094 + 0.6931 = 2.3025$

$\ln 12 = \ln(2^2 \cdot 3) = \ln 2^2 + \ln 3 = 2 \ln 2 + \ln 3 \approx 2(0.6931) + 1.0986 = 2.4848$

$\ln 15 = \ln(5 \cdot 3) = \ln 5 + \ln 3 \approx 1.6094 + 1.0986 = 2.7080$

$\ln 16 = \ln 2^4 = 4 \ln 2 \approx 4(0.6931) = 2.7724$

$\ln 18 = \ln(3^2 \cdot 2) = \ln 3^2 + \ln 2 = 2 \ln 3 + \ln 2 \approx 2(1.0986) + 0.6931 = 2.8903$

$\ln 20 = \ln(5 \cdot 2^2) = \ln 5 + \ln 2^2 = \ln 5 + 2 \ln 2 \approx 1.6094 + 2(0.6931) = 2.9956$

101.
$$\begin{aligned} x(x + 5) &= 24 \\ x^2 + 5x - 24 &= 0 \\ (x + 8)(x - 3) &= 0 \\ x &= -8, 3 \end{aligned}$$

103.
$$\begin{aligned} x - 4\sqrt{x} + 3 &= 0 \\ (\sqrt{x} - 3)(\sqrt{x} - 1) &= 0 \\ \sqrt{x} = 3 &\Rightarrow x = 9 \\ \sqrt{x} = 1 &\Rightarrow x = 1 \end{aligned}$$

Solutions: $x = 1, 9$

105.
$$\begin{aligned} 5^x &= 625 \\ 5^x &= 5^4 \\ x &= 4 \end{aligned}$$

107.
$$\begin{aligned} e^{x+5} &= e^2 \\ x + 5 &= 2 \\ x &= -3 \end{aligned}$$

109.
$$\begin{aligned} x^2 - 5x &= 0 \\ x(x - 5) &= 0 \\ x &= 0, 5 \end{aligned}$$

111.
$$\begin{aligned} x^2 &= 3x + 4 \\ x^2 - 3x - 4 &= 0 \\ (x - 4)(x + 1) &= 0 \\ x &= 4, \ -1 \end{aligned}$$

113.
$$\begin{aligned} x^2 + 8x + 16 &= 0 \\ (x + 4)^2 &= 0 \\ x &= -4 \end{aligned}$$

115. (a) $(1, 2)$: $2 \overset{?}{=} \sqrt{5 - 1}$

$$2 \overset{?}{=} \sqrt{4}$$

$$2 = 2$$

Yes, the point *is* on the graph.

(b) $(5, 0)$: $0 \overset{?}{=} \sqrt{5 - 5}$

$$0 = 0$$

Yes, the point *is* on the graph.

117. (a) $(2, 3)$: $3 \overset{?}{=} |2 - 1| + 2$

$3 \overset{?}{=} 1 + 2$

$3 = 3$

Yes, the point *is* on the graph.

(b) $(-1, 0)$: $0 \overset{?}{=} |-1 - 1| + 2$

$0 \overset{?}{=} 2 + 2$

$0 \neq 4$

No, the point *is not* on the graph.

119. (a) $(4, 10)$: $10 \overset{?}{=} 4^{3/2} + 2$

$10 \overset{?}{=} 8 + 2$

$10 = 10$

Yes, the point *is* on the graph.

(b) $(-4, -6)$: $-6 = (-4)^{3/2} + 2 \Rightarrow (-4)^{3/2}$ is undefined.

No, the point *is not* on the graph.

Section 3.4 Exponential and Logarithmic Equations

1. (a) $x = y$

(b) $x = y$

(c) x

(d) x

3. To solve $5^x = 125$, rewrite 125 as 5^3.

Then by the One-to-One Property, $x = 3$.

5. $4^{2x-7} = 64$

(a) $x = 2$

$4^{2(2)-7} = 4^{-3} = \frac{1}{64} \neq 64$

No, $x = 2$ *is not* a solution.

(b) $x = \frac{1}{2}(\log_4 64 + 7)$

$4^{2(1/2(\log_4 64+7))-7} = 64$

$4^{(\log_4 64+7)-7} = 64$

$4^{(3+7)-7} = 64$

$4^3 = 64$

Yes, $x = \frac{1}{2}(\log_4 64 + 7)$ *is* a solution.

7. $\log_2(x + 3) = 10$

(a) $x = 1021$

$\log_2(1021 + 3) = \log_2(1024)$

Because $2^{10} = 1024$, $x = 1021$ *is* a solution.

(b) $x = 10^2 - 3 = 97$

$\log_2(97 + 3) = \log_2(100)$

Because $2^{10} \neq 100$, $10^2 - 3$ *is not* a solution.

9. $4^x = 16$

$4^x = 4^2$

$x = 2$

11. $\ln x - \ln 2 = 0$

$\ln x = \ln 2$

$x = 2$

13. $e^x = 2$

$\ln e^x = \ln 2$

$x = \ln 2$

$x \approx 0.693$

15. $\ln x = -1$

$e^{\ln x} = e^{-1}$

$x = e^{-1}$

$x \approx 0.368$

17. $\log_4 x = 3$

$4^{\log_4 x} = 4^3$

$x = 4^3$

$x = 64$

19. $f(x) = g(x)$

$2^x = 8$

$2^x = 2^3$

$x = 3$

Point of intersection:

$(3, 8)$

21. $e^x = e^{x^2-2}$

$x = x^2 - 2$

$0 = x^2 - x - 2$

$0 = (x + 1)(x - 2)$

$x = -1, x = 2$

23.
$$\begin{aligned}4\left(3^x\right) &= 20\\ 3^x &= 5\\ \log_3 3^x &= \log_3 5\\ x &= \log_3 5 = \frac{\log 5}{\log 3} \text{ or } \frac{\ln 5}{\ln 3}\\ x &\approx 1.465\end{aligned}$$

25.
$$\begin{aligned}e^x - 8 &= 31\\ e^x &= 39\\ \ln e^x &= \ln 39\\ x &= \ln 39 \approx 3.664\end{aligned}$$

27.
$$\begin{aligned}3^{2-x} &= 400\\ \ln 3^{2-x} &= \ln 400\\ (2 - x)\ln 3 &= \ln 400\\ 2\ln 3 - x\ln 3 &= \ln 400\\ -x\ln 3 &= \ln 400 - 2\ln 3\\ x\ln 3 &= 2\ln 3 - \ln 400\\ x &= \frac{2\ln 3 - \ln 400}{\ln 3}\\ x &= 2 - \frac{\ln 400}{\ln 3} \approx -3.454\end{aligned}$$

29.
$$\begin{aligned}8\left(10^{3x}\right) &= 12\\ 10^{3x} &= \tfrac{12}{8}\\ \log 10^{3x} &= \log\left(\tfrac{3}{2}\right)\\ 3x &= \log\left(\tfrac{3}{2}\right)\\ x &= \tfrac{1}{3}\log\left(\tfrac{3}{2}\right)\\ x &\approx 0.059\end{aligned}$$

31.
$$\begin{aligned}e^{3x} &= 12\\ 3x &= \ln 12\\ x &= \frac{\ln 12}{3} \approx 0.828\end{aligned}$$

33.
$$\begin{aligned}7 - 2e^x &= 5\\ -2e^x &= -2\\ e^x &= 1\\ x &= \ln 1 = 0\end{aligned}$$

35.
$$\begin{aligned}6\left(2^{3x-1}\right) - 7 &= 9\\ 6\left(2^{3x-1}\right) &= 16\\ 2^{3x-1} &= \frac{8}{3}\\ \log_2 2^{3x-1} &= \log_2\left(\frac{8}{3}\right)\\ 3x - 1 &= \log_2\left(\frac{8}{3}\right) = \frac{\log(8/3)}{\log 2} \text{ or } \frac{\ln(8/3)}{\ln 2}\\ x &= \frac{1}{3}\left[\frac{\log(8/3)}{\log 2} + 1\right] \approx 0.805\end{aligned}$$

37.
$$\begin{aligned}4^x &= 5^{x^2}\\ \ln 4^x &= \ln 5^{x^2}\\ x\ln 4 &= x^2\ln 5\\ x^2\ln 5 - x\ln 4 &= 0\\ x\left(x\ln 5 - \ln 4\right) &= 0\\ x &= 0\\ x\ln 5 - \ln 4 = 0 \Rightarrow x &= \frac{\ln 4}{\ln 5} \approx 0.861\end{aligned}$$

39.
$$\begin{aligned}e^{2x} - 4e^x - 5 &= 0\\ \left(e^x + 1\right)\left(e^x - 5\right) &= 0\\ e^x = -1 \quad \text{or} \quad e^x &= 5\\ \text{(No solution)} \quad x &= \ln 5 \approx 1.609\end{aligned}$$

41.
$$\begin{aligned}\frac{1}{1 - e^x} &= 5\\ 1 &= 5\left(1 - e^x\right)\\ \frac{1}{5} &= 1 - e^x\\ \frac{1}{5} - 1 &= -e^x\\ -\frac{4}{5} &= -e^x\\ \frac{4}{5} &= e^x\\ \ln\frac{4}{5} &= \ln e^x\\ \ln\frac{4}{5} &= x\\ x &\approx -0.223\end{aligned}$$

43.
$$\begin{aligned}\left(1+\frac{0.065}{365}\right)^{365t} &= 4\\ \ln\left(1+\frac{0.065}{365}\right)^{365t} &= \ln 4\\ 365t\ln\left(1+\frac{0.065}{365}\right) &= \ln 4\\ t = \frac{\ln 4}{365\ln\left(1+\frac{0.065}{365}\right)} &\approx 21.330\end{aligned}$$

45.
$$\begin{aligned}\ln x &= -3\\ x &= e^{-3} \approx 0.050\end{aligned}$$

47.
$$\begin{aligned}2.1 &= \ln 6x\\ e^{2.1} &= 6x\\ \frac{e^{2.1}}{6} &= x\\ 1.361 &\approx x\end{aligned}$$

49.
$$\begin{aligned}3 - 4\ln x &= 11\\ -4\ln x &= 8\\ \ln x &= -2\\ x = e^{-2} &= \frac{1}{e^2} \approx 0.135\end{aligned}$$

51.
$$\begin{aligned}6\log_3(0.5x) &= 11\\ \log_3(0.5x) &= \tfrac{11}{6}\\ 3^{\log_3(0.5x)} &= 3^{11/6}\\ 0.5x &= 3^{11/6}\\ x &= 2\left(3^{11/6}\right) \approx 14.988\end{aligned}$$

53.
$$\begin{aligned}\ln(x+5) &= \ln(x-1) - \ln(x+1)\\ \ln(x+5) &= \ln\left(\frac{x-1}{x+1}\right)\\ x + 5 &= \frac{x-1}{x+1}\\ (x+5)(x+1) &= x - 1\\ x^2 + 6x + 5 &= x - 1\\ x^2 + 5x + 6 &= 0\\ (x+2)(x+3) &= 0\\ x = -2 \quad\text{or}\quad x &= -3\end{aligned}$$

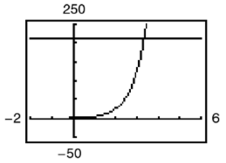
Both of these solutions are extraneous, so the equation has no solution.

55.
$$\begin{aligned}\log(3x+4) &= \log(x-10)\\ 3x + 4 &= x - 10\\ 2x &= -14\\ x &= -7\end{aligned}$$

The negative value is extraneous.
The equation has no solution.

57.
$$\begin{aligned}\log_4 x - \log_4(x-1) &= \frac{1}{2}\\ \log_4\left(\frac{x}{x-1}\right) &= \frac{1}{2}\\ 4^{\log_4[x/(x-1)]} &= 4^{1/2}\\ \frac{x}{x-1} &= 4^{1/2}\\ x &= 2(x-1)\\ x &= 2x - 2\\ -x &= -2\\ x &= 2\end{aligned}$$

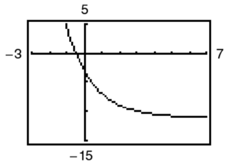
59. $f(x) = 5^x - 212$



Algebraically:
$$\begin{aligned}5^x &= 212\\ \ln 5^x &= \ln 212\\ x\ln 5 &= \ln 212\\ x &= \frac{\ln 212}{\ln 5}\\ x &\approx 3.328\end{aligned}$$

The zero is $x \approx 3.328$.

61. $g(x) = 8e^{-2x/3} - 11$



Algebraically:
$$\begin{aligned}8e^{-2x/3} &= 11\\ e^{-2x/3} &= 1.375\\ -\frac{2x}{3} &= \ln 1.375\\ x &= -1.5\ln 1.375\\ x &\approx -0.478\end{aligned}$$

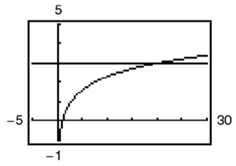
The zero is $x \approx -0.478$.

63. $y_1 = 3$

$y_2 = \ln x$

From the graph,

$x \approx 20.086$ when $y = 3$.



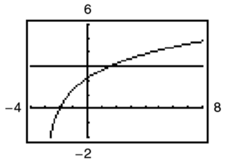
Algebraically:
$$\begin{aligned}3 - \ln x &= 0\\ \ln x &= 3\\ x &= e^3 \approx 20.086\end{aligned}$$

65. $y_1 = 2\ln(x+3)$

$y_2 = 3$

From the graph, $x \approx 1.482$ when $y = 3$.

Algebraically:



$$\begin{aligned} 2\ln(x+3) &= 3 \\ \ln(x+3) &= \tfrac{3}{2} \\ x+3 &= e^{3/2} \\ x &= e^{3/2} - 3 \approx 1.482 \end{aligned}$$

67. (a)
$$\begin{aligned} r &= 0.025 \\ A &= Pe^{rt} \\ 5000 &= 2500e^{0.025t} \\ 2 &= e^{0.025t} \\ \ln 2 &= 0.025t \\ \frac{\ln 2}{0.025} &= t \\ t &\approx 27.73 \text{ years} \end{aligned}$$

(b)
$$\begin{aligned} r &= 0.025 \\ A &= Pe^{rt} \\ 7500 &= 2500e^{0.025t} \\ 3 &= e^{0.025t} \\ \ln 3 &= 0.025t \\ \frac{\ln 3}{0.025} &= t \\ t &\approx 43.94 \text{ years} \end{aligned}$$

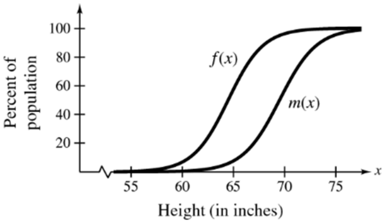
69.
$$\begin{aligned} 2x^2e^{2x} + 2xe^{2x} &= 0 \\ \left(2x^2 + 2x\right)e^{2x} &= 0 \\ 2x^2 + 2x &= 0 \qquad \left(\text{because } e^{2x} \neq 0\right) \\ 2x(x+1) &= 0 \\ x &= 0, -1 \end{aligned}$$

71.
$$\begin{aligned} -xe^{-x} + e^{-x} &= 0 \\ (-x+1)e^{-x} &= 0 \\ -x + 1 &= 0 \qquad \left(\text{because } e^{-x} \neq 0\right) \\ x &= 1 \end{aligned}$$

73.
$$\begin{aligned} \frac{1 + \ln x}{2} &= 0 \\ 1 + \ln x &= 0 \\ \ln x &= -1 \\ x &= e^{-1} = \frac{1}{e} \approx 0.368 \end{aligned}$$

75.
$$\begin{aligned} 2x\ln x + x &= 0 \\ x(2\ln x + 1) &= 0 \\ 2\ln x + 1 &= 0 \qquad (\text{because } x > 0) \\ \ln x &= -\tfrac{1}{2} \\ x &= e^{-1/2} \approx 0.607 \end{aligned}$$

77. (a)



From the graph you see horizontal asymptotes at $y = 0$ and $y = 100$.

These represent the lower and upper percent bounds; the range falls between 0% and 100%.

(b) Males:
$$\begin{aligned} 50 &= \frac{100}{1 + e^{-0.5536(x-69.51)}} \\ 1 + e^{-0.5536(x-69.51)} &= 2 \\ e^{-0.5536(x-69.51)} &= 1 \\ -0.5536(x - 69.51) &= \ln 1 \\ -0.5536(x - 69.51) &= 0 \\ x &= 69.51 \end{aligned}$$

The average height of an American male is 69.51 inches.

Females:
$$\begin{aligned} 50 &= \frac{100}{1 + e^{-0.5834(x-64.49)}} \\ 1 + e^{-0.5834(x-64.49)} &= 2 \\ e^{-0.5834(x-64.49)} &= 1 \\ -0.5834(x - 64.49) &= \ln 1 \\ -0.5834(x - 64.49) &= 0 \\ x &= 64.49 \end{aligned}$$

The average height of an American female is 64.49 inches.

79. $N = 5.5 \cdot 10^{0.23x}$

When $N = 78$:
$$\begin{aligned} 78 &= 5.5 \cdot 10^{0.23x} \\ \frac{78}{5.5} &= 10^{0.23x} \\ \log_{10}\frac{78}{5.5} &= 0.23x \\ x &= \frac{\log_{10}(78/5.5)}{0.23} \approx 5.008 \text{ years} \end{aligned}$$

The beaver population will reach 78 in about 5 years.

81. $P = 161 \ln t + 236, 9 \le t \le 19$

Let $P = 660$.

$660 = 161 \ln t + 236$

$424 = 161 \ln t$

$\ln t = \dfrac{424}{161}$

$t = e^{424/161} \approx 13.9$

The population exceeded 660 thousand in 2013.

83. $A = Pe^{rt}$

(a) $A = (2P)e^{rt} = 2(Pe^{rt})$ This doubles your money.

(b) $A = Pe^{(2r)t} = Pe^{rt}e^{rt} = e^{rt}(Pe^{rt})$

(c) $A = Pe^{r(2t)} = Pe^{rt}e^{rt} = e^{rt}(Pe^{rt})$

Doubling the interest rate yields the same result as doubling the number of years.

If $2 > e^{rt}$ (i.e., $rt < \ln 2$), then doubling your investment would yield the most money. If $rt > \ln 2$, then doubling either the interest rate or the number of years would yield more money.

85. $\log_a(uv) = \log_a u + \log_a v$

True by Property 1 in Section 3.3.

87. $\log_a(u - v) = \log_a u - \log_a v$

False.

$$\begin{aligned} 1.95 &= \log(100 - 10) \\ &\ne \log 100 - \log 10 = 1 \end{aligned}$$

89. Yes, a logarithmic equation can have more than one extraneous solution. See Exercise 57.

91. Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 1$

93.

$$\begin{aligned} (2e^{-3})^2 &= 4^x \\ 4e^{-6} &= 4^x \\ \ln 4e^{-6} &= \ln 4^x \\ \ln 4 + \ln e^{-6} &= x \ln 4 \\ 2 \ln 2 - 6 &= 2x \ln 2 \\ x &= \frac{2 \ln 2 - 6}{2 \ln 2} \\ x &= \frac{\ln 2 - 3}{\ln 2} \approx -3.328 \end{aligned}$$

95.

$$\begin{aligned} (2k)^4 &= k^2(8 + \ln 3) \\ 16k^4 &= k^2(8 + \ln 3) \\ 16k^2 &= 8 + \ln 3 \\ k^2 &= \frac{8 + \ln 3}{16} \\ k &= \frac{\pm\sqrt{8 + \ln 3}}{4}, k = 0 \end{aligned}$$

97. $\dfrac{(5x^{-3})^2}{2^{-2}} = \dfrac{5^2 \cdot 2^2}{x^6} = \dfrac{100}{x^6}$

Section 3.5 Exponential and Logarithmic Models

1. $y = a + b \ln x$; $y = a + b \log x$

3. An exponential growth model increases over time. An exponential decay model decreases over time.

5. (a)

$$\begin{aligned} A &= Pe^{rt} \\ \frac{A}{e^{rt}} &= P \end{aligned}$$

(b)

$$\begin{aligned} A &= Pe^{rt} \\ \frac{A}{P} &= e^{rt} \\ \ln \frac{A}{P} &= \ln e^{rt} \\ \ln \frac{A}{P} &= rt \\ \frac{\ln(A/P)}{r} &= t \end{aligned}$$

7. Because $A = 1000e^{0.035t}$, the time to double is given by $2000 = 1000e^{0.035t}$ and you have

$$\begin{aligned} 2 &= e^{0.035t} \\ \ln 2 &= \ln e^{0.035t} \\ \ln 2 &= 0.035t \\ t &= \frac{\ln 2}{0.035} \approx 19.8 \text{ years.} \end{aligned}$$

Amount after 10 years: $A = 1000e^{0.35} \approx \$1419.07$

9. Because $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, you have

$$\begin{aligned} 1500 &= 750e^{7.75r} \\ 2 &= e^{7.75r} \\ \ln 2 &= \ln e^{7.75r} \\ \ln 2 &= 7.75r \\ r &= \frac{\ln 2}{7.75} \approx 0.089438 = 8.9438\%. \end{aligned}$$

Amount after 10 years: $A = 750e^{0.089438(10)} \approx \$1834.37$

11. Because $A = Pe^{0.045t}$ and $A = 10{,}000.00$ when $t = 10$, you have

$$\begin{aligned} 10{,}000.00 &= Pe^{0.045(10)} \\ \frac{10{,}000.00}{e^{0.045(10)}} &= P \approx \$6376.28. \end{aligned}$$

The time to double is given by $t = \dfrac{\ln 2}{0.045} \approx 15.40$ years.

13. $P = 1000, r = 0.1, A = 2000$

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ 2000 &= 1000\left(1 + \frac{0.1}{n}\right)^{nt} \\ 2 &= \left(1 + \frac{0.1}{n}\right)^{nt} \end{aligned}$$

(a) $n = 1$

$$\begin{aligned} (1 + 0.1)^t &= 2 \\ (1.1)^t &= 2 \\ \ln(1.1)^t &= \ln 2 \\ t \ln 1.1 &= \ln 2 \\ t &= \frac{\ln 2}{\ln 1.1} \approx 7.27 \text{ years} \end{aligned}$$

(b) $n = 12$

$$\begin{aligned} \left(1 + \frac{0.1}{12}\right)^{12t} &= 2 \\ \ln\left(\frac{12.1}{12}\right)^{12t} &= \ln 2 \\ 12t \ln\left(\frac{12.1}{12}\right) &= \ln 2 \\ 12t &= \frac{\ln 2}{\ln(12.1/12)} \\ t &= \frac{\ln 2}{12 \ln(12.1/12)} \approx 6.96 \text{ years} \end{aligned}$$

(c) $n = 365$

$$\begin{aligned} \left(1 + \frac{0.1}{365}\right)^{365t} &= 2 \\ \ln\left(\frac{365.1}{365}\right)^{365t} &= \ln 2 \\ 365t \ln\left(\frac{365.1}{365}\right) &= \ln 2 \\ 365t &= \frac{\ln 2}{\ln(365.1/365)} \\ t &= \frac{\ln 2}{365 \ln(365.1/365)} \\ &\approx 6.93 \text{ years} \end{aligned}$$

(d) Compounded continuously

15. (a)
$$\begin{aligned} 3P &= Pe^{rt} \\ 3 &= e^{rt} \\ \ln 3 &= rt \\ \frac{\ln 3}{r} &= t \end{aligned}$$

r	2%	4%	6%	8%	10%	12%
$t = \frac{\ln 3}{r}$ (years)	54.93	27.47	18.31	13.73	10.99	9.16

(b)
$$\begin{aligned} 3P &= P(1 + r)^t \\ 3 &= (1 + r)^t \\ \ln 3 &= \ln (1 + r)^t \\ \frac{\ln 3}{\ln (1 + r)} &= t \end{aligned}$$

r	2%	4%	6%	8%	10%	12%
$t = \frac{\ln 3}{\ln (1 + r)}$ (years)	55.48	28.01	18.85	14.27	11.53	9.69

17. $y = 4147e^{0.294t}$ $(t = 1 \leftrightarrow \text{April})$

Month	April	May	June	July	August
Hits	5611	7402	9978	13,454	18,105
Model	5564	7466	10,018	13,442	18,036

The values given by the model are close to the original data values.

Let $y = 50{,}000$.
$$\begin{aligned} 4147e^{0.294t} &= 50{,}000 \\ e^{0.294t} &\approx 12.0569 \\ 0.294t &= \ln 12.0569 \\ t &= \frac{\ln 12.0569}{0.294} \approx 8.5 \text{ (November)} \end{aligned}$$

The website will receive over 50,000 hits in November.

19.
$$\begin{aligned} y &= ae^{bx} \\ 1 &= ae^{b(0)} \Rightarrow 1 = a \\ 10 &= e^{b(3)} \\ \ln 10 &= 3b \\ \frac{\ln 10}{3} &= b \Rightarrow b \approx 0.7675 \end{aligned}$$

So, $y = e^{0.7675x}$.

21.
$$\begin{aligned} y &= ae^{bx} \\ 5 &= ae^{b(0)} \Rightarrow 5 = a \\ 1 &= 5e^{b(4)} \\ \frac{1}{5} &= e^{4b} \\ \ln\left(\frac{1}{5}\right) &= 4b \\ \frac{\ln(1/5)}{4} &= b \Rightarrow b \approx -0.4024 \end{aligned}$$

So, $y = 5e^{-0.4024x}$.

23. $a = 10, y = \frac{1}{2}(10) = 5, t = 1599$
$$\begin{aligned} y &= ae^{-bt} \\ 5 &= 10e^{-b(1599)} \\ 0.5 &= e^{-1599b} \\ \ln 0.5 &= \ln e^{-1599b} \\ \ln 0.5 &= -1599b \\ b &= -\frac{\ln 0.5}{1599} \end{aligned}$$

Given an initial quantity of 10 grams, after 1000 years, you have

$$y = 10e^{-[-(\ln 0.5)/1599](1000)} \approx 6.48 \text{ grams.}$$

25. $y = 2, a = 2(2) = 4, t = 5715$
$$\begin{aligned} y &= ae^{-bt} \\ 2 &= 4e^{-b(5715)} \\ 0.5 &= e^{-5715b} \\ \ln 0.5 &= \ln e^{-5715b} \\ \ln 0.5 &= -5715b \\ b &= -\frac{\ln 0.5}{5715} \end{aligned}$$

Given 2 grams after 1000 years, the initial amount is

$$\begin{aligned} 2 &= ae^{-[-(\ln 0.5)/5715](1000)} \\ a &\approx 2.26 \text{ grams.} \end{aligned}$$

27. (a) *Bulgaria*: $(19, 7), (29, 6.5)$

Let $y = ae^{bt}$ So, $7 = ae^{19b}$ and $6.5 = ae^{29b}$.

$$\frac{7}{e^{19b}} = a \Rightarrow 6.5 = \frac{7}{e^{19b}}e^{29b}$$

$$\frac{6.5}{7} = e^{10b}$$

$$\ln\left(\frac{6.5}{7}\right) = 10b$$

$$\frac{1}{10}\ln\left(\frac{6.5}{7}\right) = b$$

$$-0.00741 \approx b$$

Because $b \approx -0.00741$,

$$a = \frac{7}{e^{19(-0.00741)}} \approx 8.1.$$

So, $y = 8.1e^{-0.00741t}$.

In 2039 $(t = 39)$, $y = 8.1e^{-0.00741(39)} \approx 6.1$ million people.

Canada: $(19, 37.4), (29, 40)$

Let $y = ae^{bt}$ So, $37.4 = ae^{19b}$ and $40 = ae^{29b}$.

$$\frac{37.4}{e^{19b}} = a \Rightarrow 40 = \frac{37.4}{e^{19b}}e^{29b}$$

$$\frac{40}{37.4} = e^{10b}$$

$$\ln\left(\frac{40}{37.4}\right) = 10b$$

$$\frac{1}{10}\ln\left(\frac{40}{37.4}\right) = b$$

$$0.00672 \approx b$$

Because $b \approx 0.00672$,

$$a = \frac{37.4}{e^{19(0.00672)}} \approx 32.9.$$

So, $y = 32.9e^{0.00672t}$.

In 2039, $(t = 39)$, $y = 32.9e^{0.00672(39)} \approx 42.8$ million people

China: $(19, 1389.6), (29, 1405.7)$

Let $y = ae^{bt}$ So, $1389.6 = ae^{19b}$ and $1405.7 = ae^{29b}$.

$$\frac{1389.6}{e^{19b}} = a \Rightarrow 1405.7 = \frac{1389.6}{e^{19b}}e^{29b}$$

$$\frac{1405.7}{1389.6} = e^{10b}$$

$$\ln\left(\frac{1405.7}{1389.6}\right) = 10b$$

$$\frac{1}{10}\ln\left(\frac{1405.7}{1389.6}\right) = b$$

$$0.00115 \approx b$$

Because $b \approx 0.00115$,

$$a = \frac{1389.6}{e^{19(0.00115)}} \approx 1359.6.$$

So, $y = 1359.6e^{0.00115t}$.

In 2039 $(t = 39)$, $y = 1359.6e^{0.00115(39)} \approx 1422.0$ million people.

United Kingdom: $(19, 65.4), (29, 68.2)$

Let $y = ae^{bt}$ so, $65.4 = ae^{19b}$ and $68.2 = ae^{29b}$.

$$\frac{65.4}{e^{19b}} = a \Rightarrow 68.2 = \frac{65.4}{e^{19b}}e^{29b}$$

$$\frac{68.2}{65.4} = e^{10b}$$

$$\ln\left(\frac{68.2}{65.4}\right) = 10b$$

$$\frac{1}{10}\ln\left(\frac{68.2}{65.4}\right) = b$$

$$0.00419 \approx b$$

Because $b \approx 0.00419$,

$$a = \frac{65.4}{e^{19(0.00419)}} \approx 60.4.$$

So, $y = 60.4e^{0.00419t}$.

In 2039 $(t = 39)$, $y = 60.4e^{0.00419(39)} \approx 71.1$ million people.

United States: $(19, 330.3), (29, 353)$

Let $y = ae^{bt}$ so, $330.3 = ae^{19b}$ and $353 = ae^{29b}$.

$$\frac{330.3}{e^{19b}} = a \Rightarrow 353 = \frac{330.3}{e^{19b}}e^{29b}$$

$$\frac{353}{330.3} = e^{10b}$$

$$\ln\left(\frac{353}{330.3}\right) = 10b$$

$$\frac{1}{10}\ln\left(\frac{353}{330.3}\right) = b$$

$$0.00665 \approx b$$

Because $b \approx 0.00665$,

$$a = \frac{330.3}{e^{19(0.00665)}} \approx 291.1.$$

So, $y = 291.1e^{0.00665t}$.

In 2039 $(t = 39)$, $y = 291.1e^{0.00665(39)}$ ≈ 377.3 million people.

(b) In the model $y = ae^{bt}$, the constant b gives the growth rate. The greater the growth rate, the greater the value of b.

29. $y = ae^{bt}$

When $t = 3, y = 100$: When $t = 5, y = 400$:

$$100 = ae^{3b} \qquad 400 = ae^{5b}$$

$$\frac{100}{e^{3b}} = a$$

Substitute $\frac{100}{e^{3b}}$ for a in the equation on the right.

$$\begin{aligned} 400 &= \frac{100}{e^{3b}}e^{5b} \\ 400 &= 100e^{2b} \\ 4 &= e^{2b} \\ \ln 4 &= 2b \\ \ln 2^2 &= 2b \\ 2\ln 2 &= 2b \\ \ln 2 &= b \end{aligned}$$

$$a = \frac{100}{e^{3b}} = \frac{100}{e^{3\ln 2}} = \frac{100}{e^{\ln 2^3}} = \frac{100}{2^3} = \frac{100}{8} = 12.5$$

$$y = 12.5e^{(\ln 2)t}$$

After 6 hours, there are $y = 12.5e^{(\ln 2)(6)} = 800$ bacteria.

31. $(0, 575), (2, 275)$

(a) $m = \dfrac{275 - 575}{2 - 0} = -150$

$V = -150t + 575$

(b) Since $V = 575$, when

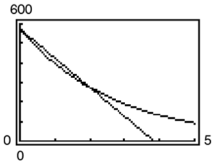
$t = 0, 575 = ae^{(b)(0)} \rightarrow a = 575$

Then $275 = 575e^{k(2)}$

$\ln\left(\dfrac{275}{575}\right) = 2k \Rightarrow k \approx -0.3688$

$V = 575e^{-0.3688t}$

(c)



The exponential model depreciates faster in the first two years.

(d)

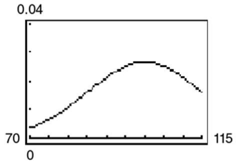
t	1	3
$V = -150t + 575$	\$425	\$125
$V = 575e^{-0.3688t}$	\$397.65	\$190.18

(e) Answers will vary. *Sample Answer:* The slope of the linear model means that the laptop depreciates \$150 per year, then loses all value late in the third year. The exponential model depreciates faster in the first three years but maintains value longer.

33. $R = \dfrac{1}{10^{12}}e^{-t/8223}$

$$\begin{aligned} R &= \frac{1}{8^{14}} \\ \frac{1}{10^{12}}e^{-t/8223} &= \frac{1}{8^{14}} \\ e^{-t/8223} &= \frac{10^{12}}{8^{14}} \\ -\frac{t}{8223} &= \ln\left(\frac{10^{12}}{8^{14}}\right) \\ t &= -8223\ln\left(\frac{10^{12}}{8^{14}}\right) \approx 12{,}180 \text{ years old} \end{aligned}$$

35. $y = 0.0266e^{-(x-100)^2/450}, 70 \le x \le 116$



The average IQ score of an adult student is 100.

37. $y = \dfrac{336{,}011}{1 + 293e^{-0.236t}}$ $(t = 5 \leftrightarrow 1985)$

(a) 1998 $(t = 18)$:

$$y = \frac{336{,}011}{1 + 293e^{-0.236(18)}} \approx 64{,}770 \text{ sites}$$

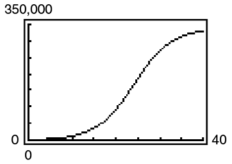
2008 $(t = 28)$:

$$y = \frac{336{,}011}{1 + 293e^{-0.236(28)}} \approx 290{,}797 \text{ sites}$$

2015 $(t = 35)$:

$$y = \frac{336{,}011}{1 + 293e^{-0.236(35)}} \approx 312{,}340 \text{ sites}$$

(b)



(c) From the graph, the number of cell sites reached 240,000 in 2007.

(d) Let $y = 240{,}000$.

$$\begin{aligned} 240{,}000 &= \frac{336{,}011}{1 + 293e^{-0.236t}} \\ 1 + 293e^{-0.236t} &= \frac{336{,}011}{240{,}000} \\ 1 + 293e^{-0.236t} &\approx 1.40 \\ 293e^{-0.236t} &= 0.40 \\ e^{-0.236t} &= \frac{0.40}{293} \\ -0.236t &= \ln\left(\frac{0.40}{293}\right) \\ t &= -\frac{1}{0.236}\ln\left(\frac{0.40}{293}\right) \approx 27.95 \end{aligned}$$

The number of cell sites reached 240,000 in 2007.

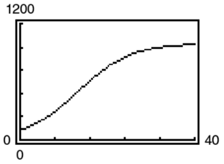
39. $p(t) = \dfrac{1000}{1 + 9e^{-0.1656t}}$

(a) $p(5) = \dfrac{1000}{1 + 9e^{-0.1656(5)}} \approx 203$ animals

(b)

$$\begin{aligned} 500 &= \frac{1000}{1 + 9e^{-0.1656t}} \\ 1 + 9e^{-0.1656t} &= 2 \\ 9e^{-0.1656t} &= 1 \\ e^{-0.1656t} &= \frac{1}{9} \\ t &= -\frac{\ln(1/9)}{0.1656} \approx 13 \text{ months} \end{aligned}$$

(c)



The horizontal asymptotes are $p = 0$ and $p = 1000$.

The asymptote with the larger p-value, $p = 1000$, indicates that the population size will approach 1000 as time increases.

41. $\beta = 10 \log \dfrac{I}{I_0}$ where $I_0 = 10^{-12}$ watt/m^2.

(a) $\beta = 10 \log \dfrac{10^{-10}}{10^{-12}} = 10 \log 10^2 = 20$ decibels

(b) $\beta = 10 \log \dfrac{10^{-5}}{10^{-12}} = 10 \log 10^7 = 70$ decibels

(c) $\beta = 10 \log \dfrac{10^{-8}}{10^{-12}} = 10 \log 10^4 = 40$ decibels

(d) $\beta = 10 \log \dfrac{10^{-3}}{10^{-12}} = 10 \log 10^9 = 90$ decibels

43.

$$\begin{aligned} \beta &= 10 \log \frac{I}{I_0} \\ \frac{\beta}{10} &= \log \frac{I}{I_0} \\ 10^{\beta/10} &= 10^{\log I/I_0} \\ 10^{\beta/10} &= \frac{I}{I_0} \\ I &= I_0 10^{\beta/10} \end{aligned}$$

$$\% \text{ decrease} = \frac{I_0 10^{9.3} - I_0 10^{8.0}}{I_0 10^{9.3}} \times 100 \approx 95\%$$

45. $R = \log \dfrac{I}{I_0} = \log I$ because $I_0 = 1$.

(a)
$$\begin{aligned} R &= 7.6 \\ 7.6 &= \log I \\ 10^{7.6} &= 10^{\log I} \\ 39{,}810{,}717 &\approx I \end{aligned}$$

(b)
$$\begin{aligned} R &= 5.6 \\ 5.6 &= \log I \\ 10^{5.6} &= 10^{\log I} \\ 10^{5.6} &= I \\ 398{,}107 &\approx I \end{aligned}$$

(c)
$$\begin{aligned} R &= 6.6 \\ 6.6 &= \log I \\ 10^{6.6} &= 10^{\log I} \\ 3{,}981{,}072 &\approx I \end{aligned}$$

47. $\text{pH} = -\log\left[\text{H}^+\right]$

$-\log\left(2.3 \times 10^{-5}\right) \approx 4.64$

49.
$$\begin{aligned} 5.8 &= -\log\left[\text{H}^+\right] \\ -5.8 &= \log\left[\text{H}^+\right] \\ 10^{-5.8} &= 10^{\log\left[\text{H}^+\right]} \\ 10^{-5.8} &= \left[\text{H}^+\right] \end{aligned}$$

$\left[\text{H}^+\right] \approx 1.58 \times 10^{-6}$ moles per liter

51.
$$\begin{aligned} 2.9 &= -\log\left[\text{H}^+\right] \\ -2.9 &= \log\left[\text{H}^+\right] \end{aligned}$$

$\left[\text{H}^+\right] = 10^{-2.9}$ for the apple juice

$$\begin{aligned} 8.0 &= -\log\left[\text{H}^+\right] \\ -8.0 &= \log\left[\text{H}^+\right] \end{aligned}$$

$\left[\text{H}^+\right] = 10^{-8}$ for the drinking water

$\dfrac{10^{-2.9}}{10^{-8}} = 10^{5.1}$ times the hydrogen ion concentration of drinking water

53. $t = -10 \ln \dfrac{T - 70}{98.6 - 70}$

At 9:00 A.M. you have:

$t = -10 \ln \dfrac{85.7 - 70}{98.6 - 70} \approx 6$ hours

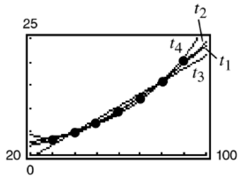
From this you can conclude that the person died at 3:00 A.M.

55. $t_1 = 40.757 + 0.556s - 15.817 \ln s$

$t_2 = 1.2259 + 0.0023s^2$

(a) Linear model: $t_3 = 0.2729s - 6.0143$

Exponential model: $t_4 = 1.5385e^{0.02913s}$ or $t_4 = 1.5385(1.0296)^s$



(b)

s	30	40	50	60	70	80	90
t_1	3.6	4.6	6.7	9.4	12.5	15.9	19.6
t_2	3.3	4.9	7.0	9.5	12.5	15.9	19.9
t_3	2.2	4.9	7.6	10.4	13.1	15.8	18.5
t_4	3.7	4.9	6.6	8.8	11.8	15.8	21.2

Note: Table values will vary slightly depending on the model used for t_4.

(c) Model t_1: $S_1 = |3.4 - 3.6| + |5 - 4.6| + |7 - 6.7| + |9.3 - 9.4| + |12 - 12.5| + |15.8 - 15.9| + |20 - 19.6| = 2.0$

Model t_2: $S_2 = |3.4 - 3.3| + |5 - 4.9| + |7 - 7| + |9.3 - 9.5| + |12 - 12.5| + |15.8 - 15.9| + |20 - 19.9| = 1.1$

Model t_3: $S_3 = |3.4 - 2.2| + |5 - 4.9| + |7 - 7.6| + |9.3 - 10.4| + |12 - 13.1| + |15.8 - 15.8| + |20 - 18.5| = 5.6$

Model t_4: $S_4 = |3.4 - 3.7| + |5 - 4.9| + |7 - 6.6| + |9.3 - 8.8| + |12 - 11.8| + |15.8 - 15.8| + |20 - 21.2| = 2.7$

The quadratic model, t_2, best fits the data.

57. False. The domain can be the set of real numbers for a logistic growth function.

59. True. The graph of a Gaussian model will never have a zero.

61. A Gaussian model will have the maximum value occur at the average value of the independent variable.

63. An exponential growth model will have a graph which shows a steadily increasing rate of growth.

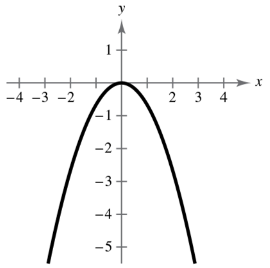
65. $$\frac{x}{x-3}+\frac{3}{x+1}=\frac{x(x+1)+3(x-3)}{(x-3)(x+1)}$$
$$=\frac{x^2+x+3x-9}{(x-3)(x+1)}$$
$$=\frac{x^2+4x-9}{(x-3)(x+1)}$$

67. $$\frac{x^2+x}{2x}\cdot\frac{x-1}{x+1}=\frac{x(x+1)(x-1)}{2x(x+1)}$$
$$=\frac{x-1}{2},\quad x\neq 0,\ -1$$

69. When $x < 0$ and $y < 0$, the point (x, y) is located in Quadrant III.

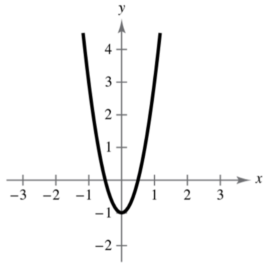
71. When $x + y = 0$ $(y = -x < 0)$ and $x - 5 > 0$ $(x > 5)$, the point (x, y) is located in Quadrant IV.

73. $h(x) = -\frac{2}{3}x^2$



The graph of h is a reflection in the x-axis and a vertical shrink of the graph of $y = x^2$.

75. $g(x) = (2x)^2 - 1 = 4x^2 - 1$



The graph of g is a horizontal shrink and a vertical shift 1 unit downward of the graph of $y = x^2$.

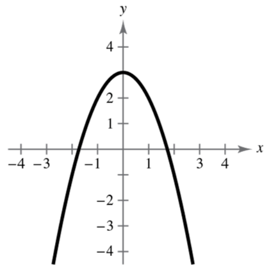
77. $f(x) = 1 + 2x^3$

Domain: All real numbers x

79. $f(x) = \frac{1}{2x^2 + 6x} = \frac{1}{2x(x + 3)}$

Domain: All real numbers x such that $x \neq 0$ and $x \neq -3$

81.



Review Exercises for Chapter 3

1. $f(x) = 0.3^x$

$f(1.5) = 0.3^{1.5} \approx 0.164$

3. $f(x) = 2^x$

$f\left(\frac{2}{3}\right) = 2^{2/3} \approx 1.587$

5. $f(x) = 7(0.2^x)$

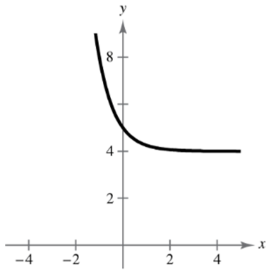
$f(-\sqrt{11}) = 7(0.2^{-\sqrt{11}})$

≈ 1456.529

7. $f(x) = 4^{-x} + 4$

Horizontal asymptote: $y = 4$

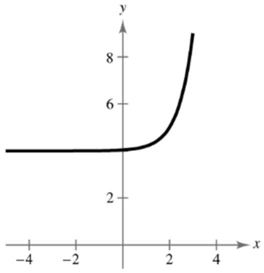
x	−1	0	1	2	3
$f(x)$	8	5	4.25	4.063	4.016



9. $f(x) = 5^{x-2} + 4$

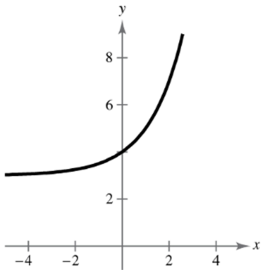
Horizontal asymptote: $y = 4$

x	−1	0	1	2	3
$f(x)$	4.008	4.04	4.2	5	9



11. $f(x) = \left(\frac{1}{2}\right)^{-x} + 3 = 2^x + 3$

Horizontal asymptote: $y = 3$



13.
$$\left(\frac{1}{3}\right)^{x-3} = 9$$
$$\left(\frac{1}{3}\right)^{x-3} = 3^2$$
$$\left(\frac{1}{3}\right)^{x-3} = \left(\frac{1}{3}\right)^{-2}$$
$$x - 3 = -2$$
$$x = 1$$

15.
$$e^{3x-5} = e^7$$
$$3x - 5 = 7$$
$$3x = 12$$
$$x = 4$$

17. $f(x) = 5^x$, $g(x) = 5^x + 1$

Because $g(x) = f(x) + 1$, the graph of g can be obtained by shifting the graph of f one unit upward.

19. $f(x) = 3^x$, $g(x) = 1 - 3^x$

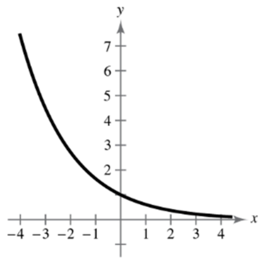
Because $g(x) = 1 - f(x)$, the graph of g can be obtained by reflecting the graph of f in the x-axis and shifting the graph one unit upward. (**Note:** This is equivalent to shifting the graph of f one unit upward and then reflecting the graph in the x-axis.)

21.
$$f(x) = e^x$$
$$f(3.4) = e^{3.4} \approx 29.964$$

23. $f(x) = e^x$
$$f\left(\frac{3}{5}\right) = e^{3/5} \approx 1.822$$

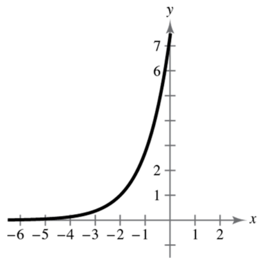
25. $h(x) = e^{-x/2}$

x	−2	−1	0	1	2
$h(x)$	2.72	1.65	1	0.61	0.37



27. $f(x) = e^{x+2}$

x	−3	−2	−1	0	1
$f(x)$	0.37	1	2.72	7.39	20.09



29. $F(t) = 1 - e^{-t/3}$

(a) $F(1) \approx 0.283$

(b) $F(2) \approx 0.487$

(c) $F(5) \approx 0.811$

31. $P = \$5000, r = 3\%, t = 10$ years

Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt} = 5000\left(1 + \frac{0.03}{n}\right)^{10n}$

Compounded continuously: $A = Pe^{rt} = 5000e^{0.03(10)}$

n	1	2	4	12	365	Continuous
A	\$6719.58	\$6734.28	\$6741.74	\$6746.77	\$6749.21	\$6749.29

33. $3^3 = 27$

$\log_3 27 = 3$

35. $e^{0.8} = 2.2255\ldots$

$\ln 2.2255\ldots = 0.8$

37. $f(x) = \log x$

$f(1000) = \log 1000$

$= \log 10^3 = 3$

39. $g(x) = \log_2 x$

$g\left(\frac{1}{4}\right) = \log_2 \frac{1}{4}$

$= \log_2 2^{-2} = -2$

41. $\log_4(x + 7) = \log_4 14$

$x + 7 = 14$

$x = 7$

43. $\ln(x + 9) = \ln 4$

$x + 9 = 4$

$x = -5$

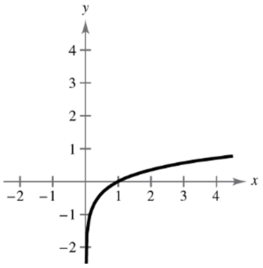
45. $g(x) = \log_7 x \Rightarrow x = 7^y$

Domain: $(0, \infty)$

x-intercept: $(1, 0)$

Vertical asymptote: $x = 0$

x	$\frac{1}{7}$	1	7	49
$g(x)$	-1	0	1	2



47. $f(x) = 4 - \log(x + 5)$

Domain: $(-5, \infty)$

Because $4 - \log(x + 5) = 0 \Rightarrow \log(x + 5) = 4$

$x + 5 = 10^4$

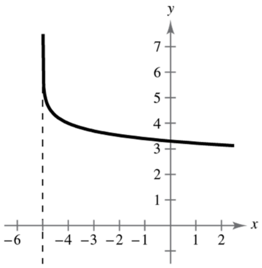
$x = 10^4 - 5$

$= 9995.$

x-intercept: $(9995, 0)$

Vertical asymptote: $x = -5$

x	-4	-3	-2	-1	0	1
$f(x)$	4	3.70	3.52	3.40	3.30	3.22



49. $f(22.6) = \ln 22.6 \approx 3.118$

51. $f\left(\sqrt{e}\right) = \frac{1}{2} \ln \sqrt{e} = 0.25$

53. $f(x) = \ln x + 6 = 6 + \ln x$

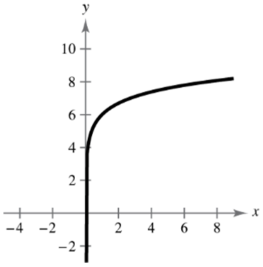
Domain: $(0, \infty)$

$$\begin{aligned} \ln x + 6 &= 0 \\ \ln x &= -6 \\ x &= e^{-6} \end{aligned}$$

x-intercept: $\left(e^{-6}, 0\right)$

Vertical asymptote: $x = 0$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$f(x)$	4.613	5.037	6	6.693	7.098



55. $f(x) = \ln(x - 6)$

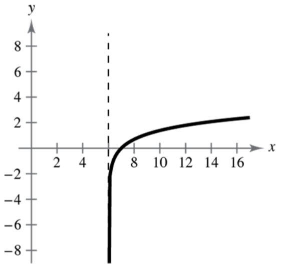
Domain: $(6, \infty)$

$$\begin{aligned} \ln(x - 6) &= 0 \\ x - 6 &= e^{0} \\ x - 6 &= 1 \\ x &= 7 \end{aligned}$$

x-intercept: $(7, 0)$

Vertical asymptote: $x = 6$

x	6.5	7	8	9	10
$f(x)$	−0.693	0	0.693	1.099	1.386



57. $M = m - 5\log\left(\dfrac{d}{10}\right)$

Let $m = 2.08$ and $M = 1.3$ and solve for d.

$$\begin{aligned} 1.3 &= 2.08 - 5\log\left(\frac{d}{10}\right) \\ -0.78 &= -5\log\left(\frac{d}{10}\right) \\ 0.156 &= \log\left(\frac{d}{10}\right) \\ 10^{0.156} &= 10^{\log(d/10)} \\ 10^{0.156} &= \frac{d}{10} \\ 10 \cdot 10^{0.156} &= d \\ d &= 10^{1.156} \approx 14.32 \text{ parsecs} \end{aligned}$$

59. (a) $\log_2 6 = \dfrac{\log 6}{\log 2} \approx 2.585$

(b) $\log_2 6 = \dfrac{\ln 6}{\ln 2} \approx 2.585$

61. (a) $\log_{1/2} 5 = \dfrac{\log 5}{\log(1/2)} \approx -2.322$

(b) $\log_{1/2} 5 = \dfrac{\ln 5}{\ln(1/2)} \approx -2.322$

63. $\log_2 \frac{5}{3} = \log_2 5 - \log_2 3$

65.
$$\begin{aligned} \log_2 \tfrac{9}{5} &= \log_2 9 - \log_2 5 \\ &= \log_2 3^2 - \log_2 9 \\ &= 2\log_2 3 - \log_2 5 \end{aligned}$$

67.
$$\begin{aligned} \log 7x^2 &= \log 7 + \log x^2 \\ &= \log 7 + 2\log x \end{aligned}$$

69.
$$\begin{aligned} \log_3 \frac{9}{\sqrt{x}} &= \log_3 9 - \log_3 \sqrt{x} \\ &= \log_3 3^2 - \log_3 x^{1/2} \\ &= 2 - \frac{1}{2}\log_3 x \end{aligned}$$

71.
$$\begin{aligned} \ln x^2y^2z &= \ln x^2 + \ln y^2 + \ln z \\ &= 2\ln x + 2\ln y + \ln z \end{aligned}$$

73. $\ln 7 + \ln x = \ln(7x)$

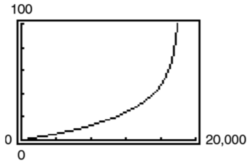
75. $$\log x - \frac{1}{2}\log y = \log x - \log y^{1/2}$$
$$= \log\left(\frac{x}{\sqrt{y}}\right)$$

77. $$\frac{1}{2}\log_3 x - 2\log_3(y+8) = \log_3 x^{1/2} - \log_3(y+8)^2$$
$$= \log_3 \sqrt{x} - \log_3(y+8)^2$$
$$= \log_3 \frac{\sqrt{x}}{(y+8)^2}$$

79. $$t = 50\log\frac{18{,}000}{18{,}000 - h}$$

(a) Domain: $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

(c) As the plane approaches its absolute ceiling, it climbs at a slower rate, so the time required increases.

(d) $50\log\frac{18{,}000}{18{,}000 - 4000} \approx 5.46$ minutes

81. $$5^x = 125$$
$$5^x = 5^3$$
$$x = 3$$

83. $$e^x = 3$$
$$x = \ln 3 \approx 1.099$$

85. $$\ln x = 4$$
$$x = e^4 \approx 54.598$$

87. $$e^{4x} = e^{x^2+3}$$
$$4x = x^2 + 3$$
$$0 = x^2 - 4x + 3$$
$$0 = (x-1)(x-3)$$
$$x = 1, x = 3$$

89. $$2^x - 3 = 29$$
$$2^x = 32$$
$$2^x = 2^5$$
$$x = 5$$

91. $$\ln 3x = 8.2$$
$$e^{\ln 3x} = e^{8.2}$$
$$3x = e^{8.2}$$
$$x = \frac{e^{8.2}}{3} \approx 1213.650$$

93. $$\ln x + \ln(x-3) = 1$$
$$\ln[x(x-3)] = 1$$
$$\ln(x^2 - 3x) = 1$$
$$e^{\ln(x^2-3x)} = e^1$$
$$x^2 - 3x - e = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-e)}}{2(1)}$$
$$x = \frac{3 \pm \sqrt{9+4e}}{2}$$
$$x = \frac{3 + \sqrt{9+4e}}{2} \approx 3.729$$
$x = \frac{3 - \sqrt{9+4e}}{2}$ is extraneous since the domain of the $\ln x$ term is $x > 0$.

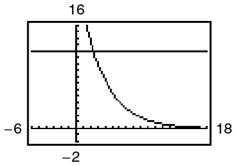
95.
$$\begin{aligned}\log_8(x-1) &= \log_8(x-2) - \log_8(x+2)\\ \log_8(x-1) &= \log_8\left(\frac{x-2}{x+2}\right)\\ x-1 &= \frac{x-2}{x+2}\\ (x-1)(x+2) &= x-2\\ x^2+x-2 &= x-2\\ x^2 &= 0\\ x &= 0\end{aligned}$$

Because $x = 0$ is not in the domain of $\log_8(x-1)$ or of $\log_8(x-2)$, it is an extraneous solution. The equation has no solution.

97.
$$\begin{aligned}\log(1-x) &= -1\\ 1-x &= 10^{-1}\\ 1-\tfrac{1}{10} &= x\\ x &= 0.900\end{aligned}$$

99. $25e^{-0.3x} = 12$

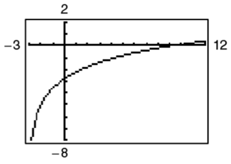
Graph $y_1 = 25e^{-0.3x}$ and $y_2 = 12$.



The graphs intersect at $x \approx 2.447$.

101. $2\ln(x+3) - 5 = 0$

Graph $y_1 = 2\ln(x+3) - 5$.



The x-intercept is at $x \approx 9.182$.

$$\begin{aligned}2\ln(x+3) &= 5\\ \ln(x+3) &= \tfrac{5}{2}\\ x+3 &= e^{5/2}\\ x &= e^{5/2} - 3 \approx 9.182\end{aligned}$$

103. $P = 8500,\ A = 3(8500) = 25{,}500,\ r = 1.5\%$

$$\begin{aligned}A &= Pe^{rt}\\ 25{,}500 &= 8500e^{0.015t}\\ 3 &= e^{0.015t}\\ \ln 3 &= 0.015t\\ t &= \frac{\ln 3}{0.015} \approx 73.2 \text{ years}\end{aligned}$$

105. $y = 3e^{-2x/3}$

Exponential decay model

Matches graph (e).

106. $y = 4e^{2x/3}$

Exponential growth model

Matches graph (b).

107. $y = \ln(x+3)$

Logarithmic model

Vertical asymptote: $x = -3$

Graph includes $(-2, 0)$

Matches graph (f).

108. $y = 7 - \log(x+3)$

Logarithmic model

Vertical asymptote: $x = -3$

Matches graph (d).

109. $y = 2e^{-(x+4)^{2}/3}$

Gaussian model

Matches graph (a).

110. $y = \dfrac{6}{1 + 2e^{-2x}}$

Logistics growth model

Matches graph (c).

111. $y = ae^{bx}$

Using the point $(0, 2)$, you have

$$2 = ae^{b(0)}$$
$$2 = ae^{0}$$
$$2 = a(1)$$
$$2 = a$$

Then, using the point $(4, 3)$, you have

$$3 = 2e^{b(4)}$$
$$3 = 2e^{4b}$$
$$\tfrac{3}{2} = e^{4b}$$
$$\ln \tfrac{3}{2} = 4b$$
$$\tfrac{1}{4}\ln\left(\tfrac{3}{2}\right) = b$$

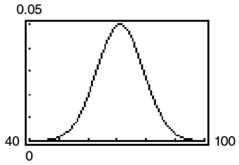
So, $y = 2e^{\frac{1}{4}\ln\left(\frac{3}{2}\right)x}$

or

$y = 2e^{0.1014x}$

113. $y = 0.0499e^{-(x-71)^2/128}, \ 40 \le x \le 100$

Graph $y_1 = 0.0499e^{-(x-71)^2/128}$.



The average test score is 71.

115. $$\beta = 10\log\left(\frac{I}{10^{-12}}\right)$$
$$\frac{\beta}{10} = \log\left(\frac{I}{10^{-12}}\right)$$
$$10^{\beta/10} = \frac{I}{10^{-12}}$$
$$I = 10^{\beta/10-12}$$

(a) $\beta = 60$

$I = 10^{60/10-12}$

$= 10^{-6}$ watt/m^2

(b) $\beta = 135$

$I = 10^{135/10-12}$

$= 10^{1.5}$

$= 10\sqrt{10}$ watts/m^2

(c) $\beta = 1$

$I = 10^{1/10-12}$

$= 10^{\frac{1}{10}} \times 10^{-12}$

$\approx 1.259 \times 10^{-12}$ watt/m^2

117. True. By the inverse properties, $\log_b b^{2x} = 2x$.

119. $y = \log_a x \Rightarrow a^y = x$, so, for example, if $a = -2$, there is no value of y for which $(-2)^y = -4$. If $a = 1$, then every power of a is equal to 1, so x could only be 1. So, $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

Problem Solving for Chapter 3

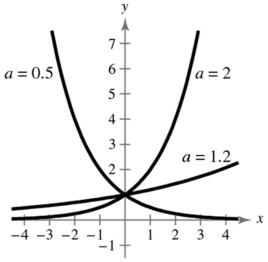
1. $y = a^x$

$y_1 = 0.5^x$

$y_2 = 1.2^x$

$y_3 = 2.0^x$

$y_4 = x$



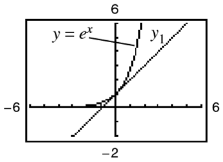
The curves $y = 0.5^x$ and $y = 1.2^x$ cross the line $y = x$. From checking the graphs it appears that $y = x$ will cross $y = a^x$ for $0 \le a \le 1.44$.

3. The exponential function, $y = e^x$, increases at a faster rate than the polynomial $y = x^n$.

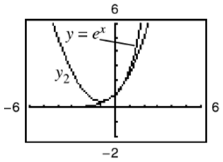
5. (a) $f(u + v) = a^{u+v} = a^u \cdot a^v = f(u) \cdot f(v)$

(b) $f(2x) = a^{2x} = \left(a^x\right)^2 = \left[f(x)\right]^2$

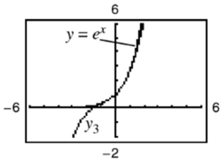
7. (a)



(b)

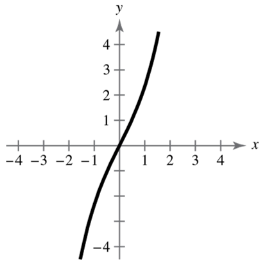


(c)



9.

$$f(x) = e^x - e^{-x}$$
$$y = e^x - e^{-x}$$
$$x = e^y - e^{-y}$$
$$x = \frac{e^{2y} - 1}{e^y}$$
$$xe^y = e^{2y} - 1$$
$$e^{2y} - xe^y - 1 = 0$$
$$e^y = \frac{x \pm \sqrt{x^2 + 4}}{2} \quad \text{Quadratic Formula}$$



Choosing the positive quantity for e^y you have

$$y = \ln\left(\frac{x + \sqrt{x^2 + 4}}{2}\right). \text{ So,}$$

$$f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2 + 4}}{2}\right).$$

11. Answer (c). $y = 6\left(1 - e^{-x^2/2}\right)$

The graph passes through (0, 0) and neither (a) nor (b) pass through the origin. Also, the graph has y-axis symmetry and a horizontal asymptote at $y = 6$.

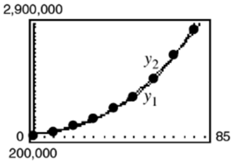
13. $y_1 = c_1\left(\frac{1}{2}\right)^{t/k_1}$ and $y_2 = c_2\left(\frac{1}{2}\right)^{t/k_2}$

$$c_1\left(\frac{1}{2}\right)^{t/k_1} = c_2\left(\frac{1}{2}\right)^{t/k_2}$$
$$\frac{c_1}{c_2} = \left(\frac{1}{2}\right)^{(t/k_2 - t/k_1)}$$
$$\ln\left(\frac{c_1}{c_2}\right) = \left(\frac{t}{k_2} - \frac{t}{k_1}\right)\ln\left(\frac{1}{2}\right)$$
$$\ln c_1 - \ln c_2 = t\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\ln\left(\frac{1}{2}\right)$$
$$t = \frac{\ln c_1 - \ln c_2}{\left[(1/k_2) - (1/k_1)\right]\ln(1/2)}$$

15. (a) $y_1 \approx 252{,}606(1.0310)^t$

(b) $y_2 \approx 400.88t^2 - 1464.6t + 291{,}782$

(c)



(d) The exponential model is a better fit for the data, but neither would be reliable to predict the population of the United States in 2025. The exponential model approaches infinity rapidly.

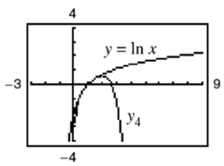
17.

$$(\ln x)^2 = \ln x^2$$
$$(\ln x)^2 - 2\ln x = 0$$
$$\ln x(\ln x - 2) = 0$$
$$\ln x = 0 \quad \text{or} \quad \ln x = 2$$
$$x = 1 \quad \text{or} \quad x = e^2$$

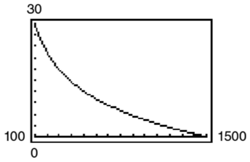
19. $y_4 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

The pattern implies that

$$\ln x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \tfrac{1}{4}(x - 1)^4 + \dots.$$

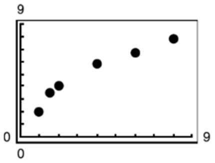


21. $y = 80.4 - 11 \ln x$



$y(300) = 80.4 - 11 \ln 300 \approx 17.7 \text{ ft}^3/\text{min}$

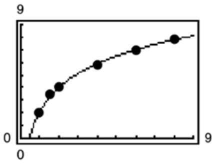
23. (a)



(b) The data could best be modeled by a logarithmic model.

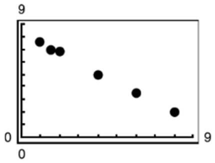
(c) The shape of the curve looks much more logarithmic than linear or exponential.

(d) $y \approx 2.1518 + 2.7044 \ln x$



(e) The model is a good fit to the actual data.

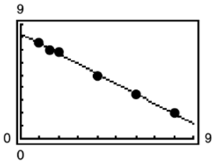
25. (a)



(b) The data could best be modeled by a linear model.

(c) The shape of the curve looks much more linear than exponential or logarithmic.

(d) $y \approx -0.7884x + 8.2566$



(e) The model is a good fit to the actual data.

Practice Test for Chapter 3

1. Solve for x: $x^{3/5} = 8$.

2. Solve for x: $3^{x-1} = \frac{1}{81}$.

3. Graph $f(x) = 2^{-x}$.

4. Graph $g(x) = e^x + 1$.

5. If \$5000 is invested at 9% interest, find the amount after three years if the interest is compounded
   (a) monthly.
   (b) quarterly.
   (c) continuously.

6. Write the equation in logarithmic form: $7^{-2} = \frac{1}{49}$.

7. Solve for x: $x - 4 = \log_2 \frac{1}{64}$.

8. Given $\log_b 2 = 0.3562$ and $\log_b 5 = 0.8271$, evaluate $\log_b \sqrt[4]{8/25}$.

9. Write $5 \ln x - \frac{1}{2} \ln y + 6 \ln z$ as a single logarithm.

10. Using your calculator and the change of base formula, evaluate $\log_9 28$.

11. Use your calculator to solve for N: $\log_{10} N = 0.6646$

12. Graph $y = \log_4 x$.

13. Determine the domain of $f(x) = \log_3(x^2 - 9)$.

14. Graph $y = \ln(x - 2)$.

15. True or false: $\dfrac{\ln x}{\ln y} = \ln(x - y)$

16. Solve for x: $5^x = 41$

17. Solve for x: $x - x^2 = \log_5 \frac{1}{25}$

18. Solve for x: $\log_2 x + \log_2(x - 3) = 2$

19. Solve for x: $\dfrac{e^x + e^{-x}}{3} = 4$

20. Six thousand dollars is deposited into a fund at an annual interest rate of 13%. Find the time required for the investment to double if the interest is compounded continuously.

CHAPTER 4
Trigonometry

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CHAPTER 4
Trigonometry

Section 4.1 Radian and Degree Measure

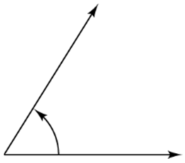
1. coterminal

3. linear, angular

5. Two complementary angles sum to $\frac{\pi}{2}$.

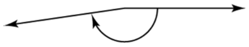
Two supplementary angles sum to π.

7.



The angle shown is approximately 1 radian.

9.

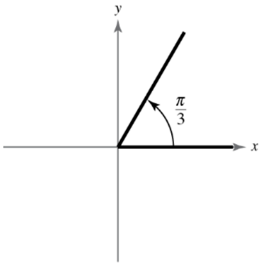


The angle shown is approximately -3 radians.

11. (a) Because $0 < \frac{\pi}{4} < \frac{\pi}{2}, \frac{\pi}{4}$ lies in Quadrant I.

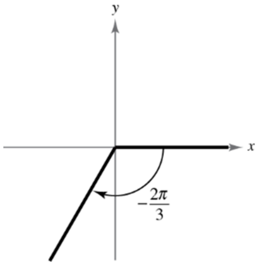
(b) Because $-\frac{5\pi}{4}$ is coterminal with $\frac{3\pi}{4}$ and $\frac{\pi}{2} < \frac{3\pi}{4} < \pi, -\frac{5\pi}{4}$ lies in Quadrant II.

13. (a) $\frac{\pi}{3}$



The angle is not quadrantal. It lies in Quadrant I.

(b) $-\frac{2\pi}{3}$



The angle is not quadrantal. It lies in Quadrant III.

15. *Sample answers*:

(a) $\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$

$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$

(b) $-\frac{5\pi}{6} + 2\pi = \frac{7\pi}{6}$

$-\frac{5\pi}{6} - 2\pi = -\frac{17\pi}{6}$

17. (a) Complement: $\frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

Supplement: $\pi - \frac{\pi}{12} = \frac{11\pi}{12}$

(b) Complement: Not possible, $\frac{11\pi}{12}$ is greater than $\frac{\pi}{2}$.

Supplement: $\pi - \frac{11\pi}{12} = \frac{\pi}{12}$

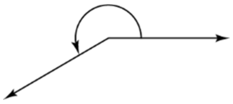
19. (a) Complement: $\frac{\pi}{2} - 1 \approx 0.57$

Supplement: $\pi - 1 \approx 2.14$

(b) Complement: Not possible, 2 is greater than $\frac{\pi}{2}$.

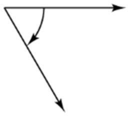
Supplement: $\pi - 2 \approx 1.14$

21.



The angle shown is approximately 210°.

23.

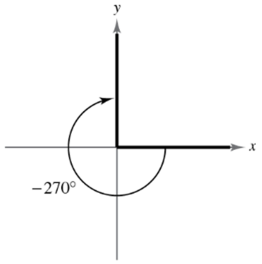


The angle shown is approximately $-60°$.

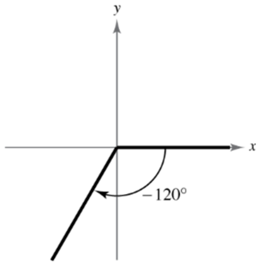
25. (a) Because $90° < 130° < 180°, 130°$ lies in Quadrant II.

(b) Because $-8.3°$ is coterminal with $351.7°$ and $270° < 351.7° < 360°, -8.3°$ lies in Quadrant IV.

27. (a) $-270°$



(b) $-120°$



29. (a) Coterminal angles for $120°$

$120° + 360° = 480°$

$120° - 360° = -240°$

(b) Coterminal angles for $-210°$

$-210° + 360° = 150°$

$-210° - 360° = -570°$

31. (a) Complement: $90° - 18° = 72°$

Supplement: $180° - 18° = 162°$

(b) Complement: $90° - 85° = 5°$

Supplement: $180° - 85° = 95°$

33. (a) Complement: $90° - 24° = 66°$

Supplement: $180° - 24° = 156°$

(b) Complement: Not possible. $126°$ is greater than $90°$.

Supplement: $180° - 126° = 54°$

35. (a) $120° = 120°\left(\frac{\pi}{180°}\right) = -\frac{2\pi}{3}$

(b) $-20° = -20\left(\frac{\pi}{180°}\right) = -\frac{\pi}{9}$

37. (a) $\frac{3\pi}{2} = \frac{3\pi}{2}\left(\frac{180°}{\pi}\right) = 270°$

(b) $-\frac{7\pi}{6} = -\frac{7\pi}{6}\left(\frac{180°}{\pi}\right) = -210°$

39. $45° = 45\left(\frac{\pi}{180°}\right) \approx 0.785$ radian

41. $-0.54° = -0.54°\left(\frac{\pi}{180°}\right) \approx -0.009$ radian

43. $\frac{5\pi}{11} = \frac{5\pi}{11}\left(\frac{180°}{\pi}\right) \approx 81.818°$

45. $-4.2\pi = -4.2\pi\left(\frac{180°}{\pi}\right) = -756°$

47. (a) $54°45' = 54° + \left(\frac{45}{60}\right)^{\circ} = 54.75°$

(b) $-128°30' = -128^{\circ} - \left(\frac{30}{60}\right)^{\circ} = -128.5°$

49. (a) $240.6° = 240° + 0.6(60)' = 240°36'$

(b) $-145.8° = -\left[145° + 0.8(60')\right] = -145°48'$

51. $r = 15$ inches, $\theta = 120°$

$s = r\theta$

$s = 15(120°)\left(\frac{\pi}{180°}\right) = 10\pi$ inches

≈ 31.42 inches

53. $r = 80$ kilometers, $s = 150$ kilometers

$s = r\theta$

$150 = 80\theta$

$\theta = \frac{150}{80} = \frac{15}{8}$ radians

55. $s = r\theta$

$28 = 7\theta$

$\theta = 4$ radians

57. $r = 6$ inches, $\theta = \frac{\pi}{3}$

$A = \frac{1}{2}r^2\theta = \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) = 6\pi \text{ in.}^2 \approx 18.85 \text{ in.}^2$

59. The angle in degrees should be multiplied by $\frac{\pi}{180°}$.

$20° = (20°)\left(\frac{\pi \text{ rad}}{180°}\right) = \frac{\pi}{9}$ radians.

61. $\theta = 41^\circ 15' 50'' - 32^\circ 47' 9'' \approx 8.47806^\circ \approx 0.14797$ radian

$s = r\theta \approx 4000(0.14782) \approx 592$ miles

63. $\theta = \dfrac{s}{r} = \dfrac{2.5}{6} = \dfrac{25}{60} = \dfrac{5}{12}$ radian $\approx 23.87^\circ$

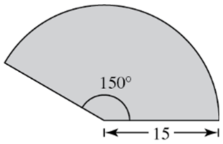
65. (a) 4 rpm $= 4(2\pi)$ radians/minute $= 8\pi \approx 25$ radians/minute

(b) $r = 25$ ft

$\dfrac{r\theta}{t} = 200\pi$ ft/minute

Linear speed $\approx 25(25.13)$ ft/minute ≈ 628.3 ft/minute

67.
$$\begin{aligned} A &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}(15)^2(150^\circ)\left(\frac{\pi}{180^\circ}\right) \\ &= 93.75\pi \text{ m}^2 \\ &\approx 294.52 \text{ m}^2 \end{aligned}$$



69. False. $\dfrac{180^\circ}{\pi}$ is in degree measure.

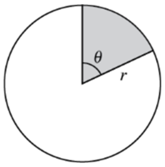
71. True. If α and β are coterminal angles, then $\alpha = \beta + n(360^\circ)$ or $\alpha = \beta + n(2\pi)$, where n is an integer. The difference between α and β is $\alpha - \beta = n(360^\circ)$, or $\alpha - \beta = n(2\pi)$ if expressed in radians.

73. Area of circle $= \pi r^2$

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Measure of central angle of sector}}{\text{Measure of central angle of circle}}$$

$$\frac{\text{Area of sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Area of sector} = \left(\pi r^2\right)\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$$



75.
$$\begin{aligned} x^2 &= 7^2 + 24^2 \\ x^2 &= 625 \\ x &= 25 \end{aligned}$$

77.
$$\begin{aligned} x^2 &= 13^2 - 5^2 \\ x^2 &= 144 \\ x &= 12 \end{aligned}$$

79.
$$\begin{aligned} \frac{10}{4+i} &= \frac{10}{4+i} \bullet \frac{4-i}{4-i} \\ &= \frac{10(4-i)}{16+1} \\ &= \frac{40-10i}{17} \\ &= \frac{40}{17} - \frac{10}{17}i \end{aligned}$$

81.
$$\begin{aligned} \frac{-8i}{(2-4i)^2} &= \frac{-8i}{-12-16i} \\ &= \frac{2i}{3+4i} \\ &= \frac{2i}{3+4i} \bullet \frac{3-4i}{3-4i} \\ &= \frac{8+6i}{9+16} \\ &= \frac{8}{25} + \frac{6}{25}i \end{aligned}$$

Section 4.2 Trigonometric Functions: The Unit Circle

1. unit circle

3. odd; even

5. $x = \frac{12}{13}, y = \frac{5}{13}$

$$\sin t = y = \frac{5}{13} \qquad \csc t = \frac{1}{y} = \frac{13}{5}$$

$$\cos t = x = \frac{12}{13} \qquad \sec t = \frac{1}{x} = \frac{13}{12}$$

$$\tan t = \frac{y}{x} = \frac{5}{12} \qquad \cot t = \frac{x}{y} = \frac{12}{5}$$

7. $x = -\frac{4}{5}, y = -\frac{3}{5}$

$$\sin t = y = -\frac{3}{5} \qquad \csc t = \frac{1}{y} = -\frac{5}{3}$$

$$\cos t = x = -\frac{4}{5} \qquad \sec t = \frac{1}{x} = -\frac{5}{4}$$

$$\tan t = \frac{y}{x} = \frac{3}{4} \qquad \cot t = \frac{x}{y} = \frac{4}{3}$$

9. $t = \frac{\pi}{2}$ corresponds to the point $(x, y) = (0, 1)$.

11. $t = \frac{5\pi}{6}$ corresponds to the point

$(x, y) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

13. $t = \frac{\pi}{4}$ corresponds to the point

$(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$\sin \frac{\pi}{4} = y = \frac{\sqrt{2}}{2}$$

$$\cos \frac{\pi}{4} = x = \frac{\sqrt{2}}{2}$$

$$\tan \frac{\pi}{4} = \frac{y}{x} = 1$$

15. $t = -\frac{\pi}{6}$ corresponds to $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

$$\sin -\frac{\pi}{6} = y = -\frac{1}{2}$$

$$\cos -\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\tan -\frac{\pi}{6} = \frac{y}{x} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

17. $t = -\frac{7\pi}{4}$ corresponds to the point

$(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$\sin\left(-\frac{7\pi}{4}\right) = y = \frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{7\pi}{4}\right) = x = \frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{7\pi}{4}\right) = \frac{y}{x} = 1$$

19. $t = \frac{11\pi}{6}$ corresponds to the point

$(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

$$\sin \frac{11\pi}{6} = y = -\frac{1}{2}$$

$$\cos \frac{11\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\tan \frac{11\pi}{6} = \frac{y}{x} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

21. $t = -\frac{3\pi}{2}$ corresponds to the point $(x, y) = (0, 1)$.

$$\sin\left(-\frac{3\pi}{2}\right) = y = 1$$

$$\cos\left(-\frac{3\pi}{2}\right) = x = 0$$

$$\tan\left(-\frac{3\pi}{2}\right) = \frac{y}{x} \text{ is undefined.}$$

23. $t = \frac{2\pi}{3}$ corresponds to the point $(x, y) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\sin\frac{2\pi}{3} = y = \frac{\sqrt{3}}{2}$ $\qquad$ $\csc\frac{2\pi}{3} = \frac{1}{y} = \frac{2\sqrt{3}}{3}$

$\cos\frac{2\pi}{3} = x = -\frac{1}{2}$ $\qquad$ $\sec\frac{2\pi}{3} = \frac{1}{x} = -2$

$\tan\frac{2\pi}{3} = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$ $\qquad$ $\cot\frac{2\pi}{3} = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3}$

25. $t = \frac{4\pi}{3}$ corresponds to the point $(x, y) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$\sin\frac{4\pi}{3} = y = -\frac{\sqrt{3}}{2}$ $\qquad$ $\csc\frac{4\pi}{3} = \frac{1}{y} = -\frac{2\sqrt{3}}{3}$

$\cos\frac{4\pi}{3} = x = -\frac{1}{2}$ $\qquad$ $\sec\frac{4\pi}{3} = \frac{1}{x} = -2$

$\tan\frac{4\pi}{3} = \frac{y}{x} = \sqrt{3}$ $\qquad$ $\cot\frac{4\pi}{3} = \frac{x}{y} = \frac{\sqrt{3}}{3}$

27. $t = -\frac{5\pi}{3}$ corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\sin\left(-\frac{5\pi}{3}\right) = y = \frac{\sqrt{3}}{2}$ $\qquad$ $\csc\left(-\frac{5\pi}{3}\right) = \frac{1}{y} = \frac{2}{\sqrt{3}}$

$\cos\left(-\frac{5\pi}{3}\right) = x = \frac{1}{2}$ $\qquad$ $\sec\left(-\frac{5\pi}{3}\right) = \frac{1}{x} = 2$

$\tan\left(-\frac{5\pi}{3}\right) = \frac{y}{x} = \sqrt{3}$ $\qquad$ $\cot\left(-\frac{5\pi}{3}\right) = \frac{x}{y} = \frac{\sqrt{3}}{3}$

29. $t = -\frac{\pi}{2}$ corresponds to the point $(x, y) = (0, -1)$.

$\sin\left(-\frac{\pi}{2}\right) = y = -1$ $\qquad$ $\csc\left(-\frac{\pi}{2}\right) = \frac{1}{y} = -1$

$\cos\left(-\frac{\pi}{2}\right) = x = 0$ $\qquad$ $\sec\left(-\frac{\pi}{2}\right) = \frac{1}{x}$ is undefined.

$\tan\left(-\frac{\pi}{2}\right) = \frac{y}{x}$ is undefined. $\qquad$ $\cot\left(-\frac{\pi}{2}\right) = \frac{x}{y} = 0$

31. $\sin 4\pi = \sin 0 = 0$

33. $\cos\frac{7\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$

35. $\sin\frac{19\pi}{6} = \sin\frac{7\pi}{6} = -\frac{1}{2}$

37. $\sin t = \frac{1}{2}$

(a) $\sin(-t) = -\sin t = -\frac{1}{2}$

(b) $\csc(-t) = -\csc t = -2$

39. $\cos(-t) = -\dfrac{1}{5}$

(a) $\cos t = \cos(-t) = -\dfrac{1}{5}$

(b) $\sec(-t) = \dfrac{1}{\cos(-t)} = -5$

41. $\sin t = \frac{4}{5}$

(a) $\sin(\pi - t) = \sin t = \frac{4}{5}$

(b) $\sin(t + \pi) = -\sin t = -\frac{4}{5}$

43. $\sin 0.6 \approx 0.5646$

45. $\tan\dfrac{\pi}{8} \approx 0.4142$

47. $\sec 3.1 = \dfrac{1}{\cos 3.1} \approx -1.0009$

49. $y(t) = \frac{1}{2}\cos 6t$

(a) $y(0) = \frac{1}{2}\cos 0 = 0.5$ foot

(b) $y\left(\frac{1}{4}\right) = \frac{1}{2}\cos\frac{3}{2} \approx 0.04$ foot

(c) $y\left(\frac{1}{2}\right) = \frac{1}{2}\cos 3 \approx -0.49$ foot

51. False. $\sin(-t) = -\sin t$ means the function is odd, not that the sine of a negative angle is a negative number.

For example: $\sin\left(-\dfrac{3\pi}{2}\right) = -\sin\left(\dfrac{3\pi}{2}\right) = -(-1) = 1.$

Even though the angle is negative, the sine value is positive.

53. True. $\tan a = \tan(a - 6\pi)$ because the period of the tangent function is π.

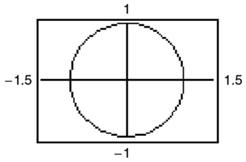
55. (a) The points have y-axis symmetry.

(b) $\sin t_1 = \sin(\pi - t_1)$ because they have the same y-value.

(c) $\cos(\pi - t_1) = -\cos t_1$ because the x-values have the opposite signs.

57. The calculator was in degree mode instead of radian mode. $\tan(\pi/2)$ is undefined.

59. (a)



Circle of radius 1 centered at $(0, 0)$

(b) The t-values represent the central angle in radians. The x- and y-values represent the location in the coordinate plane.

(c) $-1 \le x \le 1, -1 \le y \le 1$

61. Center: $(17,7)$, Radius: 6

$(x - 17)^2 + (y - 7)^2 = 6^2 = 36$

63. Endpoints of diameter: $(-4, -7), (-3, -5)$

Center is midpoint of diameter:

$$\left(\frac{-4-3}{2}, \frac{-7-5}{2}\right) = \left(\frac{-7}{2}, -6\right)$$

Radius is one-half length of diameter:

$$r = \frac{1}{2}\sqrt{(-4+3)^2 + (-7+5)^2} = \frac{1}{2}\sqrt{5}$$

Circle: $\left(x + \dfrac{7}{2}\right)^2 + (y + 6)^2 = \dfrac{5}{4}$

65. $y^2 = 6 - x$

Let $y = 0$.

$0^2 = 6 - x$

$x = 6$

x-intercept: $(6, 0)$

Let $x = 0$.

$y^2 = 6 - (0)$

$y^2 = 6$

$y = \pm\sqrt{6}$

y-intercepts: $\left(0, \pm\sqrt{6}\right)$

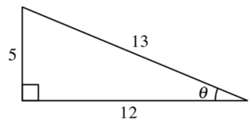
67. $y = 4x^2 - 6x - 4 = 2(x - 2)(2x + 1)$

x-intercepts: $(2, 0), \left(-\frac{1}{2}, 0\right)$

y-intercept: $(0, -4)$

Section 4.3 Right Triangle Trigonometry

1.



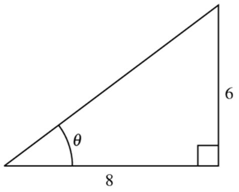
(a) The side opposite θ has length 5.

(b) The side adjacent to θ has length 12.

(c) The hypotenuse has length 13.

3. Complementary

5. $\text{hyp} = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

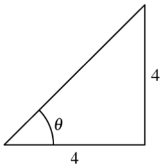


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{6}{10} = \frac{3}{5} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{10}{6} = \frac{5}{3}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{8}{10} = \frac{4}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{10}{8} = \frac{5}{4}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{6}{8} = \frac{3}{4} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{8}{6} = \frac{4}{3}$$

7. $\text{hyp} = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

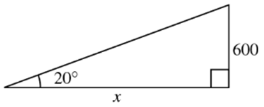


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4\sqrt{2}}{4} = \sqrt{2}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4\sqrt{2}}{4} = \sqrt{2}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{4} = 1 \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{4}{4} = 1$$

9. $\text{adj} = \sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = \sqrt{1600} = 40$

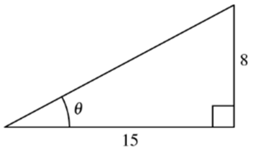


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{9}{41} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{41}{9}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{40}{41} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{41}{40}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{9}{40} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{40}{9}$$

11. $\text{hyp} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$

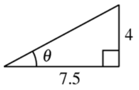


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{17}{8}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{15}{17} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{17}{15}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{8}{15} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{15}{8}$$

$\text{hyp} = \sqrt{7.5^2 + 4^2} = \frac{17}{2}$



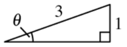
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{(17/2)} = \frac{8}{17} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{(17/2)}{4} = \frac{17}{8}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{7.5}{(17/2)} = \frac{15}{17} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{(17/2)}{7.5} = \frac{17}{15}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{7.5} = \frac{8}{15} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{7.5}{4} = \frac{15}{8}$$

The function values are the same because the triangles are similar, and corresponding sides are proportional.

13. $\text{adj} = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$

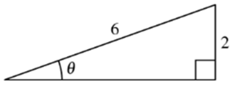


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{3} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = 3$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2\sqrt{2}}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = 2\sqrt{2}$$

$\text{adj} = \sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{6} = \frac{1}{3} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{6}{2} = 3$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{6}{4\sqrt{2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{5\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

The function values are the same since the triangles are similar and the corresponding sides are proportional.

15. Given: $\cos\theta = \frac{15}{17} = \frac{\text{adj}}{\text{hyp}}$

$$(\text{opp})^2 + 15^2 = 17^2$$
$$\text{opp} = \sqrt{289 - 225}$$
$$\text{opp} = \sqrt{64} = 8$$

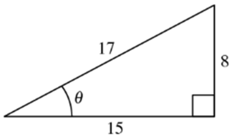
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{8}{15}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{17}{8}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{17}{15}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{15}{8}$$



17. Given: $\sec\theta = \frac{6}{5} = \frac{\text{hyp}}{\text{adj}}$

$$(\text{opp})^2 + 5^2 = 6^2$$
$$\text{opp} = \sqrt{36 - 25} = \sqrt{11}$$

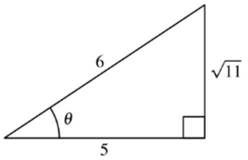
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{11}}{6}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{6}$$

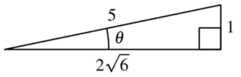
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{11}}{5}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{6\sqrt{11}}{11}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{5\sqrt{11}}{11}$$



19. Given: $\sin\theta = \dfrac{1}{5} = \dfrac{\text{opp}}{\text{hyp}}$



$$1^2 + (\text{adj})^2 = 5^2$$
$$\text{adj} = \sqrt{24} = 2\sqrt{6}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2\sqrt{6}}{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{6}}{12}$$

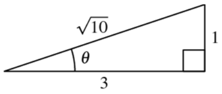
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = 5$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5\sqrt{6}}{12}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = 2\sqrt{6}$$

21. Given: $\cot\theta = 3 = \dfrac{3}{1} = \dfrac{\text{adj}}{\text{opp}}$

$$1^2 + 3^2 = (\text{hyp})^2$$
$$\text{hyp} = \sqrt{10}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{10}}{10}$$

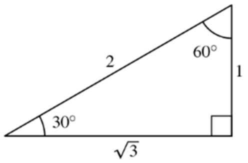
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3\sqrt{10}}{10}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \sqrt{10}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{10}}{3}$$

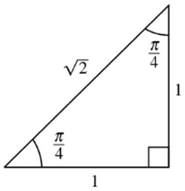
23.



$$30° = 30°\left(\frac{\pi}{180°}\right) = \frac{\pi}{6}\text{ radian}$$

$$\tan 30° = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

25.



$$\frac{\pi}{4} = \frac{\pi}{4}\left(\frac{180°}{\pi}\right) = 45°$$

$$\sec\frac{\pi}{4} = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

27. (a) $\sin 20° \approx 0.3420$

(b) $\cos 70° \approx 0.3420$

29. (a) $\sin 14.21° \approx 0.2455$

(b) $\csc 14.21° = \dfrac{1}{\sin 14.21°} \approx 4.0737$

(b) $\sec 79.56° = \dfrac{1}{\cos 79.56°} \approx 5.5186$

31. (a) $\cos 4°\,50'\,15'' = \cos\left(4 + \dfrac{50}{60} + \dfrac{15}{3600}\right)^{\circ} \approx 0.9964$

(b) $\sec 4°\,50'\,15'' = \dfrac{1}{\cos 4°50'15''} \approx 1.0036$

33. (a) $\cot 17°\,15' = \cot\left(17 + \dfrac{15}{60}\right)^{\circ} = \dfrac{1}{\tan 17.25°} \approx 3.2205$

(b) $\tan 17°\,15' = \tan\left(17 + \dfrac{15}{60}\right)^{\circ} = \tan 17.25° \approx 0.3105$

35. $\sin 60° = \dfrac{\sqrt{3}}{2}, \cos 60° = \dfrac{1}{2}$

(a) $\sin 30° = \cos 60° = \dfrac{1}{2}$

(b) $\cos 30° = \sin 60° = \dfrac{\sqrt{3}}{2}$

(c) $\tan 60° = \dfrac{\sin 60°}{\cos 60°} = \sqrt{3}$

(d) $\cot 60° = \dfrac{\cos 60°}{\sin 60°} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

37. $\cos\theta = \dfrac{1}{3}$

(a) $$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta + \left(\frac{1}{3}\right)^2 &= 1\\ \sin^2\theta &= \frac{8}{9}\\ \sin\theta &= \frac{2\sqrt{2}}{3}\end{aligned}$$

(b) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = 2\sqrt{2}$

(c) $\sec\theta = \dfrac{1}{\cos\theta} = 3$

(d) $\csc(90° - \theta) = \sec\theta = 3$

39. $\cot\alpha = 3$

(a) $\tan\alpha = \dfrac{1}{\cot\alpha} = \dfrac{1}{3}$

(b) $$\begin{aligned}\csc^2\alpha &= 1 + \cot^2\alpha\\ \csc^2\alpha &= 1 + 3^2\\ \csc^2\alpha &= 10\\ \csc\alpha &= \sqrt{10}\end{aligned}$$

(c) $\cot(90° - \alpha) = \tan\alpha = \dfrac{1}{3}$

(d) $\csc\alpha = \sqrt{10}$

$$\sin\alpha = \frac{1}{\csc\alpha} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

41. $\tan\theta\cot\theta = \tan\theta\left(\dfrac{1}{\tan\theta}\right) = 1$

43. $\tan\alpha\cos\alpha = \left(\dfrac{\sin\alpha}{\cos\alpha}\right)\cos\alpha = \sin\alpha$

45. $(1 + \sin\theta)(1 - \sin\theta) = 1 - \sin^2\theta = \cos^2\theta$

47. $$\begin{aligned}(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) &= \sec^2\theta - \tan^2\theta\\ &= (1 + \tan^2\theta) - \tan^2\theta\\ &= 1\end{aligned}$$

49. $$\begin{aligned}\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\\ &= \frac{1}{\sin\theta\cos\theta}\\ &= \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\\ &= \csc\theta\sec\theta\end{aligned}$$

51. (a) $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = 30° = \dfrac{\pi}{6}$

(b) $\csc\theta = 2 \Rightarrow \theta = 30° = \dfrac{\pi}{6}$

53. (a) $\sec\theta = 2 \Rightarrow \theta = 60° = \dfrac{\pi}{3}$

(b) $\cot\theta = 1 \Rightarrow \theta = 45° = \dfrac{\pi}{4}$

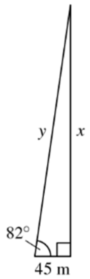
55. (a) $\csc\theta = \dfrac{2\sqrt{3}}{3} \Rightarrow \theta = 60° = \dfrac{\pi}{3}$

(b) $\sin\theta = \dfrac{\sqrt{2}}{2} \Rightarrow \theta = 45° = \dfrac{\pi}{4}$

57. $$\begin{aligned}\cos 60° &= \frac{x}{18}\\ x &= 18\cos 60° = 18\left(\frac{1}{2}\right) = 9\\ \sin 60° &= \frac{y}{18}\\ y &= 18\sin 60° = 18\frac{\sqrt{3}}{2} = 9\sqrt{3}\end{aligned}$$

59. $$\begin{aligned}\tan 60° &= \frac{32}{x}\\ \sqrt{3} &= \frac{32}{x}\\ \sqrt{3}x &= 32\\ x &= \frac{32}{\sqrt{3}} = \frac{32\sqrt{3}}{3}\\ \sin 60° &= \frac{32}{r}\\ r &= \frac{32}{\sin 60°}\\ r &= \frac{32}{\frac{\sqrt{3}}{2}} = \frac{64\sqrt{3}}{3}\end{aligned}$$

61.



$$\tan 82° = \frac{x}{45}$$
$$x = 45\tan 82°$$

Height of the building:

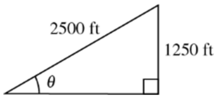
$123 + 45\tan 82° \approx 443.2$ meters

Distance between friends:

$$\cos 82° = \frac{45}{y} \Rightarrow y = \frac{45}{\cos 82°} \approx 323.34 \text{ meters}$$

63. $\sin\theta = \dfrac{1250}{2500} = \dfrac{1}{2}$

$\theta = 30^\circ = \dfrac{\pi}{6}$

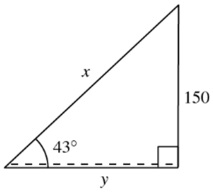


65. (a) $\sin 43^\circ = \dfrac{150}{x}$

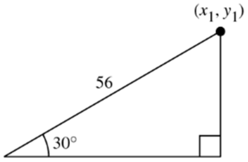
$x = \dfrac{150}{\sin 43^\circ} \approx 219.9$ ft

(b) $\tan 43^\circ = \dfrac{150}{y}$

$y = \dfrac{150}{\tan 43^\circ} \approx 160.9$ ft



67.



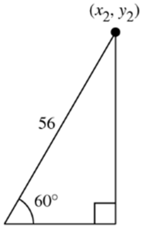
$\sin 30^\circ = \dfrac{y_1}{56}$

$y_1 = (\sin 30^\circ)(56) = \left(\dfrac{1}{2}\right)(56) = 28$

$\cos 30^\circ = \dfrac{x_1}{56}$

$x_1 = \cos 30^\circ(56) = \dfrac{\sqrt{3}}{2}(56) = 28\sqrt{3}$

$(x_1, y_1) = \left(28\sqrt{3}, 28\right)$



$\sin 60^\circ = \dfrac{y_2}{56}$

$y_2 = \sin 60^\circ(56) = \left(\dfrac{\sqrt{3}}{2}\right)(56) = 28\sqrt{3}$

$\cos 60^\circ = \dfrac{x_2}{56}$

$x_2 = (\cos 60^\circ)(56) = \left(\dfrac{1}{2}\right)(56) = 28$

$(x_2, y_2) = \left(28, 28\sqrt{3}\right)$

69. $x \approx 9.397,\ y \approx 3.420$

$\sin 20^\circ = \dfrac{y}{10} \approx 0.34$

$\cos 20^\circ = \dfrac{x}{10} \approx 0.94$

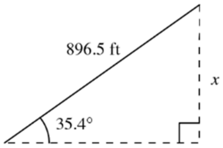
$\tan 20^\circ = \dfrac{y}{x} \approx 0.36$

$\cot 20^\circ = \dfrac{x}{y} \approx 2.75$

$\sec 20^\circ = \dfrac{10}{x} \approx 1.06$

$\csc 20^\circ = \dfrac{10}{y} \approx 2.92$

71. (a)



$\sin 35.4^\circ = \dfrac{x}{896.5}$

$x = 896.5 \sin 35.4^\circ \approx 519.33$ feet

(b) Because the top of the incline is 1693.5 feet above sea level and the vertical rise of the inclined plane is 519.33 feet, the elevation of the lower end of the inclined plan is about
$1693.5 - 519.33 = 1174.17$ feet.

73. $\sin 60^\circ \csc 60^\circ = 1$

True.

$\csc x = \dfrac{1}{\sin x} \Rightarrow \sin 60^\circ \csc 60^\circ = \sin 60^\circ\left(\dfrac{1}{\sin 60^\circ}\right)$

$= 1$

75. False, $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2} \neq 1$

77. False, $\dfrac{\sin 60^\circ}{\sin 30^\circ} = \dfrac{\cos 30^\circ}{\sin 30^\circ} = \cot 30^\circ \approx 1.7321$

$\sin 2^\circ \approx 0.0349$

79. Yes. Given $\tan\theta$, $\sec\theta$ can be found from the identity $1 + \tan^2\theta = \sec^2\theta$.

81. $\cos 60^\circ = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{1}{2}$

83. The point $(1, 3)$ lies in Quadrant I.

85. The point $(-5, 4)$ lies in Quadrant II.

87. $y = -x$ lies in Quadrants II and IV.

Sample points: $(-1, 1), (1, -1)$

89. $2x - y = 0 \Leftrightarrow y = 2x$ lies in Quadrant I and III.

Sample points: $(1, 2), (-1, -2)$

91. (a) Coterminal angles for $\theta = 20°$

$20° + 360° = 380°$

$20° - 360° = -340°$

(b) Coterminal angles for $\theta = -30°$

$-30° + 360° = 330°$

$-30° - 360° = -390°$

93. $f(x) = \dfrac{1}{x^3 - x}$

$$f(-x) = \frac{1}{(-x)^3 - (-x)} = \frac{1}{-x^3 + x} = \frac{-1}{x^3 - x} = -f(x)$$

f is odd and symmetric with respect to the origin.

95. $f(x) = e^{x^2}$

$f(-x) = e^{(-x)^2} = e^{x^2} = f(x)$

f is even and symmetric with respect to the *y*-axis.

97. $f(x) = -\log(-x)$

$f(-x) = -\log x \neq f(x) \neq -f(x)$

Note: Domain is $x < 0$

f is neither even nor odd and has no symmetry.

Section 4.4 Trigonometric Functions of Any Angle

1. $\dfrac{y}{r}$

3. $\dfrac{y}{x}$

5. $\cos\theta$

7. Reference angle

9. (a) $(x, y) = (4, 3)$

$r = \sqrt{16 + 9} = 5$

$\sin\theta = \dfrac{y}{r} = \dfrac{3}{5}$ $\quad\csc\theta = \dfrac{r}{y} = \dfrac{5}{3}$

$\cos\theta = \dfrac{x}{r} = \dfrac{4}{5}$ $\quad\sec\theta = \dfrac{r}{x} = \dfrac{5}{4}$

$\tan\theta = \dfrac{y}{x} = \dfrac{3}{4}$ $\quad\cot\theta = \dfrac{x}{y} = \dfrac{4}{3}$

(b) $(x, y) = (-8, 15)$

$r = \sqrt{64 + 225} = 17$

$\sin\theta = \dfrac{y}{r} = \dfrac{15}{17}$ $\quad\csc\theta = \dfrac{r}{y} = \dfrac{17}{15}$

$\cos\theta = \dfrac{x}{r} = -\dfrac{8}{17}$ $\quad\sec\theta = \dfrac{r}{x} = -\dfrac{17}{8}$

$\tan\theta = \dfrac{y}{x} = -\dfrac{15}{8}$ $\quad\cot\theta = \dfrac{x}{y} = -\dfrac{8}{15}$

11. (a) $(x, y) = (-\sqrt{3}, -1)$

$r = \sqrt{3 + 1} = 2$

$\sin\theta = \dfrac{y}{r} = -\dfrac{1}{2}$ $\quad\csc\theta = \dfrac{r}{y} = -2$

$\cos\theta = \dfrac{x}{r} = -\dfrac{\sqrt{3}}{2}$ $\quad\sec\theta = \dfrac{r}{x} = -\dfrac{2\sqrt{3}}{3}$

$\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{3}$ $\quad\cot\theta = \dfrac{x}{y} = \sqrt{3}$

(b) $(x, y) = (4, -1)$

$r = \sqrt{16 + 1} = \sqrt{17}$

$\sin\theta = \dfrac{y}{r} = -\dfrac{1}{\sqrt{17}} = -\dfrac{\sqrt{17}}{17}$ $\quad\csc\theta = \dfrac{r}{y} = -\sqrt{17}$

$\cos\theta = \dfrac{x}{r} = \dfrac{4}{\sqrt{17}} = \dfrac{4\sqrt{17}}{17}$ $\quad\sec\theta = \dfrac{r}{x} = \dfrac{\sqrt{17}}{4}$

$\tan\theta = \dfrac{y}{x} = -\dfrac{1}{4}$ $\quad\cot\theta = \dfrac{x}{y} = -4$

13. $(x, y) = (5, 12)$

$r = \sqrt{25 + 144} = 13$

$\sin\theta = \frac{y}{r} = \frac{12}{13}$ $\qquad \csc\theta = \frac{r}{y} = \frac{13}{12}$

$\cos\theta = \frac{x}{r} = \frac{5}{13}$ $\qquad \sec\theta = \frac{r}{x} = \frac{13}{5}$

$\tan\theta = \frac{y}{x} = \frac{12}{5}$ $\qquad \cot\theta = \frac{x}{y} = \frac{5}{12}$

15. $x = -5, y = -2$

$r = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$

$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{29}} = -\frac{2\sqrt{29}}{29}$

$\cos\theta = \frac{x}{r} = \frac{-5}{\sqrt{29}} = -\frac{5\sqrt{29}}{29}$

$\tan\theta = \frac{y}{x} = \frac{-2}{-5} = \frac{2}{5}$

$\csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{-2} = -\frac{\sqrt{29}}{2}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{-5} = -\frac{\sqrt{29}}{5}$

$\cot\theta = \frac{x}{y} = \frac{-5}{-2} = \frac{5}{2}$

17. $(x, y) = (-5.4, 7.2)$

$r = \sqrt{29.16 + 51.84} = 9$

$\sin\theta = \frac{y}{r} = \frac{7.2}{9} = \frac{4}{5}$ $\qquad \csc\theta = \frac{r}{y} = \frac{9}{7.2} = \frac{5}{4}$

$\cos\theta = \frac{x}{r} = -\frac{5.4}{9} = -\frac{3}{5}$ $\qquad \sec\theta = \frac{r}{x} = -\frac{9}{5.4} = -\frac{5}{3}$

$\tan\theta = \frac{y}{x} = -\frac{7.2}{5.4} = -\frac{4}{3}$ $\qquad \tan\theta = \frac{x}{y} = -\frac{5.4}{7.2} = -\frac{3}{4}$

19. $\sin\theta > 0 \Rightarrow \theta$ lies in Quadrant I or in Quadrant II.

$\cos\theta > 0 \Rightarrow \theta$ lies in Quadrant I or in Quadrant IV.

$\sin\theta > 0$ and $\cos\theta > 0 \Rightarrow \theta$ lies in Quadrant I.

21. $\sin\theta > 0 \Rightarrow \theta$ lies in Quadrant I or in Quadrant II.

$\cos\theta < 0 \Rightarrow \theta$ lies in Quadrant II or in Quadrant III.

$\sin\theta > 0$ and $\cos\theta < 0 \Rightarrow \theta$ lies in Quadrant II.

23. $\tan\theta > 0$ and $\sin\theta > 0 \Rightarrow \theta$ is in Quadrant I $\Rightarrow x > 0$ and $y > 0$.

$\tan\theta = \frac{y}{x} = \frac{15}{8} \Rightarrow r = 17$

$\sin\theta = \frac{y}{r} = \frac{15}{7}$ $\qquad \sec\theta = \frac{r}{x} = \frac{17}{8}$

$\cos\theta = \frac{x}{r} = \frac{8}{17}$ $\qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$

$\csc\theta = \frac{r}{y} = \frac{17}{15}$

25. $\sin\theta = 0.6 = \frac{3}{5}$ and θ in Quadrant II

$x^2 + y^2 = r^2$

$\sin\theta = \frac{3}{5}, \qquad x = -\sqrt{r^2 - y^2}$

$x = -\sqrt{5^2 - 3^2} = -4$

$\cos\theta = \frac{x}{r} = -\frac{4}{5}$ $\qquad \sec\theta = \frac{r}{x} = -\frac{5}{4}$

$\tan\theta = \frac{y}{x} = -\frac{3}{4}$ $\qquad \cot\theta = \frac{x}{y} = -\frac{4}{3}$

$\csc\theta = \frac{r}{y} = \frac{5}{3}$

27. $\cot\theta = \frac{x}{y} = -\frac{3}{1} = \frac{3}{-1}$

$\cos\theta > 0 \Rightarrow \theta$ is in Quadrant IV $\Rightarrow x$ is positive;

$x = 3, y = -1, r = \sqrt{10}$

$\sin\theta = \frac{y}{r} = -\frac{\sqrt{10}}{10}$ $\qquad \csc\theta = \frac{r}{y} = -\sqrt{10}$

$\cos\theta = \frac{x}{r} = \frac{3\sqrt{10}}{10}$ $\qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{3}$

$\tan\theta = \frac{y}{x} = -\frac{1}{3}$

29. $\cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2} + \pi n$

$\csc\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2} + 2\pi n$

$y = 1, x = 0, r = 1$

$\sin\theta = \dfrac{y}{r} = 1$

$\tan\theta = \dfrac{y}{x}$ is undefined

$\sec\theta$ is undefined

$\cot\theta = 0.$

31. $\cot\theta$ is undefined, $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2} \Rightarrow y = 0 \Rightarrow \theta = \pi$

$\sin\theta = 0$ $\qquad$ $\csc\theta$ is undefined.

$\cos\theta = -1$ $\qquad$ $\sec\theta = -1$

$\tan\theta = 0$ $\qquad$ $\cot\theta$ is undefined.

33. To find a point on the terminal side of θ, use any point on the line $y = -x$ that lies in Quadrant II. $(-1, 1)$ is one such point.

$x = -1, y = 1, r = \sqrt{2}$

$\sin\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\qquad$ $\csc\theta = \sqrt{2}$

$\cos\theta = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$ $\qquad$ $\sec\theta = -\sqrt{2}$

$\tan\theta = -1$ $\qquad$ $\cot\theta = -1$

35. To find a point on the terminal side of θ, use any point on the line $y = 2x$ that lies in Quadrant I. (1, 2) is one such point.

$x = 1, y = 2, y = \sqrt{5}$

$\sin\theta = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$ $\qquad$ $\csc\theta = \dfrac{\sqrt{5}}{2} = \dfrac{\sqrt{5}}{2}$

$\cos\theta = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$ $\qquad$ $\sec\theta = \dfrac{\sqrt{5}}{1} = \sqrt{5}$

$\tan\theta = \dfrac{2}{1} = 2$ $\qquad$ $\cot\theta = \dfrac{1}{2}$

37. $(x, y) = (-1, 0), r = 1$

$\sin\pi = \dfrac{y}{r} = \dfrac{0}{1} = 0$

39. $(x, y) = (0, -1), r = 1$

$\sec\dfrac{3\pi}{2} = \dfrac{r}{x} = \dfrac{1}{0} \Rightarrow$ undefined

41. $(x, y) = (0, 1), r = 1$

$\sin\dfrac{\pi}{2} = \dfrac{y}{r} = \dfrac{1}{1} = 1$

43. $(x, y) = (-1, 0), r = 1$

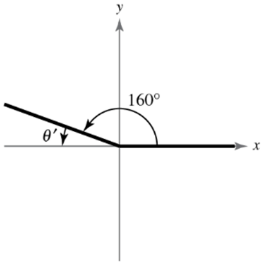
$\csc\pi = \dfrac{r}{y} = \dfrac{1}{0} \Rightarrow$ undefined

45. $(x, y) = (0, 1)$

$\cot\dfrac{9\pi}{2} = \dfrac{x}{y} = \dfrac{0}{1} = 0$

47. $\theta = 160°$

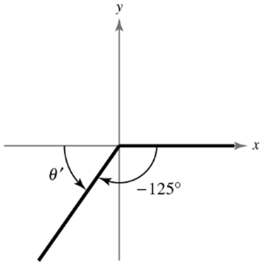
$\theta' = 180° - 160° = 20°$



49. $\theta = -125°$

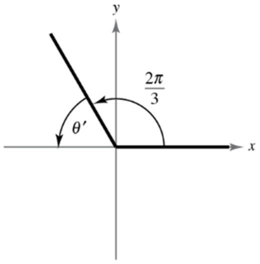
$360° - 125° = 235°$ (coterminal angle)

$\theta' = 235° - 180° = 55°$



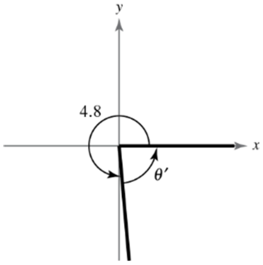
51. $\theta = \dfrac{2\pi}{3}$

$\theta' = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$



53. $\theta = 4.8$

$\theta' = 2\pi - 4.8 \approx 1.48$



55. $\theta = 225°, \theta' = 45°$, Quadrant III

$\sin 225° = -\sin 45° = -\dfrac{\sqrt{2}}{2}$

$\cos 225° = -\cos 45° = -\dfrac{\sqrt{2}}{2}$

$\tan 225° = \tan 45° = 1$

57. $\theta = 750°, \theta' = 30°$, Quadrant I

$\sin 750° = \sin 30° = \dfrac{1}{2}$

$\cos 750° = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\tan 750° = \tan 30° = \dfrac{\sqrt{3}}{3}$

59. $\theta = -120°, \theta' = 60°$, Quadrant III

$\sin(-120°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

$\cos(-120°) = -\cos 60° = -\dfrac{1}{2}$

$\tan(-120°) = \tan 60° = \sqrt{3}$

61. $\theta = \dfrac{2\pi}{3}, \theta' = \dfrac{\pi}{3}$ in Quadrant II

$\sin \dfrac{2\pi}{3} = \sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

$\cos \dfrac{2\pi}{3} = -\cos \dfrac{\pi}{3} = -\dfrac{1}{2}$

$\tan \dfrac{2\pi}{3} = -\tan \dfrac{\pi}{3} = -\sqrt{3}$

63. $\theta = -\dfrac{\pi}{6}, \theta' = \dfrac{\pi}{6}$, Quadrant IV

$\sin\left(-\dfrac{\pi}{6}\right) = -\sin \dfrac{\pi}{6} = -\dfrac{1}{2}$

$\cos\left(-\dfrac{\pi}{6}\right) = \cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\tan\left(-\dfrac{\pi}{6}\right) = -\tan \dfrac{\pi}{6} = -\dfrac{\sqrt{3}}{3}$

65. $\theta = \dfrac{11\pi}{4}, \theta' = \dfrac{\pi}{4}$, Quadrant II

$\sin \dfrac{11\pi}{4} = \sin \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

$\cos \dfrac{11\pi}{4} = -\cos \dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2}$

$\tan \dfrac{11\pi}{4} = -\tan \dfrac{\pi}{4} = -1$

67. $\theta = -\frac{17\pi}{6}, \theta' = \frac{\pi}{6}$ in Quadrant III

$$\sin\left(-\frac{17\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\cos\left(-\frac{17\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\left(-\frac{17\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

69. $\sin\theta = -\frac{3}{5}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

$$\cos^2\theta = 1 - \frac{9}{25}$$

$$\cos^2\theta = \frac{16}{25}$$

$\cos\theta > 0$ in Quadrant IV.

$$\cos\theta = \frac{4}{5}$$

71. $\tan\theta = \frac{3}{2}$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\sec^2\theta = 1 + \left(\frac{3}{2}\right)^2$$

$$\sec^2\theta = 1 + \frac{9}{4}$$

$$\sec^2\theta = \frac{13}{4}$$

$\sec\theta < 0$ in Quadrant III.

$$\sec\theta = -\frac{\sqrt{13}}{2}$$

73. $\cos\theta = \frac{5}{8}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{5}{8}\right)^2 = 1$$

$$\sin^2\theta = 1 - \frac{25}{64}$$

$$\sin^2\theta = \frac{39}{64}$$

$\sin\theta > 0$ in Quadrant I.

$$\sin\theta = \frac{\sqrt{39}}{8}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\csc\theta = \frac{8\sqrt{39}}{39}$$

75. $\sin 10° \approx 0.1736$

77. $\cos(-110°) \approx -0.3420$

79. $\cot 178° \approx -28.6363$

81. $\csc 405° = \frac{1}{\sin 405°} \approx 1.4142$

83. $\tan\left(\frac{\pi}{9}\right) \approx 0.3640$

85. $\sec\frac{11\pi}{8} = \frac{1}{\cos\frac{11\pi}{8}} \approx -2.6131$

87. $\sin(-0.65) \approx -0.6052$

89. $\csc(-10) = \frac{1}{\sin(-10)} \approx 1.8382$

91. (a) $\sin\theta = \frac{1}{2} \Rightarrow$ reference angle is $30°$ or $\frac{\pi}{6}$ and θ is in Quadrant I or Quadrant II.

Values in degrees: $30°, 150°$

Values in radian: $\frac{\pi}{6}, \frac{5\pi}{6}$

(b) $\sin\theta = \frac{1}{2} \Rightarrow$ reference angle is $30°$ or $\frac{\pi}{6}$ and θ is in Quadrant III or Quadrant IV.

Values in degrees: $210°, 330°$

Values in radians: $\frac{7\pi}{6}, \frac{11\pi}{6}$

93. (a) $\cos\theta = \frac{1}{2} \Rightarrow$ reference angle is 60° or $\frac{\pi}{3}$ and θ is in Quadrant I or IV.

Values in degrees: 60°, 300°

Values in radians: $\frac{\pi}{3}, \frac{5\pi}{3}$

(b) $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow$ reference angle is 60° or $\frac{\pi}{3}$ and θ is in Quadrant I or IV.

Values in degrees: 60°, 300°

Values in radians: $\frac{\pi}{3}, \frac{5\pi}{3}$

95. (a) $\tan\theta = 1 \Rightarrow$ reference angle is 45° or $\frac{\pi}{4}$ and θ is in Quadrant I or Quadrant III.

Values in degrees: 45°, 225°

Values in radians: $\frac{\pi}{4}, \frac{5\pi}{4}$

(b) $\cot\theta = -\sqrt{3} \Rightarrow$ reference angle is 30° or $\frac{\pi}{6}$ and θ is in Quadrant II or Quadrant IV.

Values in degrees: 150°, 330°

Values in radians: $\frac{5\pi}{6}, \frac{11\pi}{6}$

97. $\sin\theta = \frac{6}{d} \Rightarrow d = \frac{6}{\sin\theta}$

(a) $\theta = 30°$

$$d = \frac{6}{\sin 30°} = \frac{6}{1/2} = 12 \text{ miles}$$

(b) $\theta = 90°$

$$d = \frac{6}{\sin 90°} = \frac{6}{1} = 6 \text{ miles}$$

(c) $\theta = 120°$

$$d = \frac{6}{\sin 120°} = \frac{6}{\sqrt{3}/2} \approx 6.9 \text{ miles}$$

99. (a) Using a graphing utility, the equations are as follows.

$$B = 21.865\sin(0.540t - 2.343) + 60.438$$

$$F = 61.120\sin(0.325t - 0.561) + 15.787$$

(b) February $(t = 2)$: $B \approx 39.6$°F, $F \approx 21.2$°F

April $(t = 4)$: $B \approx 56.5$°F, $F \approx 57.0$°F

May $(t = 5)$: $B \approx 68.1$°F, $F \approx 69.2$°F

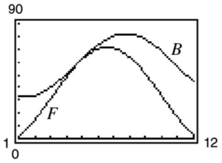
July $(t = 7)$: $B \approx 82.1$°F, $F \approx 76.3$°F

September $(t = 9)$: $B \approx 73.2$°F, $F \approx 58.7$°F

October $(t = 10)$: $B \approx 62.3$°F, $F \approx 42.5$°F

December $(t = 12)$: $B \approx 42.1$°F, $F \approx 3.8$°F

(c)



Answers will vary.

101. False. In each of the four quadrants, the sign of the secant function and the cosine function will be the same because they are reciprocals of each other.

103. Answers will vary.

105. $g(x) = \sqrt{x + 12} + 9$

(a) Parent function: $f(x) = \sqrt{x}$

(b) g is a reflection in the x-axis, a left shift of 12 units, and an upward shift of 9 units of the graph of f.

107. $(-4, 5)$ is a relative maximum.

109. $y = x^2 + x - 132$

$0 = x^2 + x - 132$

$0 = (x + 12)(x - 11)$

$x = -12, 11$

x-intercepts: $(-12, 0), (11, 0)$

111. $y = 2e^{x-5} - 4$

$0 = 2e^{x-5} - 4$

$2 = e^{x-5}$

$\ln 2 = x - 5$

$x = 5 + \ln 2$

x-intercept: $(5 + \ln 2, 0)$

Section 4.5 Graphs of Sine and Cosine Functions

1. cycle

3. The constant *d* of $y = \sin x + d$ is a vertical shift of *d* units.

5. $y = 2 \sin 5x$

Period: $\dfrac{2\pi}{5}$

Amplitude: $|2| = 2$

7. $y = \dfrac{3}{4} \cos \dfrac{\pi x}{2}$

Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{\pi/2} = 4$

Amplitude: $\left|\dfrac{3}{4}\right| = \dfrac{3}{4}$

9. $y = -\dfrac{1}{2} \sin \dfrac{5x}{4}$

Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{5/4} = \dfrac{8\pi}{5}$

Amplitude: $\left|-\dfrac{1}{2}\right| = \dfrac{1}{2}$

11. $y = -\dfrac{5}{3} \cos \dfrac{\pi x}{12}$

Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{\pi/12} = 24$

Amplitude: $\left|-\dfrac{5}{3}\right| = \dfrac{5}{3}$

13. $f(x) = \cos x$

$g(x) = \cos 5x$

The period of *g* is one-fifth the period of *f*.

15. $f(x) = \cos 2x$

$g(x) = -\cos 2x$

g is a reflection of the graph of *f* in the *x*-axis.

17. $f(x) = \sin x$

$g(x) = \sin(x - \pi)$

g is a horizontal shift to the right π units of the graph of *f*.

19. $f(x) = \sin 2x$

$f(x) = 3 + \sin 2x$

g is a vertical shift three units upward of the graph of *f*.

21. The graph of *g* has twice the amplitude as the graph of *f*. The period is the same.

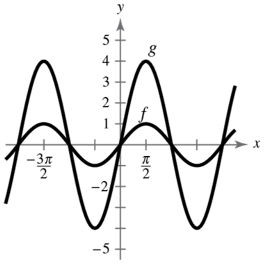
23. The graph of *g* is a horizontal shift π units to the right of the graph of *f*.

25. $f(x) = \sin x$

Period: $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$

Amplitude: 1

Symmetry: origin



Key points:	Intercept	Maximum	Intercept	Minimum	Intercept
	$(0, 0)$	$\left(\frac{\pi}{2}, 1\right)$	$(\pi, 0)$	$\left(\frac{3\pi}{2}, -1\right)$	$(2\pi, 0)$

Because $g(x) = 4 \sin x = 4f(x)$, the graph of $g(x)$ is the graph of $f(x)$, but stretched vertically by a factor of 4. The amplitude of $g(x)$ is four times the amplitude of $f(x)$.

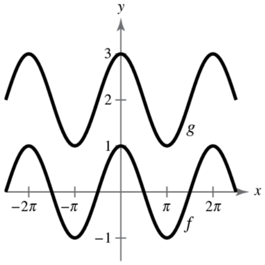
Generate key points for the graph $g(x)$ by multiplying the x-coordinate of each key point of $f(x)$ by 4.

27. $f(x) = \cos x$

Period: $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$

Amplitude: $|1| = 1$

Symmetry: y-axis



Key points:	Maximum	Intercept	Minimum	Intercept	Maximum
	$(0, 1)$	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, -1)$	$\left(\frac{3\pi}{2}, 0\right)$	$(2\pi, 1)$

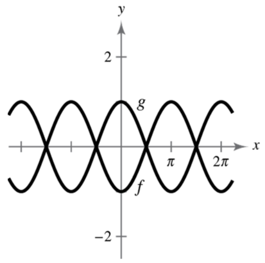
Because $g(x) = 2 + \cos x = f(x) + 2$, the graph of $g(x)$ is the graph of $f(x)$, but translated upward by two units. Generate key points of $g(x)$ by adding 2 to the y-coordinate of each key point of $f(x)$.

29. $f(x) = -\cos x$

Period: $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$

Amplitude: 1

Symmetry: y-axis



Key points:	Minimum	Intercept	Maximum	Intercept	Minimum
	$(0, -1)$	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, 1)$	$\left(\frac{3\pi}{2}, 0\right)$	$(2\pi, -1)$

Because $g(x) = -\cos(x - \pi) = f(x - \pi)$, the graph of $g(x)$ is the graph of $f(x)$, but with a phase shift (horizontal translation) of π. Generate key points for the graph of $g(x)$ by shifting each key point of $f(x)$ π units to the right.

31. $y = 5 \sin x$

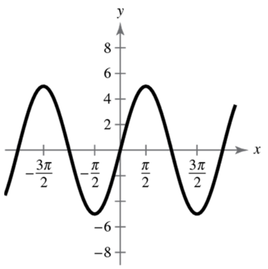
Period: 2π

Amplitude: 5

Key points:

$(0, 0), \left(\frac{\pi}{2}, 5\right), (\pi, 0),$

$\left(\frac{3\pi}{2}, -5\right), (2\pi, 0)$



33. $y = \frac{1}{3} \cos x$

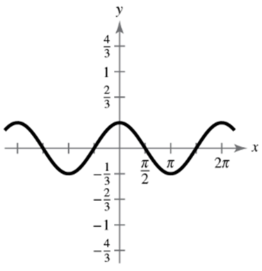
Period: 2π

Amplitude: $\frac{1}{3}$

Key points:

$\left(0, \frac{1}{3}\right), \left(\frac{\pi}{2}, 0\right), \left(\pi, -\frac{1}{3}\right),$

$\left(\frac{3\pi}{2}, 0\right), \left(2\pi, \frac{1}{3}\right)$



35. $y = \cos \dfrac{x}{2}$

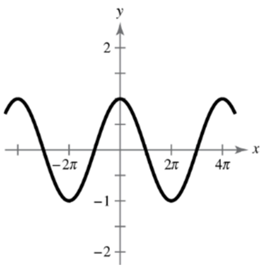
Period $\dfrac{2\pi}{1/2} = 4\pi$

Amplitude: 1

Key points:

$(0, 1), (\pi, 0), (2\pi, -1),$

$(3\pi, 0), (4\pi, 1)$



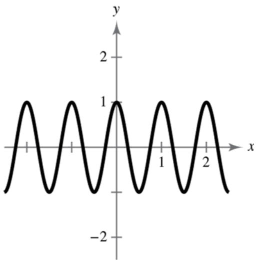
37. $y = \cos 2\pi x$

Period: $\dfrac{2\pi}{2\pi} = 1$

Amplitude: 1

Key points:

$(0, 1), \left(\dfrac{1}{4}, 0\right), \left(\dfrac{1}{2}, -1\right), \left(\dfrac{3}{4}, 0\right)$



39. $y = -\sin \dfrac{2\pi x}{3}$

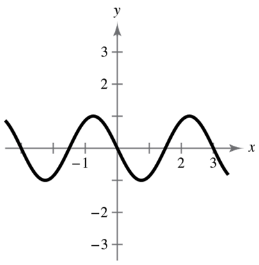
Period: $\dfrac{2\pi}{2\pi/3} = 3$

Amplitude: 1

Key points:

$(0, 0), \left(\dfrac{3}{4}, -1\right), \left(\dfrac{3}{2}, 0\right),$

$\left(\dfrac{9}{4}, 1\right), (3, 0)$



41. $y = \cos\left(x - \dfrac{\pi}{2}\right)$

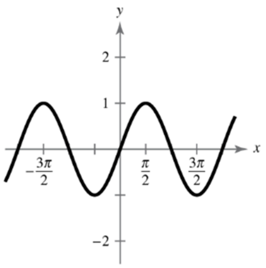
Period: 2π

Amplitude: 1

Shift: Set $x - \dfrac{\pi}{2} = 0$ and $x - \dfrac{\pi}{2} = 2\pi$

$x = \dfrac{\pi}{2} \qquad x = \dfrac{5\pi}{2}$

Key points: $\left(\dfrac{\pi}{2}, 1\right), (\pi, 0), \left(\dfrac{3\pi}{2}, -1\right), (2\pi, 0), \left(\dfrac{5\pi}{2}, 1\right)$



43. $y = 3 \sin(x + \pi)$

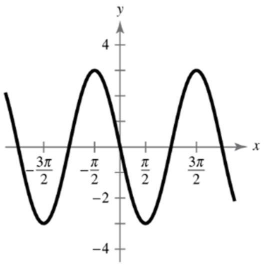
Period: 2π

Amplitude: 3

Shift: Set $x + \pi = 0$ and $x + \pi = 2\pi$

$x = -\pi \qquad x = \pi$

Key points: $(-\pi, 0), \left(-\dfrac{\pi}{2}, 3\right), (0, 0), \left(\dfrac{\pi}{2}, -3\right), (\pi, 0)$



45. $y = 2 - \sin \dfrac{2\pi x}{3}$

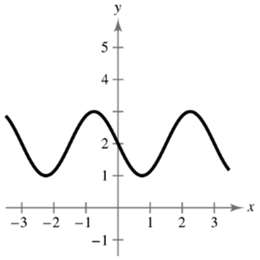
Period: $\dfrac{2\pi}{2\pi/3} = 3$

Amplitude: 1

Key points:

$(0, 2), \left(\dfrac{3}{4}, 1\right), \left(\dfrac{3}{2}, 2\right),$

$\left(\dfrac{9}{4}, 3\right), (3, 2)$



47. $y = 2 + 5 \cos 6\pi x$

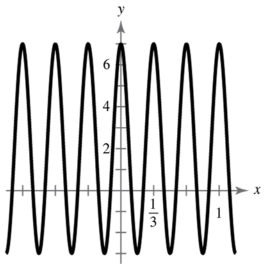
Period: $\dfrac{2\pi}{6\pi} = \dfrac{1}{3}$

Amplitude: 5

Shift: Set $6\pi x = 0$ and $6\pi x = 2\pi$

$x = 0 \qquad x = \dfrac{1}{3}$

Key points: $\left(\dfrac{1}{3}, 7\right), \left(\dfrac{5}{12}, 2\right), \left(\dfrac{1}{2}, -3\right), \left(\dfrac{7}{12}, 2\right), \left(\dfrac{1}{3}, 7\right)$



49. $y = 3\sin(x + \pi) - 3$

Period: 2π

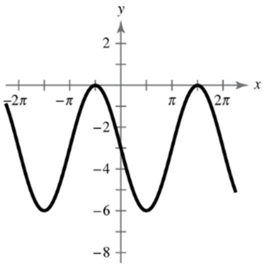
Amplitude: 3

Shift: Set $x + \pi = 0$ and $x + \pi = 2\pi$

$x = -\pi$ $\qquad$ $x = \pi$

Key points: $(-\pi, -3), \left(-\frac{\pi}{2}, 0\right), (0, -3), \left(\frac{\pi}{2}, -6\right),$

$(\pi, -3)$



51. $y = \frac{2}{3}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$

Period: $\frac{2\pi}{1/2} = 4\pi$

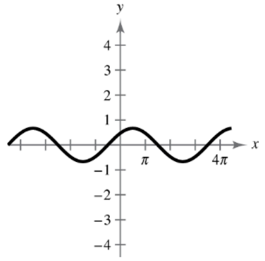
Amplitude: $\frac{2}{3}$

Shift: $\frac{x}{2} - \frac{\pi}{4} = 0$ and $\frac{\pi}{2} - \frac{\pi}{4} = 2\pi$

$x = \frac{\pi}{2}$ $\qquad$ $x = \frac{9\pi}{2}$

Key points:

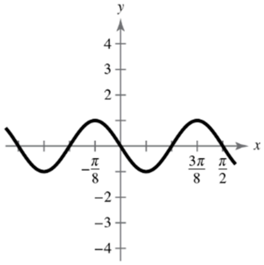
$\left(\frac{\pi}{2}, \frac{2}{3}\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{2}, \frac{-2}{3}\right), \left(\frac{7\pi}{2}, 0\right), \left(\frac{9\pi}{2}, \frac{2}{3}\right)$



53. $g(x) = \sin(4x - \pi)$

(a) $g(x)$ is obtained by a horizontal shrink and a phase shift of $\frac{\pi}{4}$. One cycle of $g(x)$ corresponds to the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

(b)

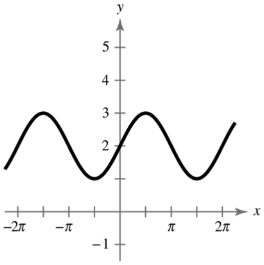


(c) $g(x) = f(4x - \pi)$ where $f(x) = \sin x$.

55. $g(x) = \cos\left(x - \frac{\pi}{2}\right) + 2$

(a) $g(x)$ is obtained by shifting $f(x)$ two units upward and a phase shift of $\frac{\pi}{2}$. One cycle of $g(x)$ corresponds to the interval $[\pi, 3\pi]$.

(b)

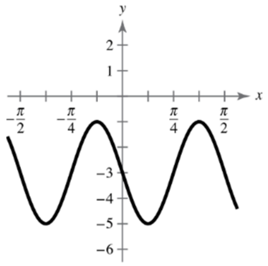


(c) $g(x) = f\left(x - \frac{\pi}{2}\right) + 2$ where $f(x) = \cos x$

57. $g(x) = 2\sin(4x - \pi) - 3$

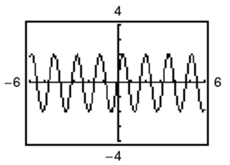
(a) $g(x)$ is obtained by a vertical stretch, a horizontal shrink, a phase shift of $\frac{\pi}{4}$, and shifting $f(x)$ three units downward. One cycle of $g(x)$ corresponds to the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

(b)

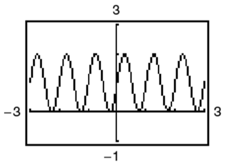


(c) $g(x) = 2f(4x - \pi) - 3$ where $f(x) = \sin x$

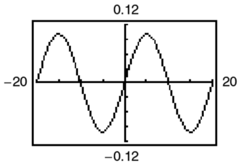
59. $y = -2\sin(4x + \pi)$



61. $y = \cos\left(2\pi x - \dfrac{\pi}{2}\right) + 1$



63. $y = -0.1\sin\left(\dfrac{\pi x}{10} + \pi\right)$



65. $f(x) = a\cos x + d$

Amplitude: $\frac{1}{2}[3 - (-1)] = 2 \Rightarrow a = 2$

$3 = 2\cos 0 + d$

$d = 3 - 2 = 1$

$a = 2, d = 1$

67. $f(x) = a\cos x + d$

Amplitude: $\frac{1}{2}[8 - 0] = 4$

Reflected in the x-axis: $a = -4$

$0 = -4\cos 0 + d$

$d = 4$

$a = -4, d = 4$

69. $y = a\sin(bx - c)$

Amplitude: $|a| = |3|$

Since the graph is reflected in the x-axis, we have $a = -3$.

Period: $\dfrac{2\pi}{b} = \pi \Rightarrow b = 2$

Phase shift: $c = 0$

$a = -3, b = 2, c = 0$

71. $y = a\sin(bx - c)$

Amplitude: $a = 2$

Period: $2\pi \Rightarrow b = 1$

Phase shift: $bx - c = 0$ when $x = -\dfrac{\pi}{4}$

$$(1)\left(-\frac{\pi}{4}\right) - c = 0 \Rightarrow c = -\frac{\pi}{4}$$

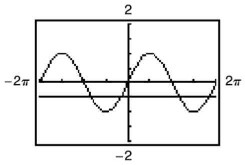
$a = 2, b = 1, c = -\dfrac{\pi}{4}$

73. $y_1 = \sin x$

$y_2 = -\dfrac{1}{2}$

In the interval $[-2\pi, 2\pi]$,

$y_1 = y_2$ when $x = -\dfrac{5\pi}{6}, -\dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.



Answers for 75 and 77 are sample answers.

75. $f(x) = 2\sin(2x - \pi) + 1$

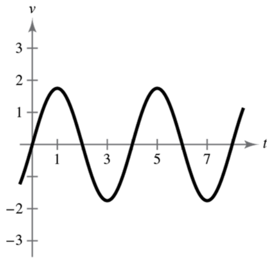
77. $f(x) = \cos(2x + 2\pi) - \frac{3}{2}$

79. $v = 1.75\sin\dfrac{\pi t}{2}$

(a) Period $= \dfrac{2\pi}{\pi/2} = 4$ seconds

(b) $\dfrac{1 \text{ cycle}}{4 \text{ seconds}} \cdot \dfrac{60 \text{ seconds}}{1 \text{ minute}} = 15$ cycles per minute

(c)



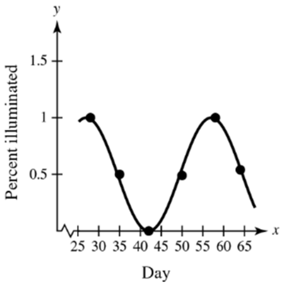
81. $P = 100 - 20\cos\dfrac{5\pi t}{3}$

(a) Period: $\dfrac{2\pi}{(5\pi)/3} = \dfrac{6}{5}$ seconds

(b) $\dfrac{1 \text{ heartbeat}}{6/5 \text{ seconds}} \cdot \dfrac{60 \text{ seconds}}{1 \text{ minute}}$

$= 50$ heartbeats per minute

83. (a) and (c)



The model fits the data well.

(b) Amplitude: $\alpha = \frac{1}{2}[\text{max. percent} - \text{min. percent}]$

$$= \frac{1}{2}[1.0 - 0] = 0.5$$

Period: $p = (\text{2nd day of } 1.0\% \ - \text{1st day of } 1\%)$

$$= 31 - 1 = 30$$

$$b = \frac{2\pi}{p} = \frac{2\pi}{30} = \frac{\pi}{15}$$

Because the zero percent occurs at day 42, $x = 42$.

$$bx - c = \pi$$

$$\frac{\pi}{15}(42) - c = \pi$$

$$\frac{14\pi}{5} - \pi = c$$

$$\frac{9\pi}{5} = c$$

The average percent of illumination is $\frac{1}{2}(1.0 - 0) = 0.5$. So, $d = 0.5$. So, the model is

$$y = 0.5\cos\left(\frac{\pi}{15}x - \frac{9\pi}{5}\right) + 0.5.$$

(d) The period is $p = 30$ days.

(e) Because March 28, 2021 is the 87th day of the year, $x = 87$. So, the percent of the moon's face illuminated is

$$y = 0.5\cos\left(\frac{\pi}{15}(87) - \frac{9\pi}{5}\right) + 0.5 = 1, \text{ or } 100\%.$$

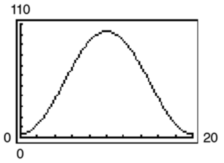
85. (a) Period $= \dfrac{2\pi}{\left(\dfrac{\pi}{10}\right)} = 20$ seconds

The wheel takes 20 seconds to revolve once.

(b) Amplitude: 50 feet

The radius of the wheel is 50 feet.

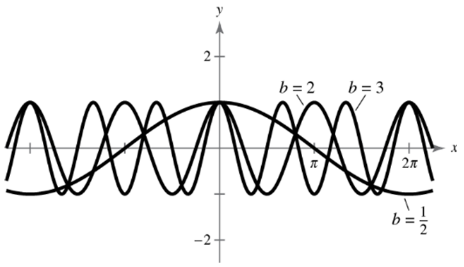
(c)



87. False. $y = \frac{1}{2}\cos 2x$ has an amplitude that is half that of $y = \cos x$. For $y = a\cos bx$, the amplitude is $|a|$.

89. These are the key points for $y = \cos x$, not $y = \sin x$.

91.



As the value of b increases, the period decreases.

$b = \frac{1}{2} \rightarrow \frac{1}{2}$ cycle

$b = 2 \rightarrow 2$ cycles

$b = 3 \rightarrow 3$ cycles

93. $y = x^2 + 4$

(a)

x	0	1	2	3	4
y	4	5	8	13	20

(b) $y = (-x)^2 + 4 \Rightarrow y = x^2 + 4$

$\Rightarrow$ y-axis symmetry

$-y = x^2 + 4 \Rightarrow y = -x^2 - 4$

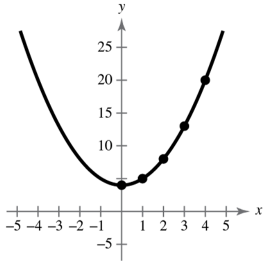
$\Rightarrow$ No x-axis symmetry

$-y = (-x)^2 + 4 \Rightarrow -y = x^2 + 4$

$\Rightarrow y = -x^2 - 4$

$\Rightarrow$ No origin symmetry

(c)



95. $y = x^3 - x$

(a)

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$-\frac{3}{8}$	0	$\frac{15}{8}$	6

(b) $y = (-x)^3 - (-x) \Rightarrow y = -x^3 + x$

$\Rightarrow$ No y-axis symmetry

$-y = x^3 - x \Rightarrow y = -x^3 + x$

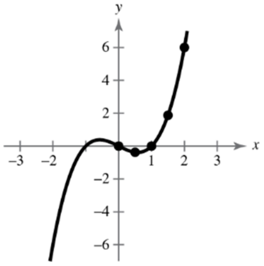
$\Rightarrow$ No x-axis symmetry

$-y = (-x)^3 - (-x) \Rightarrow -y = -x^3 + x$

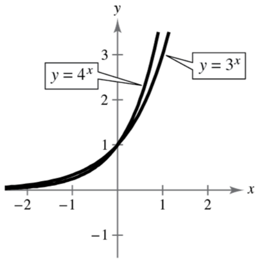
$\Rightarrow y = x^3 - x$

$\Rightarrow$ origin symmetry

(c)



97. $y = 3^x$ and $y = 4^x$



x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

(a) $4^x < 3^x$ when $x < 0$.

(b) $4^x > 3^x$ when $x > 0$.

Section 4.6 Graphs of Other Trigonometric Functions

1. odd; origin

3. damping

5. The range of both $y = \csc x$ and $y = \sec x$ is $(-\infty, -1] \cup [1, \infty)$.

7. $y = \sec 2x$

Period: $\frac{2\pi}{2} = \pi$

Matches graph (e).

8. $y = \tan \frac{x}{2}$

Period: $\frac{\pi}{b} = \frac{\pi}{1/2} = 2\pi$

Asymptotes: $x = -\pi, x = \pi$

Matches graph (c).

9. $y = \frac{1}{2} \cot \pi x$

Period: $\frac{\pi}{\pi} = 1$

Matches graph (a).

10. $y = -\csc x$

Period: 2π

Matches graph (d).

11. $y = \frac{1}{2} \sec \frac{\pi x}{2}$

Period: $\frac{2\pi}{b} = \frac{2\pi}{\pi/2} = 4$

Asymptotes: $x = -1, x = 1$

Matches graph (f).

12. $y = -2 \sec \frac{\pi x}{2}$

Period: $\frac{2\pi}{b} = \frac{2\pi}{\pi/2} = 4$

Asymptotes: $x = -1, x = 1$

Reflected in x-axis

Matches graph (b).

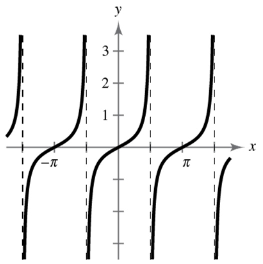
13. $y = \frac{1}{3}\tan x$

Period: π

Two consecutive asymptotes:

$x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
y	$-\frac{1}{3}$	0	$\frac{1}{3}$



15. $y = \tan\frac{\pi x}{4}$

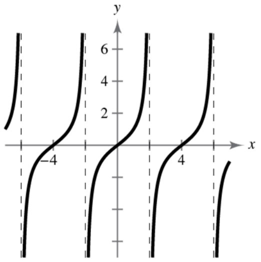
Period: $\frac{\pi}{\pi/4} = 4$

Two consecutive asymptotes:

$\frac{\pi x}{4} = -\frac{\pi}{2} \Rightarrow x = -2$

$\frac{\pi x}{4} = \frac{\pi}{2} \Rightarrow x = 2$

x	−1	0	1
y	−1	0	1



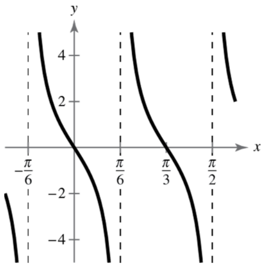
17. $y = -2\tan 3x$

Period: $\frac{\pi}{3}$

Two consecutive asymptotes:

$x = -\frac{\pi}{6}, x = \frac{\pi}{6}$

x	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$
y	0	0	0

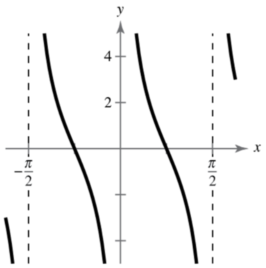


19. $y = 3\cot 2x$

Period: $\frac{\pi}{2}$

Two consecutive asymptotes:

$x = -\frac{\pi}{2}, x = \frac{\pi}{2}$



x	$-\frac{\pi}{6}$	$-\frac{\pi}{8}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$
y	$-3\sqrt{3}$	−3	3	$3\sqrt{3}$

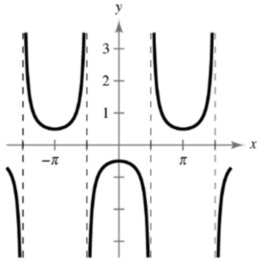
21. $y = -\frac{1}{2}\sec x$

Period: 2π

Two consecutive asymptotes:

$x = -\frac{\pi}{2}, x = \frac{\pi}{2}$

x	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$
y	−1	$-\frac{1}{2}$	−1



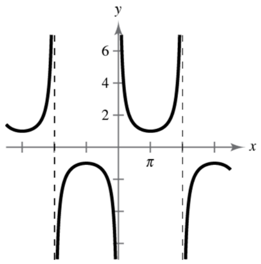
23. $y = \csc\frac{x}{2}$

Period: $\frac{2\pi}{1/2} = 4\pi$

Two consecutive asymptotes:

$x = 0, x = 2\pi$

x	$\frac{\pi}{3}$	π	$\frac{5\pi}{3}$
y	2	1	2



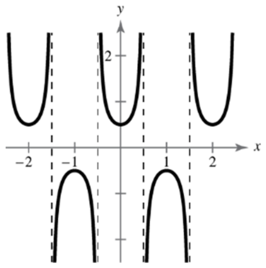
25. $y = \frac{1}{2}\sec\pi x$

Period: 2

Two consecutive asymptotes:

$x = -\frac{1}{2}, x = \frac{1}{2}$

x	−1	0	1
y	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$



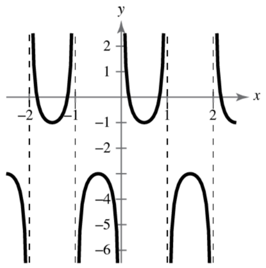
27. $y = \csc\pi x - 2$

Period: $\frac{2\pi}{\pi} = 2$

Two consecutive asymptotes:

$x = 0, x = 1$

x	0.5	1.5	2.5
y	−1	−3	−1



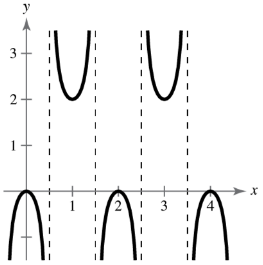
29. $y = -\sec \pi x + 1$

Period: $\dfrac{2\pi}{\pi} = 2$

Two consecutive asymptotes:

$x = -\dfrac{1}{2}, x = \dfrac{1}{2}$

x	$-\dfrac{1}{3}$	0	$\dfrac{1}{3}$
y	-1	0	1



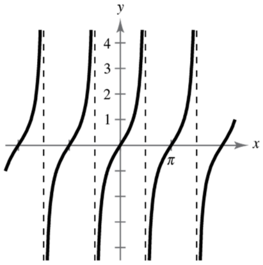
31. $y = \tan(x + \pi)$

Period: π

Two consecutive asymptotes:

$x = -\dfrac{\pi}{2}, x = \dfrac{\pi}{2}$

x	$-\dfrac{\pi}{4}$	0	$\dfrac{\pi}{4}$
y	-1	0	1



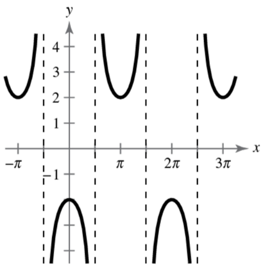
33. $y = 2\sec(x + \pi)$

Period: 2π

Two consecutive asymptotes:

$x = -\dfrac{\pi}{2}, x = \dfrac{\pi}{2}$

x	$-\dfrac{\pi}{3}$	0	$\dfrac{\pi}{3}$
y	-4	-2	-4



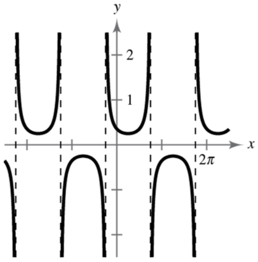
35. $y = \dfrac{1}{4}\csc\left(x + \dfrac{\pi}{4}\right)$

Period: 2π

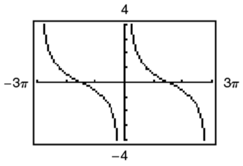
Two consecutive asymptotes:

$x = -\dfrac{\pi}{4}, x = \dfrac{3\pi}{4}$

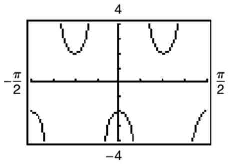
x	$-\dfrac{\pi}{12}$	$\dfrac{\pi}{4}$	$\dfrac{7\pi}{12}$
y	$\dfrac{1}{2}$	$\dfrac{1}{4}$	$\dfrac{1}{2}$



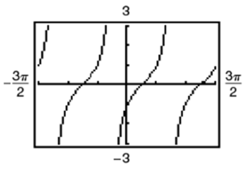
37. $y = \cot \dfrac{x}{3}$



39. $y = -2\sec 4x = \dfrac{-2}{\cos 4x}$

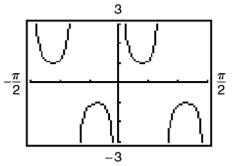


41. $y = \tan\left(x - \dfrac{\pi}{4}\right)$

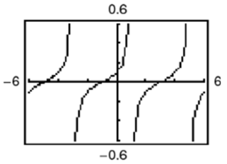


43. $y = -\csc(4x - \pi)$

$y = \dfrac{-1}{\sin(4x - \pi)}$

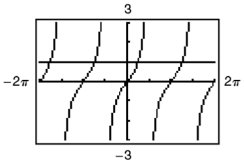


45. $y = 0.1\tan\left(\frac{\pi x}{4} + \frac{\pi}{4}\right)$



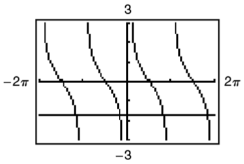
47. $\tan x = 1$

$$x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$



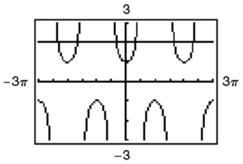
49. $\cot x = -\sqrt{3}$

$$x = -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$$

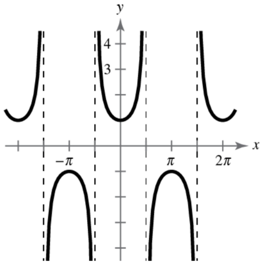


51. $\sec x = 2$

$$x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$$



53.
$$\begin{aligned} f(x) &= \sec x = \frac{1}{\cos x} \\ f(-x) &= \sec(-x) \\ &= \frac{1}{\cos(-x)} \\ &= \frac{1}{\cos x} \\ &= f(x) \end{aligned}$$



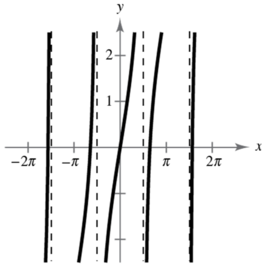
So, $f(x) = \sec x$ is an even function and the graph has y-axis symmetry.

55. The function

$$f(x) = \csc 2x = \frac{1}{\sin 2x}$$

has origin symmetry. Thus, the function is odd.

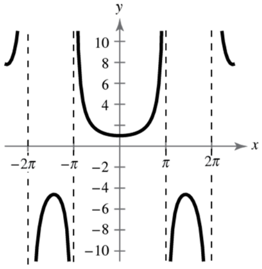
57. $f(x) = x + \tan x$



$$\begin{aligned} f(-x) &= (-x) + \tan(-x) \\ &= -x - \tan x \\ &= -(x + \tan x) \\ &= -f(x) \end{aligned}$$

So, $f(x) = x + \tan x$ is an odd function and the graph has origin symmetry.

59. $g(x) = x\csc x = \dfrac{x}{\sin x}$



$$\begin{aligned} g(-x) &= (-x)\csc(-x) \\ &= \frac{-x}{\sin(-x)} \\ &= \frac{-x}{-\sin x} \\ &= \frac{x}{\sin x} \\ &= x\csc x \\ &= g(x) \end{aligned}$$

So, $g(x) = x\csc x$ is an even function and the graph has y-axis symmetry.

61. $f(x) = |x \cos x|$

Matches graph (d).

As $x \to 0, f(x) \to 0.$

62. $f(x) = x \sin x$

Matches graph (a)

As $x \to 0, f(x) \to 0.$

63. $g(x) = |x| \sin x$

Matches graph (b).

As $x \to 0, g(x) \to 0.$

64. $g(x) = |x| \cos x$

Matches graph (c).

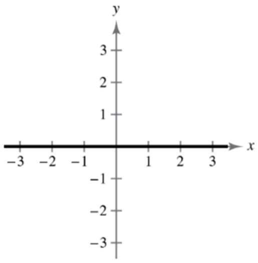
As $x \to 0, g(x) \to 0.$

65. $f(x) = \sin x + \cos\left(x + \dfrac{\pi}{2}\right)$

$g(x) = 0$

$f(x) = g(x)$

The functions are equal.

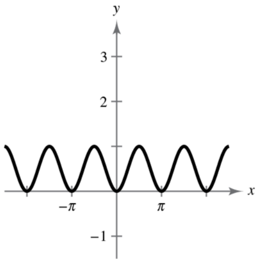


67. $f(x) = \sin^2 x$

$g(x) = \frac{1}{2}(1 - \cos 2x)$

$f(x) = g(x)$

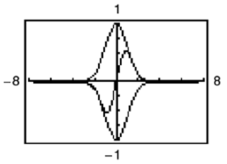
The functions are equal.



69. $g(x) = e^{-x^2/2} \sin x$

Damping factor: $e^{-x^2/2}$

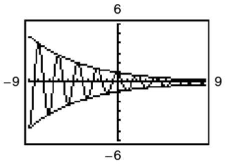
As $x \to \infty, g(x) \to 0.$



71. $f(x) = 2^{-x/4} \cos \pi x$

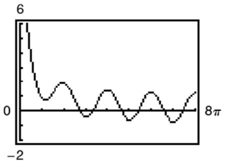
Damping factor: $y = 2^{-x/4}$

As $x \to \infty, f(x) \to 0.$



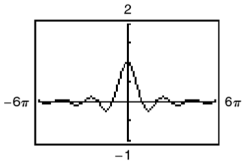
73. $y = \dfrac{6}{x} + \cos x, x > 0$

As $x \to 0, y \to \infty.$



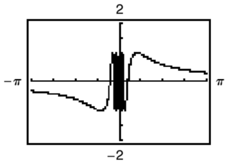
75. $g(x) = \dfrac{\sin x}{x}$

As $x \to 0, g(x) \to 1.$



77. $f(x) = \sin \dfrac{1}{x}$

As $x \to 0, f(x)$ oscillates between −1 and 1.



79. (a) Period of $\cos \dfrac{\pi t}{6} = \dfrac{2\pi}{\pi/6} = 12$

Period of $\sin \dfrac{\pi t}{6} = \dfrac{2\pi}{\pi/6} = 12$

The period of $H(t)$ is 12 months.

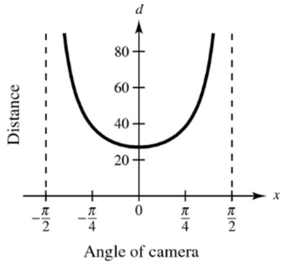
The period of $L(t)$ is 12 months.

(b) From the graph, it appears that the greatest difference between high and low temperatures occurs in the summer. The smallest difference occurs in the winter.

(c) The highest high and low temperatures appear to occur about half of a month after the time when the sun is northernmost in the sky.

81. $\cos x = \dfrac{27}{d}$

$$d = \frac{27}{\cos x} = 27 \sec x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$



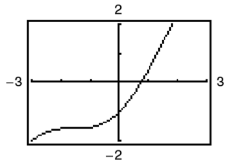
83. True. Because

$$y = \csc x = \frac{1}{\sin x},$$

for a given value of x, the y-coordinate of $\csc x$ is the reciprocal of the y-coordinate of $\sin x$.

85. $f(x) = x - \cos x$

(a)



The zero between 0 and 1 occurs at $x \approx 0.7391$.

(b)
$$\begin{aligned}
x_n &= \cos(x_{n-1})\\
x_0 &= 1\\
x_1 &= \cos 1 \approx 0.5403\\
x_2 &= \cos 0.5403 \approx 0.8576\\
x_3 &= \cos 0.8576 \approx 0.6543\\
x_4 &= \cos 0.6543 \approx 0.7935\\
x_5 &= \cos 0.7935 \approx 0.7014\\
x_6 &= \cos 0.7014 \approx 0.7640\\
x_7 &= \cos 0.7640 \approx 0.7221\\
x_8 &= \cos 0.7221 \approx 0.7504\\
x_9 &= \cos 0.7504 \approx 0.7314\\
&\ \vdots
\end{aligned}$$

This sequence appears to be approaching the zero of f: $x \approx 0.7391$.

87. $f(x) = \sqrt[3]{x},\ g(x) = x^2$

(a) $(f \circ g)(x) = f(x^2) = (x^2)^{1/3} = x^{2/3}$

(b) $(g \circ f)(x) = g(x^{1/3}) = (x^{1/3})^2 = x^{2/3}$

Domains of $f, g, f \circ g, g \circ f$: all real numbers x

89. $f(x) = x^4,\ g(x) = \sqrt{x - 1}$

(a) $(f \circ g)(x) = f\sqrt{x - 1} = (x - 1)^2,\ x \ge 1$

(b) $(g \circ f)(x) = g(x^4) = \sqrt{x^4 - 1}$

Domain of f: all real numbers x

Domains of g and $f \circ g$: all real numbers x such that $x \ge 1$

Domain of $g \circ f$: all real numbers x such that $x \ge 1$ or $x \le -1$

91. $f(x) = 9 - x^2$

f is not one-to-one $\left(f(1) = f(-1) = 8\right)$.

No inverse function

93. $h(x) = \dfrac{1}{x^3}$ is one-to-one, so it has an inverse function.

$$\begin{aligned}
y &= \frac{1}{x^3}\\
x &= \frac{1}{y^3}\\
y^3 &= \frac{1}{x}\\
y &= \frac{1}{\sqrt[3]{x}}\\
h^{-1}(x) &= \frac{1}{\sqrt[3]{x}}
\end{aligned}$$

95.
$$\begin{aligned}
f(x) &= \sqrt{3x - 6},\ x \ge 2\\
y &= \sqrt{3x - 6}\\
x &= \sqrt{3y - 6}\\
x^2 &= 3y - 6\\
3y &= x^2 + 6\\
y &= \frac{x^2 + 6}{3}\\
f^{-1}(x) &= \frac{x^2 + 6}{3},\ x \ge 0
\end{aligned}$$

97.
$$\begin{aligned}
10e^x + 2 &= 7\\
10e^x &= 5\\
e^x &= \frac{1}{2}\\
x &= \ln\frac{1}{2} \approx -0.693
\end{aligned}$$

Section 4.7 Inverse Trigonometric Functions

	Function	*Alternative Notation*	*Domain*	*Range*
1.	$y = \arcsin x$	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
3.	$y = \arctan x$	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

5. The inverse cosecant function can be denoted as $\csc^{-1} x$ or arccsc x.

7. $y = \arcsin\frac{1}{2} \Rightarrow \sin y = \frac{1}{2}$ for

$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{6}$

9. $y = \arccos 0 \Rightarrow \cos y = 0$ for

$0 \le y \le \pi \Rightarrow y = \frac{\pi}{2}$

11. $y = \arctan\frac{\sqrt{3}}{3} \Rightarrow \tan y = \frac{\sqrt{3}}{3}$ for

$-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow y = \frac{\pi}{6}$

13. It is not possible to evaluate arcsin 3. The domain of the inverse sine function is $[-1, 1]$.

15. $y = \arctan\left(-\sqrt{3}\right) \Rightarrow \tan y = -\sqrt{3}$ for

$-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow y = -\frac{\pi}{3}$

17. $y = \arccos\left(-\frac{1}{2}\right) \Rightarrow \cos y = -\frac{1}{2}$ for

$0 \le y \le \pi \Rightarrow y = \frac{2\pi}{3}$

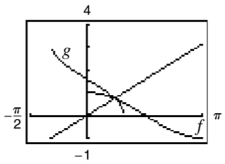
19. $y = \sin^{-1} -\frac{\sqrt{3}}{2} \Rightarrow \sin y = -\frac{\sqrt{3}}{2}$ for

$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = -\frac{\pi}{3}$

21. $f(x) = \cos x$

$g(x) = \arccos x$

$y = x$



23. $\arccos 0.37 = \cos^{-1}(0.37) \approx 1.19$

25. $\arcsin(-0.75) = \sin^{-1}(-0.75) \approx -0.85$

27. $\arctan(-3) = \tan^{-1}(-3) \approx -1.25$

29. It is not possible to evaluate $\sin^{-1} 1.36$. The domain of the inverse sine function is $[-1, 1]$.

31. It is not possible to evaluate $\arccos\left(-\frac{4}{3}\right)$. The domain of the inverse cosine function is $[-1, 1]$.

33. $\tan^{-1}\left(-\frac{95}{7}\right) \approx -1.50$

35. $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

$\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$\tan\left(\frac{\pi}{4}\right) = 1$

37. $\tan\theta = \frac{x}{4}$

$\theta = \arctan\frac{x}{4}$

39. $\sin\theta = \frac{x+2}{5}$

$\theta = \arcsin\left(\frac{x+2}{5}\right)$

41. $\sin(\arcsin 0.3) = 0.3$

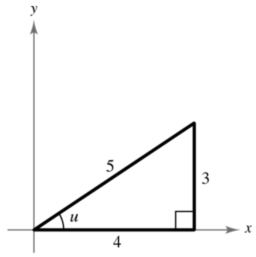
43. $\arcsin\left[\sin\left(\frac{9\pi}{4}\right)\right] = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$

Note: $\frac{9\pi}{4}$ is not in the range of the arcsin function.

45. Let $u = \arctan \frac{3}{4}$.

$\tan u = \frac{3}{4}, 0 < u < \frac{\pi}{2},$

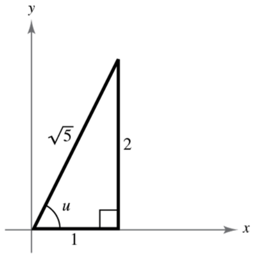
$\sin\left(\arctan \frac{3}{4}\right) = \sin u = \frac{3}{5}$



47. Let $u = \tan^{-1} 2,$

$\tan u = 2 = \frac{2}{1}, 0 < u < \frac{\pi}{2},$

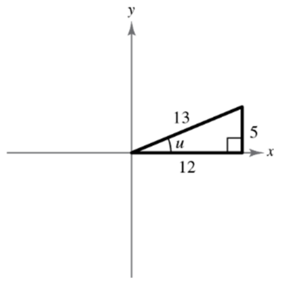
$\cos\left(\tan^{-1} 2\right) = \cos u = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$



49. Let $u = \arcsin \frac{5}{13},$

$\sin u = \frac{5}{13}, 0 < u < \frac{\pi}{2},$

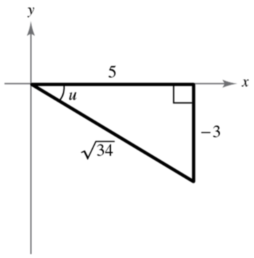
$\sec\left(\arcsin \frac{5}{13}\right) = \sec u = \frac{13}{12}.$



51. Let $u = \arctan\left(-\frac{3}{5}\right),$

$\tan u = -\frac{3}{5}, -\frac{\pi}{2} < y < 0,$

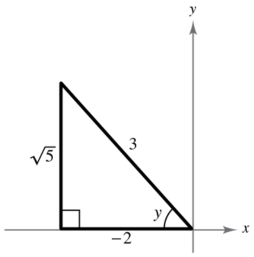
$\cot\left[\arctan\left(-\frac{3}{5}\right)\right] = \cot u = -\frac{5}{3}.$



53. Let $u = \arccos\left(-\frac{2}{3}\right).$

$\cos u = -\frac{2}{3}, \frac{\pi}{2} < u < \pi,$

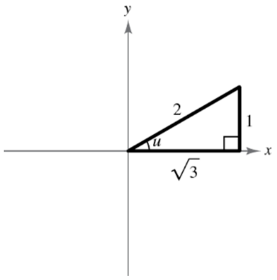
$\tan\left[\arccos\left(-\frac{2}{3}\right)\right] = \tan u = -\frac{\sqrt{5}}{2}$



55. Let $u = \cos^{-1} \frac{\sqrt{3}}{2}.$

$\cos u = \frac{\sqrt{3}}{2}, 0 < u < \frac{\pi}{2},$

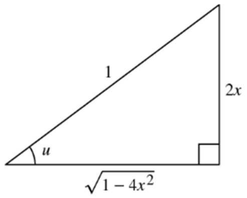
$\csc\left[\cos^{-1} \frac{\sqrt{3}}{2}\right] = \csc u = 2.$



57. Let $u = \arcsin(2x)$.

$\sin u = 2x = \dfrac{2x}{1}$,

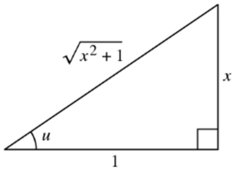
$\cos(\arcsin 2x) = \cos u = \sqrt{1 - 4x^2}$



59. Let $u = \arctan x$.

$\tan u = x = \dfrac{x}{1}$,

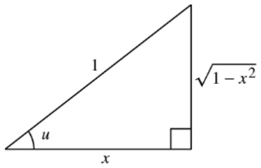
$\cot(\arctan x) = \cot u = \dfrac{1}{x}$



61. Let $u = \arccos x$.

$\cos u = x = \dfrac{x}{1}$,

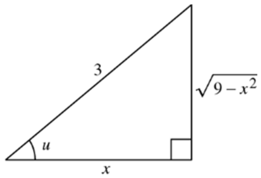
$\sin(\arccos x) = \sin u = \sqrt{1 - x^2}$



63. Let $u = \arccos\left(\dfrac{x}{3}\right)$.

$\cos u = \dfrac{x}{3}$,

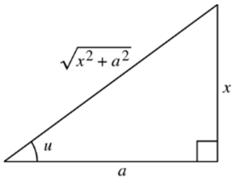
$\tan\left(\arccos \dfrac{x}{3}\right) = \tan u = \dfrac{\sqrt{9 - x^2}}{x}$



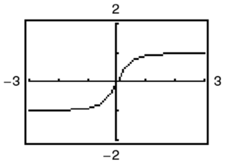
65. Let $u = \arctan \dfrac{x}{a}$.

$\tan u = \dfrac{x}{a}$,

$\csc\left(\arctan \dfrac{x}{a}\right) = \csc u = \dfrac{\sqrt{x^2 + a^2}}{a}$



67. $f(x) = \sin(\arctan 2x)$, $g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

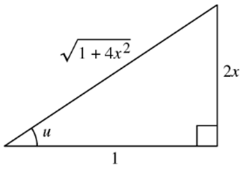


They are equal. Let $u = \arctan 2x$,

$\tan u = 2x = \dfrac{2x}{1}$,

and $\sin u = \dfrac{2x}{\sqrt{1 + 4x^2}}$.

$g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}} = f(x)$



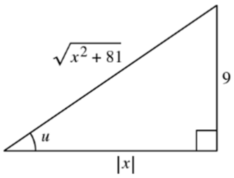
The graph has horizontal asymptotes at $y = \pm 1$.

69. Let $u = \arctan \dfrac{9}{x}$.

$\tan u = \dfrac{9}{x}$ and $\sin u = \dfrac{9}{\sqrt{x^2 + 81}}$, $x > 0$

So,

$\arctan \dfrac{9}{x} = \arcsin \dfrac{9}{\sqrt{x^2 + 81}}$, $x > 0$.

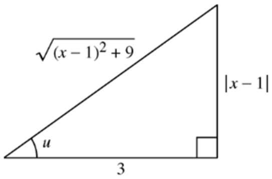


71. Let $u = \arccos \dfrac{3}{\sqrt{x^2 - 2x + 10}}$. Then,

$$\cos u = \frac{3}{\sqrt{x^2 - 2x + 10}} = \frac{3}{\sqrt{(x - 1)^2 + 9}}$$

and $\sin u = \dfrac{|x - 1|}{\sqrt{(x - 1)^2 + 9}}$.

So, $u = \arcsin \dfrac{|x - 1|}{\sqrt{x^2 - 2x + 10}}$.

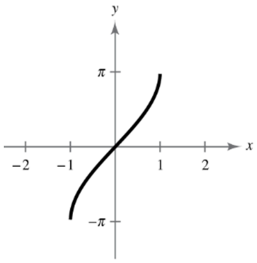


73. $g(x) = 2 \arcsin x$

Domain: $-1 \le x \le 1$

Range: $-\pi \le y \le \pi$

This is the graph of $f(x) = \arcsin(x)$ with a vertical stretch.

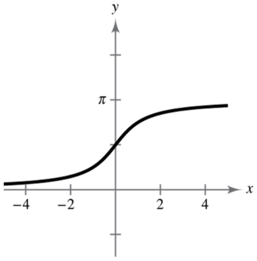


75. $f(x) = \dfrac{\pi}{2} + \arctan x$

Domain: all real numbers

Range: $0 < y \le \pi$

This is the graph of $y = \arctan x$ shifted upward $\pi/2$ units.

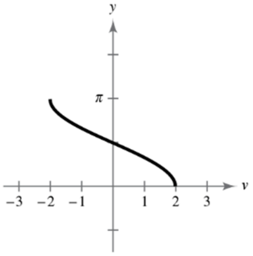


77. $h(v) = \arccos \dfrac{v}{2}$

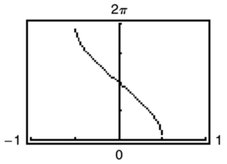
Domain: $-2 \le v \le 2$

Range: $0 \le y \le \pi$

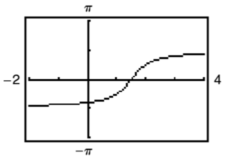
This is the graph of $h(v) = \arccos v$ with a horizontal stretch.



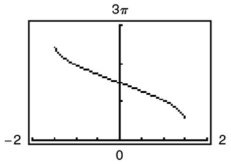
79. $f(x) = 2 \arccos(2x)$



81. $f(x) = \arctan(2x - 3)$

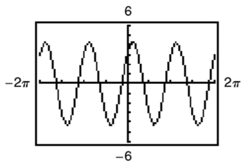


83. $f(x) = \pi - \sin^{-1} \dfrac{2x}{3} + \cos^{-1} \dfrac{2x}{3}$



85. $$\begin{aligned} f(t) = 3\cos 2t + 3\sin 2t &= \sqrt{3^2 + 3^2}\,\sin\left(2t + \arctan\frac{3}{3}\right)\\ &= 3\sqrt{2}\,\sin(2t + \arctan 1)\\ &= 3\sqrt{2}\,\sin\left(2t + \frac{\pi}{4}\right) \end{aligned}$$

The graph implies that the identity is true.



87. $\dfrac{\pi}{2}$

89. $\dfrac{\pi}{2}$

91. π

93. (a) $\sin\theta = \dfrac{5}{s}$

$\theta = \arcsin\dfrac{5}{s}$

(b) $s = 40$: $\theta = \arcsin\dfrac{5}{40} \approx 0.13$

$s = 20$: $\theta = \arcsin\dfrac{5}{20} \approx 0.25$

95. (a) $\tan\theta = \dfrac{5.5}{8.5}$

$\theta \approx 32.9°$

(b) $\tan 32.9° = \dfrac{h}{10}$

$h = 10\tan 32.9°$

$h \approx 6.49$ meters

The height is about 6.5 meters.

97. (a) $\sin\theta = \dfrac{6}{x}$

$\theta = \arcsin\dfrac{6}{x}$

(b) $x = 12$ miles

$\theta = \arcsin\dfrac{6}{12} \approx 30.0°$

$x = 7$ mile

$\theta = \arcsin\dfrac{6}{7} \approx 59.0°$

99. False. The domain of arctan x is all real x, but the domain of both arcsin x and arccos x are $-1 \le x \le 1$.

$$\arctan(-1) = -\frac{\pi}{4} \ne \frac{\arcsin(-1)}{\arcsin(-1)} = \frac{\left(-\frac{\pi}{2}\right)}{\pi} = -\frac{1}{2}$$

101. False.

$\dfrac{5\pi}{6}$ is not in the range of $\arcsin(x)$.

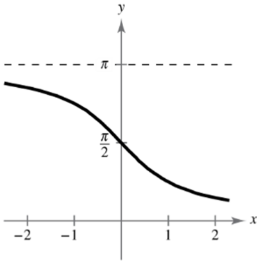
$\arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$

103. $\dfrac{5\pi}{4}$ is not in the range of the arcsin function.

105. $y = \operatorname{arccot} x$ if and only if $\cot y = x$.

Domain: $(-\infty, \infty)$

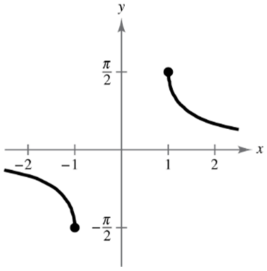
Range: $(0, \pi)$



107. $y = \operatorname{arccsc} x$ if and only if $\csc y = x$.

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$



109. $y = \operatorname{arcsec}\sqrt{2} \Rightarrow \sec y = \sqrt{2}$ and

$0 \le y < \dfrac{\pi}{2} \cup \dfrac{\pi}{2} < y \le \pi \Rightarrow y = \dfrac{\pi}{4}$

111. $y = \operatorname{arccot}(-1) \Rightarrow \cot y = -1$ and

$0 < y < \pi \Rightarrow y = \dfrac{3\pi}{4}$

113. $y = \operatorname{arccsc} 2 \Rightarrow \csc y = 2$ and

$$-\frac{\pi}{2} \le y < 0 \cup 0 < y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{6}$$

115. $y = \operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right) \Rightarrow \csc y = \frac{2\sqrt{3}}{3}$ and

$$-\frac{\pi}{2} \le y < 0 \cup 0 < y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{3}$$

117. $y = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right) \Rightarrow \cot y = \frac{-\sqrt{3}}{3}$ and

$$0 < y < \pi \Rightarrow y = \frac{2\pi}{3}$$

119. $\operatorname{arcsec} 2.54 = \arccos\left(\frac{1}{2.54}\right) \approx 1.17$

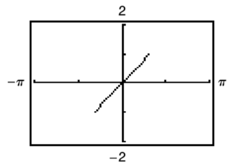
121. $\operatorname{arccsc}\left(-\frac{25}{3}\right) = \arcsin\left(-\frac{3}{25}\right) \approx -0.12$

123. $\operatorname{arccot} 5.25 = \arctan\left(\frac{1}{5.25}\right) \approx 0.19$

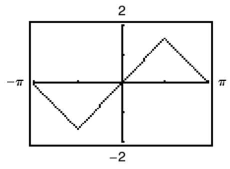
125. $\operatorname{arccot}\left(\frac{5}{3}\right) = \arctan\left(\frac{3}{5}\right) \approx 0.54$

127. $f(x) = \sin(x), f^{-1}(x) = \arcsin(x)$

(a) $f \circ f^{-1} = \sin(\arcsin x)$



$f^{-1} \circ f = \arcsin(\sin x)$



(b) The graphs coincide with the graph of $y = x$ only for certain values of x.

$f \circ f^{-1} = x$ over its entire domain, $-1 \le x \le 1$.

$f^{-1} \circ f = x$ over the region $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, corresponding to the region where $\sin x$ is one-to-one and has an inverse.

129. Area $= \arctan b - \arctan a$

(a) $a = 0, b = 1$

$$\text{Area} = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(b) $a = -1, b = 1$

$$\text{Area} = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) $a = 0, b = 3$

$$\text{Area} = \arctan 3 - \arctan 0 \approx 1.25 - 0 = 1.25$$

(d) $a = -1, b = 3$

$$\text{Area} = \arctan 3 - \arctan(-1) \approx 1.25 - \left(-\frac{\pi}{4}\right) \approx 2.03$$

131. $x^2 + 8^2 = 12^2 = 144$

$$x^2 = 144 - 64 = 80$$
$$x = \sqrt{80} = 4\sqrt{5}$$

133. $x^2 + 2^2 = 6^2$

$$x^2 = 36 - 4 = 32$$
$$x = 4\sqrt{2}$$

135. $\cos 40° = \frac{c}{5} \Rightarrow c = 5 \cos 40°$

137. $\tan 40° = \frac{5}{c} \Rightarrow c = \frac{5}{\tan 40°}$

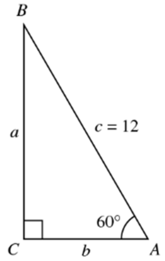
Section 4.8 Applications and Models

1. bearing

3. No. N 20° E means a direction first of due north, then 20° east of north.

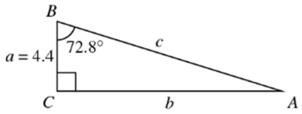
5. Given: $A = 60°, c = 12$

$$\sin A = \frac{a}{c} \Rightarrow a = c \sin A$$
$$= 12 \sin 60° = \frac{12\sqrt{3}}{2}$$
$$= 6\sqrt{3} \approx 10.39$$
$$\cos A = \frac{b}{c} \Rightarrow b = c \cos A$$
$$= 12 \cos 60° = 12\left(\frac{1}{2}\right) = 6$$
$$B = 90° - 60° = 30°$$



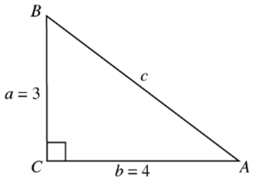
7. Given: $B = 72.8°, a = 4.4$

$$\cos B = \frac{a}{c} \Rightarrow c = \frac{a}{\cos B} = \frac{4.4}{\cos 72.8°} \approx 14.88$$
$$\tan B = \frac{b}{a} \Rightarrow b = a \tan B = 4.4 \tan 72.8° \approx 14.21$$
$$A = 90° - 72.8° = 17.2°$$



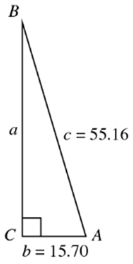
9. Given: $a = 3, b = 4$

$$a^2 + b^2 = c^2 \Rightarrow c^2 = (3)^2 + (4)^2 \Rightarrow c = 5$$
$$\tan A = \frac{a}{b} \Rightarrow A = \tan^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.87°$$
$$B = 90° - 36.87° = 53.13°$$



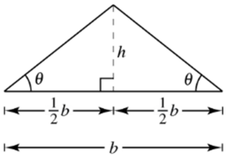
11. Given: $b = 15.70, c = 55.16$

$$a = \sqrt{55.16^2 - 15.7^2} \approx 52.88$$
$$\cos A = \frac{b}{c}$$
$$\cos A = \frac{15.7}{55.15}$$
$$A = \arccos \frac{15.7}{55.15} \approx 73.46°$$
$$B = 90° - 73.46° \approx 16.54°$$



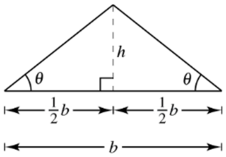
13. $\theta = 45°, b = 6$

$$\tan \theta = \frac{h}{(1/2)b} \Rightarrow h = \frac{1}{2}b \tan \theta$$
$$h = \frac{1}{2}(6) \tan 45° = 3.00 \text{ units}$$



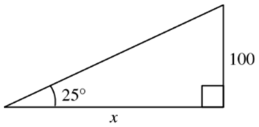
15. $\tan \theta = \frac{h}{(1/2)b} \Rightarrow h = \frac{1}{2}b \tan \theta$

$$h = \frac{1}{2}(8) \tan 32° \approx 2.50 \text{ units}$$



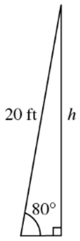
17. $\tan 25° = \frac{100}{x}$

$$x = \frac{100}{\tan 25°}$$
$$\approx 214.45 \text{ feet}$$



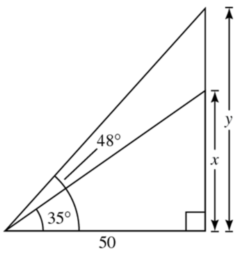
19. $\sin 80° = \frac{h}{20}$

$$20 \sin 80° = h$$
$$h \approx 19.7 \text{ feet}$$



21. Let the height of the castle $= x$ and the height of the castle and tower $= y$.

$$\tan 35° = \frac{x}{50} \text{ and } \tan 48° = \frac{y}{50}$$
$$x = 50 \tan 35° \approx 35.01 \text{ and } y = 50 \tan 48° \approx 55.53$$
$$h = y - x = 55.53 - 35.01 = 20.52.$$
$$h \approx 20.5 \text{ feet}$$

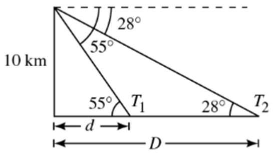


23. $\cot 55 = \dfrac{d}{10} \Rightarrow d \approx 7$ kilometers

$\cot 28° = \dfrac{D}{10} \Rightarrow D \approx 18.8$ kilometers

Distance between towns:

$D - d = 18.8 - 7 = 11.8$ kilometers



25. (a) $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$

$\sin 60° = \dfrac{h}{30}$

$h = 15\sqrt{3} \approx 25.98$ feet

(b) $\tan\theta = \dfrac{25.981}{d}$

$\theta = \arctan\left(\dfrac{25.981}{d}\right)$

(c) When $\theta = 25°$,

$\tan 25° = \dfrac{25.981}{d}$

$d = \dfrac{25.981}{\tan 25°}$

$d \approx 55.72$ feet.

When $\theta = 30°$,

$\tan 30° = \dfrac{25.981}{d}$

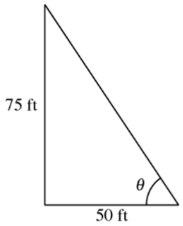
$d = \dfrac{25.981}{\tan 30°}$

$d \approx 45.00$

$45 \text{ ft} \le d \le 55.7$ feet.

27. $\tan\theta = \frac{75}{50}$

$\theta = \arctan\frac{3}{2} \approx 56.3°$



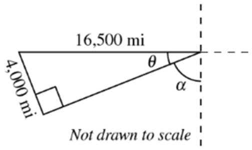
29. $12{,}500 + 4000 = 16{,}500$

$\sin\theta = \dfrac{4000}{16{,}500}$

$\theta = \arcsin\left(\dfrac{4000}{16{,}500}\right)$

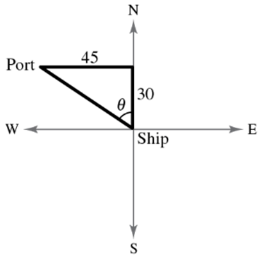
$\theta \approx 14.03°$

Angle of depression $= \alpha \approx 90° - 14.03° = 75.97°$



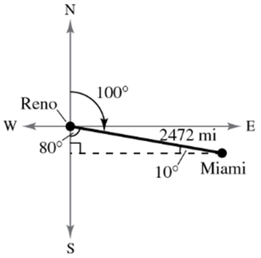
31. $\tan\theta = \frac{45}{30} \Rightarrow \theta \approx 56.3°$

Bearing: N 56.31°

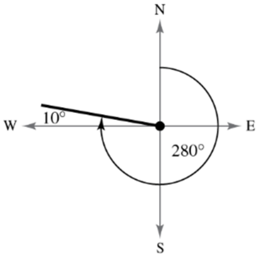


33. (a) Reno is $2472 \sin 10° = 429.26$ miles north of Miami.

Reno is $2472 \cos 10° = 2434.44$ miles west of Miami.

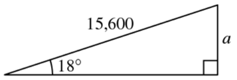


(b) The return heading is 280°.



35. (a) Because the airplane speed is

$$\left(260\,\frac{\text{ft}}{\text{sec}}\right)\left(60\,\frac{\text{sec}}{\text{min}}\right) = 15{,}600\,\frac{\text{ft}}{\text{min}},$$



after one minute its distance travelled is 15,600 feet.

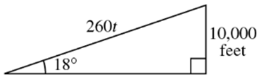
$$\sin 18^\circ = \frac{a}{15{,}600}$$

$$a = 15{,}600 \sin 18^\circ \approx 4820.7 \text{ ft}$$

(b) Let t be the time for the plane to reach 10,000 ft.

$$\sin 18^\circ = \frac{10{,}000}{260t}$$

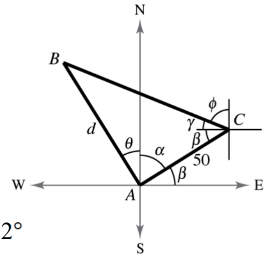
$$s = \frac{10{,}000}{260(\sin 18^\circ)} \approx 124.5 \text{ seconds}$$



37. $\theta = 32^\circ, \phi = 68^\circ$

(a) $\alpha = 90^\circ - 32^\circ = 58^\circ$

Bearing from
A to C: N 58° E



(b)

$$\beta = \theta = 32^\circ$$

$$\gamma = 90^\circ - \phi = 22^\circ$$

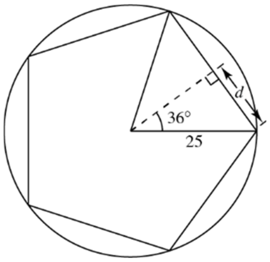
$$C = \beta + \gamma = 54^\circ$$

$$\tan C = \frac{d}{50} \Rightarrow \tan 54^\circ$$

$$= \frac{d}{50} \Rightarrow d \approx 68.82 \text{ meters}$$

39. $\sin 36^\circ = \dfrac{d}{25} \Rightarrow d \approx 14.69$

Length of side: $2d \approx 29.4$ inches

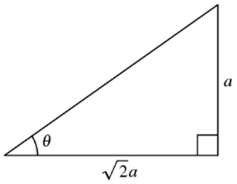


41. The diagonal of the base has a length of $\sqrt{a^2 + a^2} = \sqrt{2}a$. Now, you have

$$\tan\theta = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\theta = \arctan\frac{1}{\sqrt{2}}$$

$$\theta \approx 35.3^\circ.$$



43. Use $d = a\sin\omega t$ because $d = 0$ when $t = 0$.

$$\text{Period: } \frac{2\pi}{\omega} = 2 \Rightarrow \omega = \pi$$

So, $d = 4\sin(\pi t)$.

45. Use $d = a\cos\omega t$ because $d = 3$ when $t = 0$.

$$\text{Period: } \frac{2\pi}{\omega} = 1.5 \Rightarrow \omega = \frac{4\pi}{3}$$

So, $d = 3\cos\left(\dfrac{4\pi}{3}t\right) = 3\cos\left(\dfrac{4\pi t}{3}\right)$.

47. $d = a\sin\omega t$

$$\text{Frequency} = \frac{\omega}{2\pi}$$

$$262 = \frac{\omega}{2\pi}$$

$$\omega = 2\pi(262) = 524\pi$$

49. $d = 9\cos\frac{6\pi}{5}t$

(a) Maximum displacement = amplitude = 9

(b) $\text{Frequency} = \frac{\omega}{2\pi} = \frac{\frac{6\pi}{5}}{2\pi}$

$= \frac{3}{5}$ cycle per unit of time

(c) $d = 9\cos\frac{6\pi}{5}(5) = 9$

(d) $9\cos\frac{6\pi}{5}t = 0$

$$\cos\frac{6\pi}{5}t = 0$$

$$\frac{6\pi}{5}t = \arccos 0$$

$$\frac{6\pi}{5}t = \frac{\pi}{2}$$

$$t = \frac{5}{12}$$

51. $d = \frac{1}{4}\sin 6\pi t$

(a) Maximum displacement = amplitude = $\frac{1}{4}$

(b) $\text{Frequency} = \frac{\omega}{2\pi} = \frac{6\pi}{2\pi}$

$= 3$ cycles per unit of time

(c) $d = \frac{1}{4}\sin 30\pi \approx 0$

(d) $\frac{1}{4}\sin 6\pi t = 0$

$$\sin 6\pi t = 0$$

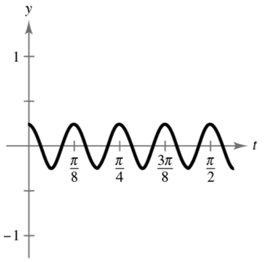
$$6\pi t = \arcsin 0$$

$$6\pi t = \pi$$

$$t = \frac{1}{6}$$

53. $y = \frac{1}{4}\cos 16t,\ t > 0$

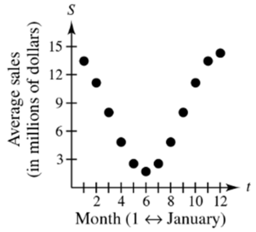
(a)



(b) Period: $\frac{2\pi}{16} = \frac{\pi}{8}$

(c) $\frac{1}{4}\cos 16t = 0$ when $16t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{32}$

55. (a)



(b) $a = \frac{1}{2}(14.3 - 1.7) = 6.3$

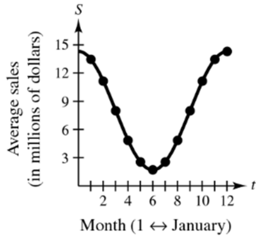
$\frac{2\pi}{b} = 12 \Rightarrow b = \frac{\pi}{6}$

Shift: $d = 14.3 - 6.3$

$= 8$

$S = d + a\cos bt$

$S = 8 + 6.3\cos\left(\frac{\pi t}{6}\right)$



Note: Another model is $S = 8 + 6.3\sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right)$.

The model is a good fit.

(c) The period is $\frac{2\pi}{(\pi/6)} = 12$. Yes, sales of outwear are seasonal.

(d) The amplitude is the maximum displacement from average sales of \$8 million.

57. False. The tower isn't vertical and so the triangle formed is not a right triangle.

59. $6t^3 - 2t + 10t^2 = 2t(3t^2 + 5t - 1)$

61. $10y^2 - 13y - 3 = (5y + 1)(2y - 3)$

63. $25z^4 - x^2 = (5z^2 - x)(5z^2 + x)$

65. $\sin\theta = \dfrac{1}{4}$

(a) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \left(\dfrac{1}{4}\right)^2 = \dfrac{15}{16}$

$\Rightarrow \cos\theta = \dfrac{\sqrt{15}}{4}$

(b) $\cot\theta + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\left(\dfrac{\sqrt{15}}{4}\right)}{\left(\dfrac{1}{4}\right)} = \sqrt{15}$

(c) $\csc\theta = \dfrac{1}{\sin\theta} = 4$

(d) $\tan(90° - \theta) = \cot\theta = \sqrt{15}$

67. $\log_2 x + \log_2 5 = \log_2(5x)$

69. $3\log x - \log(x + 2) - \log(x - 2)$

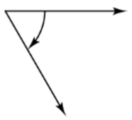
$= \log x^3 - \log(x + 2) - \log(x - 2)$

$= \log\left[\dfrac{x^3}{(x + 2)(x - 2)}\right]$

Review Exercises for Chapter 4

1. The angle is approximately 2 radians.

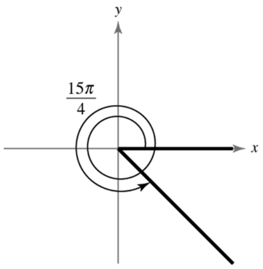
3.



The angle shown is 60°.

5. $\theta = \dfrac{15\pi}{4}$

(a)



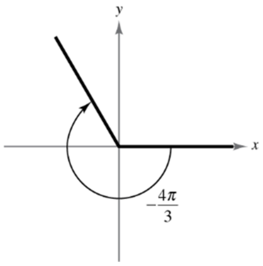
(b) Quadrant IV

(c) $\dfrac{15\pi}{4} - 2\pi = \dfrac{7\pi}{4}$

$\dfrac{7\pi}{4} - 2\pi = -\dfrac{\pi}{4}$

7. $\theta = -\dfrac{4\pi}{3}$

(a)



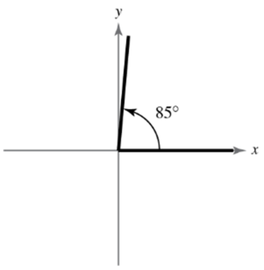
(b) Quadrant II

(c) Coterminal angles:

$-\dfrac{4\pi}{3} + 2\pi = \dfrac{2\pi}{3}$

$-\dfrac{4\pi}{3} - 2\pi = -\dfrac{10\pi}{3}$

9. (a) $\theta = 85°$



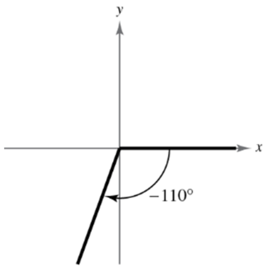
(b) The angle lies in Quadrant I.

(c) Coterminal angles: $85° + 360° = 445°$

$85° - 360° = -275°$

11. $\theta = -110°$

(a)



(b) Quadrant III

(c) Coterminal angles:

$-110° + 360° = 250°$

$-110° - 360° = -470°$

13. $138° = \dfrac{138\pi}{180} = \dfrac{23\pi}{30}$ radians

$s = r\theta = 20\left(\dfrac{23\pi}{30}\right) \approx 48.17$ inches

15. $150° = \dfrac{150\pi}{180} = \dfrac{5\pi}{6}$ radians

$A = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(20)^2\left(\dfrac{5\pi}{6}\right) = \dfrac{500\pi}{3} \approx 523.6$ square inches

17. $t = \dfrac{2\pi}{3}$ corresponds to the point $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

19. $t = \dfrac{7\pi}{6}$ corresponds to the point $(x, y) = \left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.

21. $t = \dfrac{3\pi}{4}$ corresponds to the point $(x, y) = \left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

$$\sin\frac{3\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \csc\frac{3\pi}{4} = \frac{1}{y} = \sqrt{2}$$

$$\cos\frac{3\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec\frac{3\pi}{4} = \frac{1}{x} = -\sqrt{2}$$

$$\tan\frac{3\pi}{4} = \frac{y}{x} = -1 \qquad \cot\frac{3\pi}{4} = \frac{x}{y} = -1$$

23. $\sin\dfrac{11\pi}{4} = \sin\dfrac{3\pi}{4} = \dfrac{\sqrt{2}}{2}$

25. $\cos\left(-\dfrac{17\pi}{6}\right) = \cos\dfrac{7\pi}{6} = -\dfrac{\sqrt{3}}{2}$

27. $\sec\left(\dfrac{12\pi}{5}\right) = \dfrac{1}{\cos\left(\dfrac{12\pi}{5}\right)} \approx 3.2361$

29. $\tan 33 \approx -75.3130$

31. opp = 4, adj = 5, hyp $= \sqrt{4^2 + 5^2} = \sqrt{41}$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{41}}{4}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{41}}{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{5} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{5}{4}$$

33. $\tan 33° \approx 0.6494$

35. $\cot 15° 14' = \dfrac{1}{\tan\left(15 + \frac{14}{60}\right)} \approx 3.6722$

37. $\sin\theta = \dfrac{1}{3}$

(a) $\csc\theta = \dfrac{1}{\sin\theta} = 3$

(b) $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{9}$$

$$\cos^2\theta = \frac{8}{9}$$

$$\cos\theta = \sqrt{\frac{8}{9}}$$

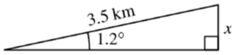
$$\cos\theta = \frac{2\sqrt{2}}{3}$$

(c) $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{3}{2\sqrt{2}} = \dfrac{3\sqrt{2}}{4}$

(d) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1/3}{\left(2\sqrt{2}\right)/3} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

39. $\sin 1.2° = \dfrac{x}{3.5}$

$x = 3.5 \sin 1.2° \approx 0.0733$ kilometer or 73.3 meters



Not drawn to scale

41. $x = 12, y = 16, r = \sqrt{144 + 256} = \sqrt{400} = 20$

$\sin\theta = \dfrac{y}{r} = \dfrac{4}{5}$ $\qquad$ $\csc\theta = \dfrac{r}{y} = \dfrac{5}{4}$

$\cos\theta = \dfrac{x}{r} = \dfrac{3}{5}$ $\qquad$ $\sec\theta = \dfrac{r}{x} = \dfrac{5}{3}$

$\tan\theta = \dfrac{y}{x} = \dfrac{4}{3}$ $\qquad$ $\cot\theta = \dfrac{x}{y} = \dfrac{3}{4}$

43. $x = 0.3, y = 0.4$

$r = \sqrt{(0.3)^2 + (0.4)^2} = 0.5$

$\sin\theta = \dfrac{y}{r} = \dfrac{0.4}{0.5} = \dfrac{4}{5} = 0.8$ $\qquad$ $\csc\theta = \dfrac{r}{y} = \dfrac{0.5}{0.4} = \dfrac{5}{4} = 1.25$

$\cos\theta = \dfrac{x}{r} = \dfrac{0.3}{0.5} = \dfrac{3}{5} = 0.6$ $\qquad$ $\sec\theta = \dfrac{r}{x} = \dfrac{0.5}{0.3} = \dfrac{5}{3} \approx 1.67$

$\tan\theta = \dfrac{y}{x} = \dfrac{0.4}{0.3} = \dfrac{4}{3} \approx 1.33$ $\qquad$ $\cot\theta = \dfrac{x}{y} = \dfrac{0.3}{0.4} = \dfrac{3}{4} = 0.75$

45. $\sec\theta = \frac{6}{5}, \tan\theta < 0 \Rightarrow \theta$ is in Quadrant IV.

$r = 6, x = 5, y = -\sqrt{36 - 25} = -\sqrt{11}$

$\sin\theta = \frac{y}{r} = -\frac{\sqrt{11}}{6}$

$\cos\theta = \frac{x}{r} = \frac{5}{6}$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{11}}{5}$

$\csc\theta = \frac{r}{y} = -\frac{6\sqrt{11}}{11}$

$\cot\theta = -\frac{5\sqrt{11}}{11}$

47. $\cos\theta = \frac{x}{r} = \frac{-2}{5} \Rightarrow y^2 = 21$

$\sin\theta > 0 \Rightarrow \theta$ is in Quadrant II $\Rightarrow y = \sqrt{21}$

$\sin\theta = \frac{y}{r} = \frac{\sqrt{21}}{5}$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{21}}{2}$

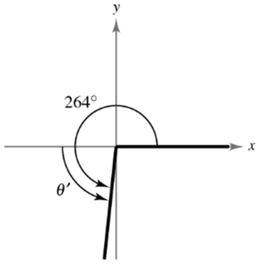
$\csc\theta = \frac{r}{y} = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$

$\sec\theta = \frac{r}{x} = \frac{5}{-2} = -\frac{5}{2}$

$\cot\theta = \frac{x}{y} = \frac{-2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$

49. $\theta = 264°$

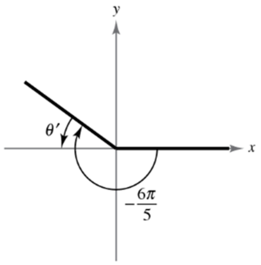
$= 264° - 180° = 84°$



51. $\theta = -\frac{6\pi}{5}$

$-\frac{6\pi}{5} + 2\pi = \frac{4\pi}{5}$

$\theta' = \pi - \frac{4\pi}{5} = \frac{\pi}{5}$



53. $\sin(-150°) = -\frac{1}{2}$

$\cos(-150°) = -\frac{\sqrt{3}}{2}$

$\tan(-150°) = \frac{-1/2}{-\sqrt{3}/2} = \frac{\sqrt{3}}{3}$

55. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \sqrt{3}$

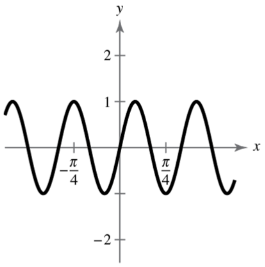
57. $\sin 106° \approx 0.9613$

59. $\tan\left(-\frac{17\pi}{15}\right) \approx -0.4452$

61. $y = \sin 6x$

Amplitude: 1

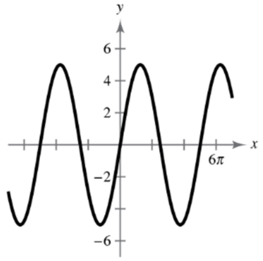
Period: $\frac{2\pi}{6} = \frac{\pi}{3}$



63. $f(x) - 5 \sin \frac{2x}{5}$

Amplitude: 5

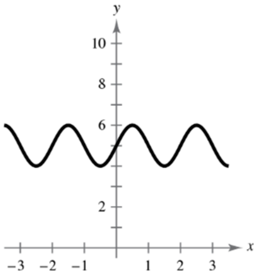
Period: $\frac{2\pi}{2/5} = 5\pi$



65. $y = 5 + \sin \pi x$

Amplitude: 1

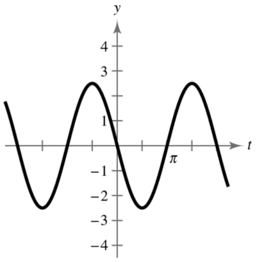
Period: $\frac{2\pi}{\pi} = 2$



67. $g(t) = \frac{5}{2} \sin(t - \pi)$

Amplitude: $\frac{5}{2}$

Period: 2π



69. $y = a \sin bx$

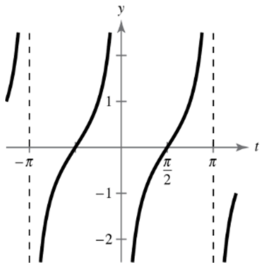
(a) $a = 2$,

$$\frac{2\pi}{b} = \frac{1}{264} \Rightarrow b = 528\pi$$

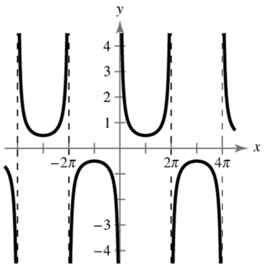
$$y = 2 \sin 528\pi x$$

(b) $f = \frac{1}{1/264} = 264$ cycles per second

71. $f(t) = \tan\left(t + \frac{\pi}{2}\right)$



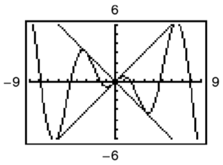
73. $f(x) = \frac{1}{2} \csc \frac{x}{2}$



75. $f(x) = x \cos x$

Damping factor: x

As $x \to \infty$, $f(x)$ oscillates.



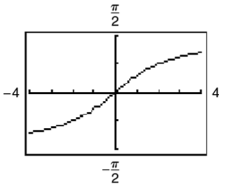
77. $\arcsin(-1) = -\frac{\pi}{2}$

79. $\operatorname{arccot} \sqrt{3} = \frac{\pi}{6}$

81. $\tan^{-1}(-1.3) \approx -0.92$ radian

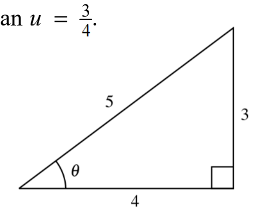
83. $\operatorname{arccot} 15.5 = \arctan \frac{1}{15.5} \approx 0.06$

85. $f(x) = \arctan\left(\frac{x}{2}\right) = \tan^{-1}\left(\frac{x}{2}\right)$



87. Let $u = \arctan \frac{3}{4}$ then $\tan u = \frac{3}{4}$.

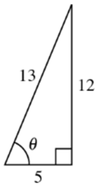
$\cos\left(\arctan \frac{3}{4}\right) = \frac{4}{5}$



89. Let $u = \arctan \frac{12}{5}$

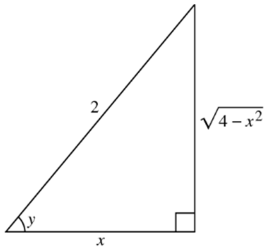
then $\tan u = \frac{12}{5}$.

$\sec\left(\arctan \frac{12}{5}\right) = \frac{13}{5}$



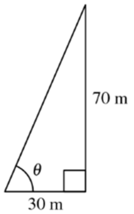
91. Let $y = \arccos\left(\frac{x}{2}\right)$. Then

$\cos y = \frac{x}{2}$ and $\tan y = \tan\left(\arccos\left(\frac{x}{2}\right)\right) = \frac{\sqrt{4 - x^2}}{x}$.



93. $\tan \theta = \frac{70}{30}$

$\theta = \arctan\left(\frac{70}{30}\right) \approx 66.8°$



95. $\sin 48° = \frac{d_1}{650} \Rightarrow d_1 \approx 483$

$\cos 25° = \frac{d_2}{810} \Rightarrow d_2 \approx 734$

$d_1 + d_2 \approx 1217$

$\cos 48° = \frac{d_3}{650} \Rightarrow d_3 \approx 435$

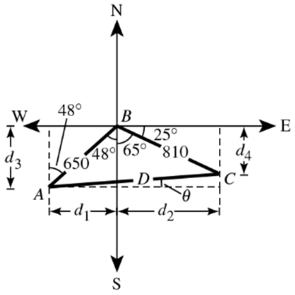
$\sin 25° = \frac{d_4}{810} \Rightarrow d_4 \approx 342$

$d_3 - d_4 \approx 93$

$\tan \theta \approx \frac{93}{1217} \Rightarrow \theta \approx 4.4°$

$\sec 4.4° \approx \frac{D}{1217} \Rightarrow D \approx 1217 \sec 4.4° \approx 1221$

The distance is 1221 miles and the bearing is 85.6°.



97. False. For each θ there corresponds exactly one value of y.

99. $f(\theta) = \sec \theta$ is undefined at the zeros of $g(\theta) = \cos \theta$ because $\sec \theta = \frac{1}{\cos \theta}$.

101. The ranges of the other four trigonometric functions are not bounded. For $y = \tan x$ and $y = \cot x$, the range is $(-\infty, \infty)$. For $y = \sec x$ and $y = \csc x$, the range is $(-\infty, -1] \cup [1, \infty)$.

Problem Solving for Chapter 4

1. (a) $8{:}57 - 6{:}45 = 2$ hours 12 minutes $= 132$ minutes

$\frac{132}{48} = \frac{11}{4}$ revolutions

$\theta = \left(\frac{11}{4}\right)(2\pi) = \frac{11\pi}{2}$ radians or 990°

(b) $s = r\theta = 47.25(5.5\pi) \approx 816.42$ feet

3. If you alter the model so that $h = 1$ when $t = 0$, you can use either a sine or a cosine model.

$a = \frac{1}{2}[\max - \min] = \frac{1}{2}[101 - 1] = 50$

$d = \frac{1}{2}[\max + \min] = \frac{1}{2}[101 + 1] = 51$

$b = 8\pi$

Cosine model: $h = 51 - 50\cos(8\pi t)$

Sine model: $h = 51 - 50\sin\left(8\pi t + \frac{\pi}{2}\right)$

Notice that you needed the horizontal shift so that the sine value was one when $t = 0$.

Another model would be: $h = 51 + 50\sin\left(8\pi t + \frac{3\pi}{2}\right)$

Here you wanted the sine value to be 1 when $t = 0$.

5. (a) $\sin 39° = \dfrac{3000}{d}$

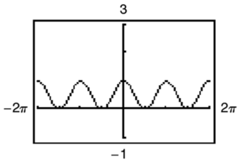
$d = \dfrac{3000}{\sin 39°} \approx 4767$ feet

(b) $\tan 39° = \dfrac{3000}{x}$

$x = \dfrac{3000}{\tan 39°} \approx 3705$ feet

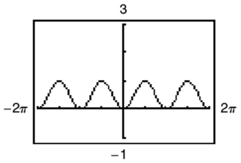
(c)
$$\tan 63° = \frac{w + 3705}{3000}$$
$$3000 \tan 63° = w + 3705$$
$$w = 3000 \tan 63° - 3705 \approx 2183 \text{ feet}$$

7. (a) $h(x) = \cos^2 x$



h is even.

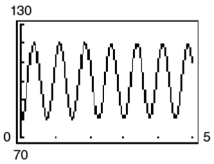
(b) $h(x) = \sin^2 x$



h is even.

9. $P = 100 - 20 \cos\left(\dfrac{8\pi}{3}t\right)$

(a)



(b) Period $= \dfrac{2\pi}{8\pi/3} = \dfrac{6}{8} = \dfrac{3}{4}$ sec

This is the time between heartbeats.

(c) Amplitude: 20

The blood pressure ranges between $100 - 20 = 80$ and $100 + 20 = 120$.

(d) Pulse rate $= \dfrac{60 \text{ sec/min}}{\frac{3}{4} \text{ sec/beat}} = 80$ beats/min

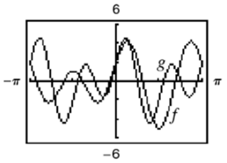
(e) Period $= \dfrac{60}{64} = \dfrac{15}{16}$ sec

$$64 = \frac{60}{2\pi/b} \Rightarrow b = \frac{64}{60} \cdot 2\pi = \frac{32}{15}\pi$$

11. $f(x) = 2 \cos 2x + 3 \sin 3x$

$g(x) = 2 \cos 2x + 3 \sin 4x$

(a)

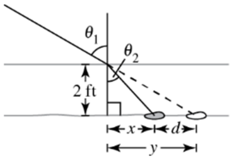


(b) The period of $f(x)$ is 2π.

The period of $g(x)$ is π.

(c) $h(x) = A \cos \alpha x + B \sin \beta x$ is periodic because the sine and cosine functions are periodic.

13.



(a)
$$\frac{\sin \theta_1}{\sin \theta_2} = 1.333$$
$$\sin \theta_2 = \frac{\sin \theta_1}{1.333} = \frac{\sin 60°}{1.333} \approx 0.6497$$
$$\theta_2 = 40.5°$$

(b) $\tan \theta_2 = \dfrac{x}{2} \Rightarrow x = 2 \tan 40.52° \approx 1.71$ feet

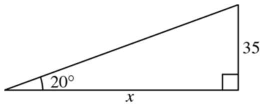
$\tan \theta_1 = \dfrac{y}{2} \Rightarrow y = 2 \tan 60° \approx 3.46$ feet

(c) $d = y - x = 3.46 - 1.71 = 1.75$ feet

(d) As you move closer to the rock, θ_1 decreases, which causes y to decrease, which in turn causes d to decrease.

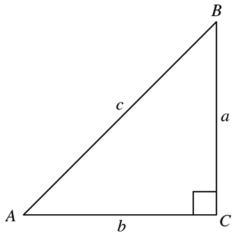
Practice Test for Chapter 4

1. Express 350° in radian measure.
2. Express $(5\pi)/9$ in degree measure.
3. Convert $135°\,14'\,12''$ to decimal form.
4. Convert $-22.569°$ to $D°\,M'\,S''$ form.
5. If $\cos\theta = \frac{2}{3}$, use the trigonometric identities to find $\tan\theta$.
6. Find θ given $\sin\theta = 0.9063$.
7. Solve for x in the figure below.



8. Find the magnitude of the reference angle for $\theta = (6\pi)/5$.
9. Evaluate $\csc 3.92$.
10. Find $\sec\theta$ given that θ lies in Quadrant III and $\tan\theta = 6$.
11. Graph $y = 3\sin\dfrac{x}{2}$.
12. Graph $y = -2\cos(x - \pi)$.
13. Graph $y = \tan 2x$.
14. Graph $y = -\csc\left(x + \dfrac{\pi}{4}\right)$.
15. Graph $y = 2x + \sin x$, using a graphing calculator.
16. Graph $y = 3x\cos x$, using a graphing calculator.
17. Evaluate $\arcsin 1$.
18. Evaluate $\arctan(-3)$.
19. Evaluate $\sin\left(\arccos\dfrac{4}{\sqrt{35}}\right)$.
20. Write an algebraic expression for $\cos\left(\arcsin\dfrac{x}{4}\right)$.

For Exercises 21–23, solve the right triangle.



21. $A = 40°, c = 12$

22. $B = 6.84°, a = 21.3$

23. $a = 5, b = 9$

24. A 20-foot ladder leans against the side of a barn. Find the height of the top of the ladder if the angle of elevation of the ladder is 67°.

25. An observer in a lighthouse 250 feet above sea level spots a ship off the shore. If the angle of depression to the ship is 5°, how far out is the ship?

CHAPTER 5

Analytic Trigonometry

Section 5.1 Using Fundamental Identities **207**

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CHAPTER 5
Analytic Trigonometry

Section 5.1 Using Fundamental Identities

1. $\cot u$

3. $\csc^2 u$

5. $\sec x = -\frac{5}{2}$, $\tan x < 0 \Rightarrow x$ is in Quadrant II.

$$\cos x = \frac{1}{\sec x} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$$

$$\sin x = \sqrt{1 - \left(-\frac{2}{5}\right)^2} = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{\sqrt{21}}{5}}{-\frac{2}{5}} = -\frac{\sqrt{21}}{2}$$

$$\csc x = \frac{1}{\sin x} = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

$$\cot x = \frac{1}{\tan x} = -\frac{2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$$

7. $\sin \theta = -\frac{3}{4}$, $\cos \theta > 0 \Rightarrow \theta$ is in Quadrant IV.

$$\cos \theta = \sqrt{1 - \left(-\frac{3}{4}\right)^2} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\frac{3}{4}}{\frac{\sqrt{7}}{4}} = -\frac{3}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{\sqrt{7}}{4}} = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{-\frac{3}{\sqrt{7}}} = -\frac{\sqrt{7}}{3}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

9. $\tan x = \frac{2}{3}$, $\cos x > 0 \Rightarrow x$ is in Quadrant I.

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\sec x = \sqrt{1 + \left(\frac{2}{3}\right)^2} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$$\csc x = \sqrt{1 + \left(\frac{3}{2}\right)^2} = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$$

$$\sin x = \frac{1}{\csc x} = \frac{1}{\frac{\sqrt{13}}{2}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\frac{\sqrt{13}}{3}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

11. $\sec x \cos x = \left(\frac{1}{\cancel{\cos x}}\right)\cancel{\cos x}$

$= 1$

Matches (c).

12. $\cot^2 x - \csc^2 x = (\csc^2 x - 1) - \csc^2 x$

$= -1$

Matches (b).

13. $\cos x(1 + \tan^2 x) = \cos x(\sec^2 x)$

$= \cos x\left(\frac{1}{\cos^2 x}\right)$

$= \frac{1}{\cos x}$

$= \sec x$

Matches (f).

14. $\cot x \sec x = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} = \frac{1}{\sin x} = \csc x$

Matches (a).

15. $\dfrac{\sec^2 x - 1}{\sin^2 x} = \dfrac{\tan^2 x}{\sin^2 x} = \dfrac{\sin^2 x}{\cos^2 x} \cdot \dfrac{1}{\sin^2 x} = \sec^2 x$

Matches (e).

16. $\dfrac{\cos^2\left[(\pi/2) - x\right]}{\cos x} = \dfrac{\sin^2 x}{\cos x} = \dfrac{\sin x}{\cos x}\sin x = \tan x \sin x$

Matches (d).

17. $$\frac{\tan\theta\cot\theta}{\sec\theta} = \frac{\tan\theta\left(\dfrac{1}{\tan\theta}\right)}{\dfrac{1}{\cos\theta}} = \frac{1}{\dfrac{1}{\cos\theta}} = \cos\theta$$

19. $$\begin{aligned}\tan^2 x - \tan^2 x \sin^2 x &= \tan^2 x\left(1 - \sin^2 x\right)\\ &= \tan^2 x \cos^2 x\\ &= \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x\\ &= \sin^2 x\end{aligned}$$

21. $$\begin{aligned}\frac{\sec^2 x - 1}{\sec x - 1} &= \frac{(\sec x + 1)(\sec x - 1)}{\sec x - 1}\\ &= \sec x + 1\end{aligned}$$

23. $$\begin{aligned}1 - 2\cos^2 x + \cos^4 x &= \left(1 - \cos^2 x\right)^2\\ &= \left(\sin^2 x\right)^2\\ &= \sin^4 x\end{aligned}$$

25. $$\begin{aligned}\cot^3 x + \cot^2 x + \cot x + 1 &= \cot^2 x(\cot x + 1) + (\cot x + 1)\\ &= (\cot x + 1)\left(\cot^2 x + 1\right)\\ &= (\cot x + 1)\csc^2 x\end{aligned}$$

27. $3\sin^2 x - 5\sin x - 2 = (3\sin x + 1)(\sin x - 2)$

29. $$\begin{aligned}\cot^2 x + \csc x - 1 &= \left(\csc^2 x - 1\right) + \csc x - 1\\ &= \csc^2 x + \csc x - 2\\ &= (\csc x - 1)(\csc x + 2)\end{aligned}$$

31. $\tan\theta\csc\theta = \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\sin\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

33. $$\begin{aligned}\sin\phi(\csc\phi - \sin\phi) &= (\sin\phi)\frac{1}{\sin\phi} - \sin^2\phi\\ &= 1 - \sin^2\phi = \cos^2\phi\end{aligned}$$

35. $$\begin{aligned}\sin\beta\tan\beta + \cos\beta &= (\sin\beta)\frac{\sin\beta}{\cos\beta} + \cos\beta\\ &= \frac{\sin^2\beta}{\cos\beta} + \frac{\cos^2\beta}{\cos\beta}\\ &= \frac{\sin^2\beta + \cos^2\beta}{\cos\beta}\\ &= \frac{1}{\cos\beta}\\ &= \sec\beta\end{aligned}$$

37. $$\begin{aligned}\frac{1 - \sin^2 x}{\csc^2 x - 1} = \frac{\cos^2 x}{\cot^2 x} = \cos^2 x \tan^2 x &= \left(\cos^2 x\right)\frac{\sin^2 x}{\cos^2 x}\\ &= \sin^2 x\end{aligned}$$

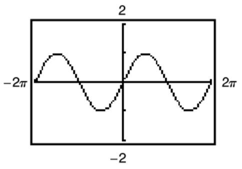
39. $$\begin{aligned}\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} &= \frac{1 - \cos x + 1 + \cos x}{(1 + \cos x)(1 - \cos x)}\\ &= \frac{2}{1 - \cos^2 x}\\ &= \frac{2}{\sin^2 x}\\ &= 2\csc^2 x\end{aligned}$$

41. $$\frac{\cos x}{1+\sin x}-\frac{\cos x}{1-\sin x}=\frac{\cos x(1-\sin x)-\cos x(1+\sin x)}{(1+\sin x)(1-\sin x)}$$
$$=\frac{\cos x-\sin x\cos x-\cos x-\sin x\cos x}{(1+\sin x)(1-\sin x)}$$
$$=\frac{-2\sin x\cos x}{1-\sin^2 x}$$
$$=\frac{-2\sin x\cos x}{\cos^2 x}$$
$$=\frac{-2\sin x}{\cos x}$$
$$=-2\tan x$$

43. $$\tan x-\frac{\sec^2 x}{\tan x}=\frac{\tan^2 x-\sec^2 x}{\tan x}$$
$$=\frac{-1}{\tan x}=-\cot x$$

45. $$\frac{\sin^2 y}{1-\cos y}=\frac{1-\cos^2 y}{1-\cos y}$$
$$=\frac{(1+\cos y)(1-\cos y)}{1-\cos y}=1+\cos y$$

47. $$y_1=\frac{\tan x+1}{\sec x+\csc x}$$
$$=\frac{\dfrac{\sin x}{\cos x}+1}{\dfrac{1}{\cos x}+\dfrac{1}{\sin x}}$$
$$=\frac{\dfrac{\sin x+\cos x}{\cos x}}{\dfrac{\sin x+\cos x}{\sin x\cos x}}$$
$$=\left(\frac{\sin x+\cos x}{\cos x}\right)\left(\frac{\sin x\cos x}{\sin x+\cos x}\right)$$
$$=\sin x$$

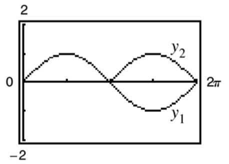


49. $\sin\theta=\sqrt{1-\cos^2\theta}$

Let $y_1=\sin x$ and $y_2=\sqrt{1-\cos^2 x}, 0\le x\le 2\pi$.

$y_1=y_2$ for $0\le x\le \pi$.

So, $\sin\theta=\sqrt{1-\cos^2\theta}$ for $0\le\theta\le\pi$.



51. Let $x=3\cos\theta$.

$$\sqrt{9-x^2}=\sqrt{9-(3\cos\theta)^2}$$
$$=\sqrt{9-9\cos^2\theta}$$
$$=\sqrt{9(1-\cos^2\theta)}$$
$$=\sqrt{9\sin^2\theta}=3\sin\theta$$

53. Let $x=2\sec\theta$.

$$\sqrt{x^2-4}=\sqrt{(2\sec\theta)^2-4}$$
$$=\sqrt{4(\sec^2\theta-1)}$$
$$=\sqrt{4\tan^2\theta}$$
$$=2\tan\theta$$

55. $$\ln|\sin x|+\ln|\cot x|=\ln|\sin x\cot x|$$
$$=\ln\left|\sin x\cdot\frac{\cos x}{\sin x}\right|$$
$$=\ln|\cos x|$$

57. $$\begin{aligned} \ln\left|\tan t\right| - \ln\left(1 - \cos^2 t\right) &= \ln\left[\frac{\left|\tan t\right|}{1 - \cos^2 t}\right] \\ &= \ln\left|\frac{\tan t}{\sin^2 t}\right| \\ &= \ln\left|\frac{\sin t}{\cos t} \cdot \frac{1}{\sin^2 t}\right| \\ &= \ln\left|\frac{1}{\cos t \sin t}\right| \\ &= \ln\left|\sec t \csc t\right| \end{aligned}$$

59. $$\begin{aligned} \mu W \cos\theta &= W \sin\theta \\ \mu &= \frac{W \sin\theta}{W \cos\theta} = \tan\theta \end{aligned}$$

61. True for all $\theta \neq n\pi$.

$$\sin\theta \cdot \csc\theta = \sin\theta\left(\frac{1}{\sin\theta}\right) = 1$$

63. True.

$$\tan u = \frac{\sin u}{\cos u}$$

$$\cot u = \frac{\cos u}{\sin u}$$

$$\sec u = \frac{1}{\cos u}$$

$$\csc u = \frac{1}{\sin u}$$

65. Because $\sin^2\theta + \cos^2\theta = 1$, then

$\cos^2\theta = 1 - \sin^2\theta$.

$$\cos\theta = \pm\sqrt{1 - \sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1 - \sin^2\theta}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{1 - \sin^2\theta}}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1 - \sin^2\theta}}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

67. Let $u = a\tan\theta$, then

$$\begin{aligned} \sqrt{a^2 + u^2} &= \sqrt{a^2 + (a\tan\theta)^2} \\ &= \sqrt{a^2 + a^2\tan^2\theta} \\ &= \sqrt{a^2\left(1 + \tan^2\theta\right)} \\ &= \sqrt{a^2\sec^2\theta} \\ &= a\sec\theta. \end{aligned}$$

69. $$\begin{aligned} \frac{x}{x + 4} \cdot \frac{x^2 - 16}{2x} &= \frac{(x + 4)(x - 4)}{(x + 4)(2)} \\ &= \frac{x - 4}{2}, \ x \neq 0, \ -4 \end{aligned}$$

71. $$\begin{aligned} \frac{12}{x^3 - x} \div \frac{3x - 3}{x + x^2} &= \frac{12}{x(x^2 - 1)} \cdot \frac{x(x + 1)}{3(x - 1)} \\ &= \frac{4(x + 1)}{(x + 1)(x - 1)^2} \\ &= \frac{4}{(x - 1)^2}, x \neq 0, -1 \end{aligned}$$

73. $(7 + 3i)(7 - 3i) = 49 - 9i^2 = 49 + 9 = 58$

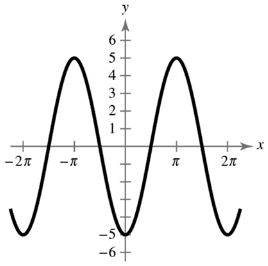
75. $\sqrt{-32} \cdot \left(-\sqrt{-32}\right) = (32i)(-32i) = 32$

77. $y = -5\cos x$

Period: 2π

Amplitude: 5

Key points: $(0, -5), \left(\frac{\pi}{2}, 0\right), (\pi, 5), \left(\frac{3\pi}{2}, 0\right), (2\pi, -5)$

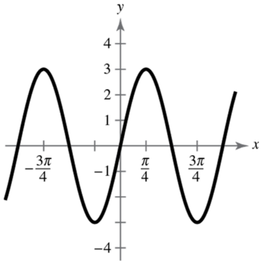


79. $y = 3\sin(2x)$

Period: $\dfrac{2\pi}{2} = \pi$

Amplitude: 3

Key points: $(0, 0), \left(\dfrac{\pi}{4}, 3\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{4}, -3\right), (\pi, 0)$



81. True. $\ln(xy) = \ln x + \ln y$

Section 5.2 Verifying Trigonometric Identities

1. identity

3. *Sample answer:* Actual equivalence of two expressions can only be verified algebraically.

5. $\tan t \cot t = \dfrac{\sin t}{\cos t} \cdot \dfrac{\cos t}{\sin t} = 1$

7. $1 + \sin^2 \alpha = 1 + (1 - \cos^2 \alpha) = 2 - \cos^2 \alpha$

9.
$$\begin{aligned}\cos^2 \beta - \sin^2 \beta &= (1 - \sin^2 \beta) - \sin^2 \beta \\ &= 1 - 2\sin^2 \beta\end{aligned}$$

11.
$$\begin{aligned}\tan\left(\frac{\pi}{2} - \theta\right)\sin\theta &= \cot\theta \sin\theta \\ &= \left(\frac{\cos\theta}{\sin\theta}\right)\sin\theta \\ &= \cos\theta\end{aligned}$$

13.
$$\begin{aligned}\sin t \csc\left(\frac{\pi}{2} - t\right) &= \sin t \sec t = \sin t\left(\frac{1}{\cos t}\right) \\ &= \frac{\sin t}{\cos t} = \tan t\end{aligned}$$

15.
$$\begin{aligned}\frac{1}{\tan x} + \frac{1}{\cot x} &= \frac{\cot x + \tan x}{\tan x \cot x} \\ &= \frac{\cot x + \tan x}{1} \\ &= \tan x + \cot x\end{aligned}$$

17.
$$\begin{aligned}\frac{1 + \sin\theta}{\cos\theta} + \frac{\cos\theta}{1 + \sin\theta} &= \frac{(1 + \sin\theta)^2 + \cos^2\theta}{\cos\theta(1 + \sin\theta)} \\ &= \frac{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1 + \sin\theta)} \\ &= \frac{2 + 2\sin\theta}{\cos\theta(1 + \sin\theta)} \\ &= \frac{2(1 + \sin\theta)}{\cos\theta(1 + \sin\theta)} \\ &= \frac{2}{\cos\theta} \\ &= 2\sec\theta\end{aligned}$$

19.
$$\begin{aligned}\frac{1}{\cos x + 1} + \frac{1}{\cos x - 1} &= \frac{\cos x - 1 + \cos x + 1}{(\cos x + 1)(\cos x - 1)} \\ &= \frac{2\cos x}{\cos^2 x - 1} \\ &= \frac{2\cos x}{-\sin^2 x} \\ &= -2 \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -2\csc x \cot x\end{aligned}$$

21. $\sec^3 y \cos y = \sec^2 y\left(\dfrac{1}{\cos y}\right)\cos y = \sec^2 y$

23. $\dfrac{\tan^2\theta}{\sec\theta} = \dfrac{(\sin\theta/\cos\theta)\tan\theta}{1/\cos\theta} = \sin\theta\tan\theta$

25. $$\frac{1}{\tan\beta} + \tan\beta = \frac{1 + \tan^2\beta}{\tan\beta} = \frac{\sec^2\beta}{\tan\beta}$$

27. $$\frac{\cot^2 t}{\csc t} = \frac{\cos^2 t/\sin^2 t}{1/\sin t} = \frac{\cos^2 t}{\sin t} = \frac{1 - \sin^2 t}{\sin t}$$

29. $$\begin{aligned}\sec x - \cos x &= \frac{1}{\cos x} - \cos x\\ &= \frac{1 - \cos^2 x}{\cos x}\\ &= \frac{\sin^2 x}{\cos x}\\ &= \sin x \cdot \frac{\sin x}{\cos x}\\ &= \sin x \tan x\end{aligned}$$

31. $$\begin{aligned}\frac{\cot x}{\sec x} + \frac{\tan x}{\csc x} &= \frac{\cos x/\sin x}{1/\cos x} + \frac{\sin x/\cos x}{1/\sin x}\\ &= \frac{\cos^2 x}{\sin x} + \frac{\sin^2 x}{\cos x}\\ &= \frac{1 - \sin^2 x}{\sin x} + \frac{1 - \cos^2 x}{\cos x}\\ &= \csc x(1 - \sin^2 x) + \sec x(1 - \cos^2 x)\\ &= \csc x - \csc x \sin^2 x + \sec x - \sec x \cos^2 x\\ &= \csc x - \csc x \cdot \frac{1}{\csc x} \cdot \sin x + \sec x - \sec x \cdot \frac{1}{\sec x} \cdot \cos x\\ &= \csc x - \sin x + \sec x - \cos x\end{aligned}$$

33. $$\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \sin^{1/2} x \cos x(1 - \sin^2 x) = \sin^{1/2} x \cos x \cdot \cos^2 x = \cos^3 x\sqrt{\sin x}$$

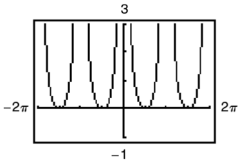
35. $$\begin{aligned}(1 + \sin y)[1 + \sin(-y)] &= (1 + \sin y)(1 - \sin y)\\ &= 1 - \sin^2 y\\ &= \cos^2 y\end{aligned}$$

37. $$\begin{aligned}\sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} &= \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta} \cdot \frac{1 + \sin\theta}{1 + \sin\theta}}\\ &= \sqrt{\frac{(1 + \sin\theta)^2}{1 - \sin^2\theta}}\\ &= \sqrt{\frac{(1 + \sin\theta)^2}{\cos^2\theta}}\\ &= \frac{1 + \sin\theta}{|\cos\theta|}\end{aligned}$$

39. $$\begin{aligned}\tan^3 x \sec^2 x - \tan^3 x &= \tan^3 x(\sec^2 x - 1)\\ &= \tan^3 x \tan^2 x\\ &= \tan^5 x\end{aligned}$$

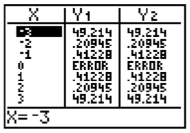
41. $$\begin{aligned}(\sin^2 x - \sin^4 x)\cos x &= \sin^2 x(1 - \sin^2 x)\cos x\\ &= \sin^2 x \cos^2 x \cos x\\ &= \sin^2 x \cos^3 x\end{aligned}$$

43. (a)



Identity

(b)

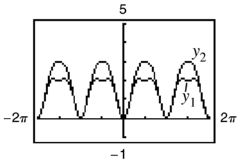


```
X     | Y1     | Y2
-3    | 49.214 | 49.214
-2    | .20945 | .20945
-1    | .41228 | .41228
0     | ERROR  | ERROR
1     | .41228 | .41228
2     | .20945 | .20945
3     | 49.214 | 49.214
X=-3
```

Identity

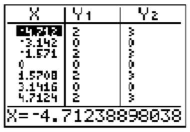
(c) $\left(1+\cot^2 x\right)\left(\cos^2 x\right) = \csc^2 x \cos^2 x = \dfrac{1}{\sin^2 x}\cdot \cos^2 x = \cot^2 x$

45. (a)



Not an identity

(b)

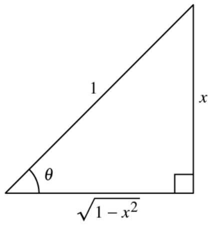


```
X      | Y1 | Y2
-4.712 | 2  | 3
-3.142 | 0  | 0
-1.571 | 2  | 3
0      | 0  | 0
1.5708 | 2  | 3
3.1416 | 0  | 0
4.7124 | 2  | 3
X=-4.71238898038
```

Not an identity

(c) $2+\cos^2 x - 3\cos^4 x = \left(1-\cos^2 x\right)\left(2+3\cos^2 x\right) = \sin^2 x\left(2+3\cos^2 x\right) \neq \sin^2 x\left(3+2\cos^2 x\right)$

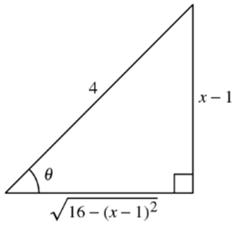
47. Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x = \dfrac{x}{1}$.



From the diagram,

$$\tan\left(\sin^{-1} x\right) = \tan\theta = \frac{x}{\sqrt{1-x^2}}.$$

49. Let $\theta = \sin^{-1}\dfrac{x-1}{4} \Rightarrow \sin\theta = \dfrac{x-1}{4}$.



From the diagram,

$$\tan\left(\sin^{-1}\frac{x-1}{4}\right) = \tan\theta = \frac{x-1}{\sqrt{16-(x-1)^2}}.$$

51. (a) $\dfrac{h\sin(90^\circ-\theta)}{\sin\theta} = \dfrac{h\cos\theta}{\sin\theta} = h\cot\theta$

(b)

θ	15°	30°	45°	60°	75°	90°
s	18.66	8.66	5	2.89	1.34	0

(c) Maximum: 15°

Minimum: 90°

(d) Noon

53. False. $(1+\csc x)(1-\csc x) = 1-\csc^2 x$

$= -\cot^2 x$, not $-\cot x^2$.

Note: $\cot^2 x$ means $(\cot x)^2$ and

$\cot x^2$ means $\cot\left(x^2\right)$.

55. $\cot(-x) \neq \cot x$

The correct substitution is $\cot(-x) = -\cot x$.

$$\frac{1}{\tan x} + \cot(-x) = \cot x - \cot x = 0$$

57. Because $\tan^2\theta + 1 = \sec^2\theta$, you have

$\tan\theta = \pm\sqrt{\sec^2\theta - 1}$.

For example, if $\theta = \dfrac{3\pi}{4}$, then

$\tan\theta = -1$ and $\sqrt{\sec^2\theta - 1} = 1$.

59.
$$\begin{aligned} x^2 - 15x - 34 &= 0 \\ (x-17)(x+2) &= 0 \\ x &= 17, -2 \end{aligned}$$

61. $(x + 1)^2 = 64$

$x + 1 = \pm 8$

$x = -9, 7$

63. (a) $\sin\theta = 0 \Rightarrow \theta = 0° = 0$ and $\theta = 180° = \pi$

(b) $\sin\theta = \dfrac{\sqrt{2}}{2} \Rightarrow \theta = 45° = \dfrac{\pi}{4}$ and $\theta = 135° = \dfrac{3\pi}{4}$

Section 5.3 Solving Trigonometric Equations

1. isolate

3. quadratic

5. $x = 0$ is not a solution of the equation $\cos x = 0$ because you know that $\cos 0 = 1 \neq 0$.

7. $\tan x - \sqrt{3} = 0$

(a) $x = \dfrac{\pi}{3}$: $\tan\dfrac{\pi}{3} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$

(b) $x = \dfrac{4\pi}{3}$: $\tan\dfrac{4\pi}{3} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$

9. $3\tan^2 2x - 1 = 0$

(a) $x = \dfrac{\pi}{12}$: $3\left[\tan\left(\dfrac{2\pi}{12}\right)\right]^2 - 1 = 3\tan^2\dfrac{\pi}{6} - 1$

$= 3\left(\dfrac{1}{\sqrt{3}}\right)^2 - 1 = 0$

(b) $x = \dfrac{5\pi}{12}$: $3\left[\tan\left(\dfrac{10\pi}{12}\right)\right]^2 - 1 = 3\tan^2\dfrac{5\pi}{6} - 1$

$= 3\left(-\dfrac{1}{\sqrt{3}}\right)^2 - 1 = 0$

11. $2\sin^2 x - \sin x - 1 = 0$

(a) $x = \dfrac{\pi}{2}$: $2\sin^2\left(\dfrac{\pi}{2}\right) - \sin\left(\dfrac{\pi}{2}\right) - 1$

$= 2 - 1 - 1 = 0$

(b) $x = \dfrac{7\pi}{6}$: $2\sin^2\left(\dfrac{7\pi}{6}\right) - \sin\left(\dfrac{7\pi}{6}\right) - 1$

$= 2\left(\dfrac{1}{4}\right) - \left(-\dfrac{1}{2}\right) - 1 = 0$

13. $\sqrt{3}\csc x - 2 = 0$

$\sqrt{3}\csc x = 2$

$\csc x = \dfrac{2}{\sqrt{3}}$

$x = \dfrac{\pi}{3} + 2n\pi$

or $x = \dfrac{2\pi}{3} + 2n\pi$

15. $\cos x + 1 = -\cos x$

$2\cos x + 1 = 0$

$\cos x = -\dfrac{1}{2}$

$x = \dfrac{2\pi}{3} + 2n\pi$ or $x = \dfrac{4\pi}{3} + 2n\pi$

17. $3\sec^2 x - 4 = 0$

$\sec^2 x = \dfrac{4}{3}$

$\sec x = \pm\dfrac{2}{\sqrt{3}}$

$x = \dfrac{\pi}{6} + n\pi$

or $x = \dfrac{5\pi}{6} + n\pi$

19. $4\cos^2 x - 1 = 0$

$\cos^2 x = \dfrac{1}{4}$

$\cos x = \pm\dfrac{1}{2}$

$x = \dfrac{\pi}{3} + n\pi$ or $x = \dfrac{2\pi}{3} + n\pi$

21. $\sin x(\sin x + 1) = 0$

$\sin x = 0$ or $\sin x = -1$

$x = n\pi$ $\qquad$ $x = \dfrac{3\pi}{2} + 2n\pi$

23.
$$\cos^3 x - \cos x = 0$$
$$\cos x(\cos^2 x - 1) = 0$$
$$\cos x = 0 \qquad \text{or} \qquad \cos^2 x - 1 = 0$$
$$x = \frac{\pi}{2} + n\pi \qquad\qquad \cos x = \pm 1$$
$$x = n\pi$$

Both of these answers can be represented as $x = \dfrac{n\pi}{2}$.

25.
$$3\tan^3 x = \tan x$$
$$3\tan^3 x - \tan x = 0$$
$$\tan x(3\tan^2 x - 1) = 0$$
$$\tan x = 0 \quad \text{or} \quad 3\tan^2 x - 1 = 0$$
$$x = n\pi \qquad\qquad \tan x = \pm\frac{\sqrt{3}}{3}$$
$$x = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$$

27.
$$\sec^2 x - \sec x = 2$$
$$\sec^2 x - \sec x - 2 = 0$$
$$(\sec x - 2)(\sec x + 1) = 0$$
$$\sec x - 2 = 0 \qquad \text{or} \qquad \sec x + 1 = 0$$
$$\sec x = 2 \qquad\qquad \sec x = -1$$
$$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi \qquad\qquad x = \pi + 2n\pi$$

29.
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$2\cos x - 1 = 0 \qquad \text{or} \qquad \cos x + 1 = 0$$
$$\cos x = \frac{1}{2} \qquad\qquad \cos x = -1$$
$$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi \qquad\qquad x = \pi + 2n\pi$$

31.
$$\sin x - 2 = \cos x - 2$$
$$\sin x = \cos x$$
$$\frac{\sin x}{\cos x} = 1$$
$$\tan x = 1$$
$$x = \tan^{-1} 1$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

33.
$$2\sin^2 x = 2 + \cos x$$
$$2 - 2\cos^2 x = 2 + \cos x$$
$$2\cos^2 x + \cos x = 0$$
$$\cos x(2\cos x + 1) = 0$$
$$\cos x = 0 \qquad \text{or} \qquad 2\cos x + 1 = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad 2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

35.
$$\sin^2 x = 3\cos^2 x$$
$$\sin^2 x - 3\cos^2 x = 0$$
$$\sin^2 x - 3(1 - \sin^2 x) = 0$$
$$4\sin^2 x = 3$$
$$\sin x = \pm\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

37.
$$2\sin x + \csc x = 0$$
$$2\sin x + \frac{1}{\sin x} = 0$$
$$2\sin^2 x + 1 = 0$$
$$\sin^2 x = -\frac{1}{2} \Rightarrow \text{No solution}$$

39.
$$\begin{aligned} \csc x + \cot x &= 1 \\ (\csc x + \cot x)^2 &= 1^2 \\ \csc^2 x + 2\csc x \cot x + \cot^2 x &= 1 \\ \cot^2 x + 1 + 2\csc x \cot x + \cot^2 x &= 1 \\ 2\cot^2 x + 2\csc x \cot x &= 0 \\ 2\cot x(\cot x + \csc x) &= 0 \end{aligned}$$

$2\cot x = 0$ or $\cot x + \csc x = 0$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\qquad$ $\dfrac{\cos x}{\sin x} = -\dfrac{1}{\sin x}$

$\left(\dfrac{3\pi}{2}\text{ is extraneous.}\right)$ $\qquad$ $\cos x = -1$

$x = \pi$

$(\pi \text{ is extraneous.})$

$x = \pi/2$ is the only solution.

41. $2\cos 2x - 1 = 0$

$$\cos 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{3} + 2n\pi \quad\text{or}\quad 2x = \frac{5\pi}{3} + 2n\pi$$

$$x = \frac{\pi}{6} + n\pi \qquad x = \frac{5\pi}{6} + n\pi$$

43. $\tan 3x - 1 = 0$

$$\tan 3x = 1$$

$$3x = \frac{\pi}{4} + n\pi$$

$$x = \frac{\pi}{12} + \frac{n\pi}{3}$$

45. $2\cos\dfrac{x}{2} - \sqrt{2} = 0$

$$\cos\frac{x}{2} = \frac{\sqrt{2}}{2}$$

$$\frac{x}{2} = \frac{\pi}{4} + 2n\pi \quad\text{or}\quad \frac{x}{2} = \frac{7\pi}{4} + 2n\pi$$

$$x = \frac{\pi}{2} + 4n\pi \qquad x = \frac{7\pi}{2} + 4n\pi$$

47. $3\tan\dfrac{x}{2} - \sqrt{3} = 0$

$$\tan\frac{x}{2} = \frac{\sqrt{3}}{3}$$

$$\frac{x}{2} = \frac{\pi}{6} + n\pi \Rightarrow x = \frac{\pi}{3} + 2n\pi$$

49. $y = \sin\dfrac{\pi x}{2} + 1$

$$\sin\left(\frac{\pi x}{2}\right) + 1 = 0$$

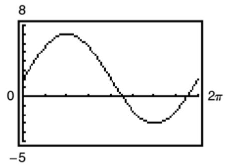
$$\sin\left(\frac{\pi x}{2}\right) = -1$$

$$\frac{\pi x}{2} = \frac{3\pi}{2} + 2n\pi$$

$$x = 3 + 4n$$

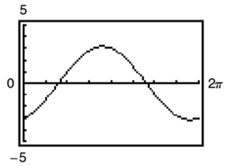
For $-2 < x < 4$, the intercepts are -1 and 3.

51. $5\sin x + 2 = 0$



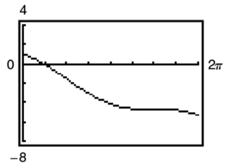
$x \approx 3.553$ and $x \approx 5.872$

53. $\sin x - 3\cos x = 0$



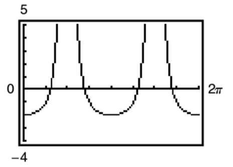
$x \approx 1.249$ and $x \approx 4.391$

55. $\cos x = x$



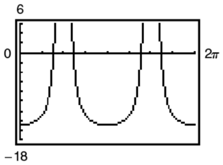
$x \approx 0.739$

57. $\sec^2 x - 3 = 0$



$x \approx 0.955$, $x \approx 2.186$, $x \approx 4.097$ and $x \approx 5.328$

59. $2\tan^2 x = 15$



$x \approx 1.221, x \approx 1.921, x \approx 4.362$ and $x \approx 5.062$

61.
$$\tan^2 x + \tan x - 12 = 0$$
$$(\tan x + 4)(\tan x - 3) = 0$$

$\tan x + 4 = 0$ or $\tan x - 3 = 0$

$\tan x = -4$ $\qquad$ $\tan x = 3$

$x = \arctan(-4) + n\pi$ $\qquad$ $x = \arctan 3 + n\pi$

63.
$$\begin{aligned} \sec^2 x - 6\tan x &= -4 \\ 1 + \tan^2 x - 6\tan x + 4 &= 0 \\ \tan^2 x - 6\tan x + 5 &= 0 \\ (\tan x - 1)(\tan x - 5) &= 0 \end{aligned}$$

$\tan x - 1 = 0$ $\qquad$ $\tan x - 5 = 0$

$\tan x = 1$ $\qquad$ $\tan x = 5$

$x = \dfrac{\pi}{4} + n\pi$ $\qquad$ $x = \arctan 5 + n\pi$

65.
$$\begin{aligned} 2\sin^2 x + 5\cos x &= 4 \\ 2(1 - \cos^2 x) + 5\cos x - 4 &= 0 \\ -2\cos^2 x + 5\cos x - 2 &= 0 \\ -(2\cos x - 1)(\cos x - 2) &= 0 \end{aligned}$$

$2\cos x - 1 = 0$ or $\cos x - 2 = 0$

$\cos x = \dfrac{1}{2}$ $\qquad$ $\cos x = 2$

$x = \dfrac{\pi}{3} + 2n\pi, \dfrac{5\pi}{3} + 2n\pi$ $\qquad$ No solution

67.
$$\begin{aligned} \cot^2 x - 9 &= 0 \\ \cot^2 x &= 9 \\ \frac{1}{9} &= \tan^2 x \\ \pm\frac{1}{3} &= \tan x \\ x &= \arctan \tfrac{1}{3} + n\pi, \arctan\left(-\tfrac{1}{3}\right) + n\pi \end{aligned}$$

69. $\sec^2 x - 4\sec x = 0$

$\sec x(\sec x - 4) = 0$

$\sec x = 0 \qquad \sec x - 4 = 0$

No solution $\qquad \sec x = 4$

$$\frac{1}{4} = \cos x$$

$$x = \arccos\frac{1}{4} + 2n\pi, -\arccos\frac{1}{4} + 2n\pi$$

71. $\csc^2 x + 3\csc x - 4 = 0$

$(\csc x + 4)(\csc x - 1) = 0$

$\csc x + 4 = 0 \qquad$ or $\qquad \csc x - 1 = 0$

$\csc x = -4 \qquad \csc x = 1$

$-\frac{1}{4} = \sin x \qquad 1 = \sin x$

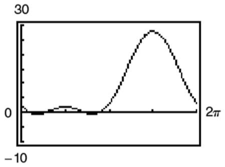
$x = \arcsin\left(\frac{1}{4}\right) + 2n\pi, \arcsin\left(-\frac{1}{4}\right) + 2n\pi \qquad x = \frac{\pi}{2} + 2n\pi$

73. $12\sin^2 x - 13\sin x + 3 = 0$

$$\sin x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(12)(3)}}{2(12)} = \frac{13 \pm 5}{24}$$

$\sin x = \frac{1}{3} \qquad$ or $\qquad \sin x = \frac{3}{4}$

$x \approx 0.3398, 2.8018 \qquad x \approx 0.8481, 2.2935$



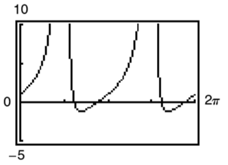
The x-intercepts occur at $x \approx 0.3398$, $x \approx 0.8481$, $x \approx 2.2935$, and $x \approx 2.8018$.

75. $\tan^2 x + 3\tan x + 1 = 0$

$$\tan x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

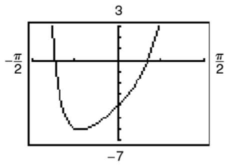
$\tan x = \frac{-3 - \sqrt{5}}{2} \qquad$ or $\qquad \tan x = \frac{-3 + \sqrt{5}}{2}$

$x \approx 1.9357, 5.0773 \qquad x \approx 2.7767, 5.9183$



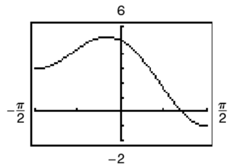
The x-intercepts occur at $x \approx 1.9357$, $x \approx 2.7767$, $x \approx 5.0773$, and $x \approx 5.9183$.

77. $3\tan^2 x + 5\tan x - 4 = 0, \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



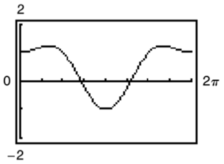
$x \approx -1.154, 0.534$

79. $4\cos^2 x - 2\sin x + 1 = 0, \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$x \approx 1.110$

81. (a) $f(x) = \sin^2 x + \cos x$



Maximum: $(1.0472, 1.25)$

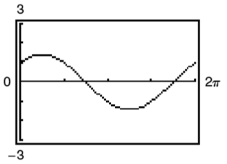
Maximum: $(5.2360, 1.25)$

Minimum: $(0, 1)$

Minimum: $(3.1416, -1)$

(b)
$$2\sin x\cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$x = 0, \pi \qquad \cos x = \frac{1}{2}$$
$$\approx 0, 3.1416 \qquad x = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$\approx 1.0472, 5.2360$$

83. (a) $f(x) = \sin x + \cos x$

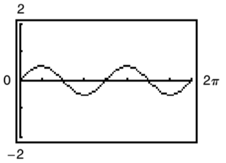


Maximum: $(0.7854, 1.4142)$

Minimum: $(3.9270, -1.4142)$

(b)
$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$
$$1 = \frac{\sin x}{\cos x}$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\approx 0.7854, 3.9270$$

85. (a) $f(x) = \sin x\cos x$



Maximum: $(0.7854, 0.5)$

Maximum: $(3.9270, 0.5)$

Minimum: $(2.3562, -0.5)$

Minimum: $(5.4978, -0.5)$

(b)
$$-\sin^2 x + \cos^2 x = 0$$
$$-\sin^2 x + 1 - \sin^2 x = 0$$
$$-2\sin^2 x + 1 = 0$$
$$\sin^2 x = \frac{1}{2}$$
$$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$\approx 0.7854, 2.3562, 3.9270, 5.4978$$

87. The graphs of $y_1 = 2\sin x$ and $y_2 = 3x + 1$ appear to have one point of intersection. This implies there is one solution to the equation $2\sin x = 3x + 1$.

89.
$$y = \frac{1}{12}(\cos 8t - 3\sin 8t)$$
$$\frac{1}{12}(\cos 8t - 3\sin 8t) = 0$$
$$\cos 8t = 3\sin 8t$$
$$\frac{1}{3} = \tan 8t$$
$$8t \approx 0.32175 + n\pi$$
$$t \approx 0.04 + \frac{n\pi}{8}$$

In the interval $0 \le t \le 1$, $t \approx 0.04, 0.43,$ and 0.83.

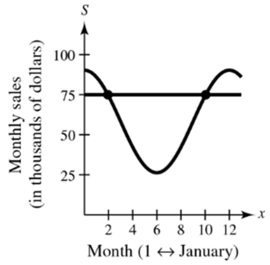
91. Graph $y_1 = 58.3 + 32\cos\left(\frac{\pi t}{6}\right)$

$y_2 = 75.$

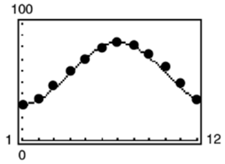
Left point of intersection: $(1.95, 75)$

Right point of intersection: $(10.05, 75)$

So, sales exceed 7500 in January, November, and December.



93. (a) and (c)



The model fits the data well.

(b) $C = a\cos(bt - c) + d$

$$a = \frac{1}{2}[\text{high} - \text{low}] = \frac{1}{2}[84.1 - 31.0] = 26.55$$

$$p = 2[\text{high time} - \text{low time}] = 2[7 - 1] = 12$$

$$b = \frac{2\pi}{p} = \frac{2\pi}{12} = \frac{\pi}{6}$$

The maximum occurs at 7, so the left end point is

$$\frac{c}{b} = 7 \Rightarrow c = 7\left(\frac{\pi}{6}\right) = \frac{7\pi}{6}$$

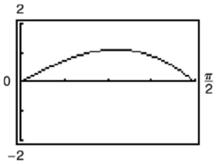
$$d = \frac{1}{2}[\text{high} + \text{low}] = \frac{1}{2}[93.6 + 62.3] = 57.55$$

$$C = 26.55\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 57.55$$

(d) The average maximum temperature is above 72°F from June through September. The average maximum temperature is below 70°F from October through May.

95. $A = 2x\cos x, 0 < x < \frac{\pi}{2}$

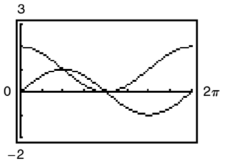
(a)



The maximum area of $A \approx 1.12$ occurs when $x \approx 0.86$.

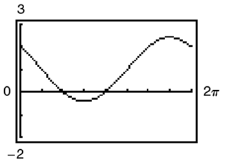
(b) $A \geq 1$ for $0.6 < x < 1.1$

97. (a)



The graphs intersect when $x = \frac{\pi}{2}$ and $x = \pi$.

(b)



The x-intercepts are $\left(\frac{\pi}{2}, 0\right)$ and $(\pi, 0)$.

(c) Both methods produce the same x-values. Answers will vary on which method is preferred.

99. $f(x) = \tan\frac{\pi x}{4}$

Because $\tan \pi/4 = 1, x = 1$ is the smallest nonnegative fixed point.

101. True. The period of $2\sin 4t - 1$ is $\frac{\pi}{2}$ and the period of $2\sin t - 1$ is 2π.

In the interval $[0, 2\pi)$ the first equation has four cycles whereas the second equation has only one cycle, so the first equation has four times the x-intercepts (solutions) as the second equation.

103. $\cot x\cos^2 x = 2\cot x$

$$\cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}$$

No solution

Because you solved this problem by first dividing by $\cot x$, you do not get the same solution as Example 3.

When solving equations, you do not want to divide each side by a variable expression that will cancel out because you may accidentally remove one of the solutions.

105.

$$\begin{aligned} x^2 - 4x &= 8 \\ x^2 - 4x + 4 &= 8 + 4 \\ (x - 2)^2 &= 12 \\ x - 2 &= \pm\sqrt{12} \\ x &= 2 \pm 2\sqrt{3} \end{aligned}$$

107.

$$\begin{aligned} \frac{1}{x^2 + 2x - 20} &= 1 \\ x^2 + 2x - 20 &= 1 \\ x^2 + 2x + 1 &= 22 \\ (x + 1)^2 &= 22 \\ x + 1 &= \pm\sqrt{22} \\ x &= -1 \pm \sqrt{22} \end{aligned}$$

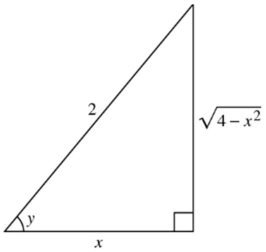
109.

$$\begin{aligned} (x - 4)^2 &= 4x \\ x^2 - 8x + 16 &= 4x \\ x^2 - 12x + 16 &= 0 \\ x &= \frac{12 \pm \sqrt{144 - 4(16)}}{2} \\ x &= \frac{12 \pm \sqrt{80}}{2} \\ x &= \frac{12 \pm 4\sqrt{5}}{2} \\ x &= 6 \pm 2\sqrt{5} \end{aligned}$$

111. $x^2 + 2x = \dfrac{2x + 2 - x^{-1}}{x^{-1}}$

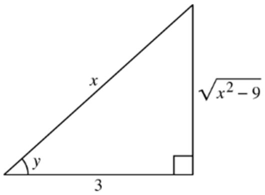
$$x^2 + 2x = 2x^2 + 2x - 1$$
$$x^2 - 1 = 0$$
$$x = \pm 1$$

113. Let $y = \arccos\dfrac{x}{2}$. Then $\cos y = \dfrac{x}{2}$.



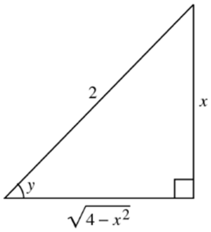
So, $\tan\left(\arccos\dfrac{x}{2}\right) = \tan y = \dfrac{\sqrt{4 - x^2}}{x}, 0 < x \le 2.$

115. Let $y = \arccos\dfrac{3}{x}$. Then $\cos y = \dfrac{3}{x}$.



So, $\csc\left(\arccos\dfrac{3}{x}\right) = \csc y = \dfrac{x}{\sqrt{x^2 - 9}}, x > 3.$

117. Let $y = \arcsin\dfrac{x}{2}$. Then $\sin y = \dfrac{x}{2}$.



So, $\sqrt{4 - x^2}\tan\left(\arcsin\dfrac{x}{2}\right) = \sqrt{4 - x^2}\tan y$

$$= \sqrt{4 - x^2}\frac{x}{\sqrt{4 - x^2}}$$
$$= x, 0 \le x < 2.$$

Section 5.4 Sum and Difference Formulas

1. $\sin u \cos v - \cos u \sin v$

3. $\dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

5. $\cos u \cos v + \sin u \sin v$

7. $\dfrac{5\pi}{12} = \dfrac{\pi}{6} + \dfrac{\pi}{4}$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

9. (a) $\cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{4}\cos\dfrac{\pi}{3} - \sin\dfrac{\pi}{4}\sin\dfrac{\pi}{3}$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

(b) $\cos\dfrac{\pi}{4} + \cos\dfrac{\pi}{3} = \dfrac{\sqrt{2}}{2} + \dfrac{1}{2} = \dfrac{\sqrt{2} + 1}{2}$

11. (a) $\sin(135^\circ - 30^\circ) = \sin 135^\circ \cos 30^\circ - \cos 135^\circ \sin 30^\circ$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

(b) $\sin 135^\circ - \cos 30^\circ = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{2} - \sqrt{3}}{2}$

13. $$\sin\frac{11\pi}{12} = \sin\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$$

$$= \sin\frac{3\pi}{4}\cos\frac{\pi}{6} + \cos\frac{3\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\frac{1}{2}$$

$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\cos\frac{11\pi}{12} = \cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$$

$$= \cos\frac{3\pi}{4}\cos\frac{\pi}{6} - \sin\frac{3\pi}{4}\sin\frac{\pi}{6}$$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\tan\frac{11\pi}{4} = \tan\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{3\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{3\pi}{4}\tan\frac{\pi}{6}}$$

$$= \frac{-1 + \frac{\sqrt{3}}{3}}{1 - (-1)\frac{\sqrt{3}}{3}}$$

$$= \frac{-3 + \sqrt{3}}{3 + \sqrt{3}}\cdot\frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

$$= \frac{-12 + 6\sqrt{3}}{6} = -2 + \sqrt{3}$$

15. $$\sin\frac{17\pi}{12} = \sin\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$$

$$= \sin\frac{9\pi}{4}\cos\frac{5\pi}{6} - \cos\frac{9\pi}{4}\sin\frac{5\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\frac{17\pi}{12} = \cos\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$$

$$= \cos\frac{9\pi}{4}\cos\frac{5\pi}{6} + \sin\frac{9\pi}{4}\sin\frac{5\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan\frac{17\pi}{12} = \tan\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$$

$$= \frac{\tan(9\pi/4) - \tan(5\pi/6)}{1 + \tan(9\pi/4)\tan(5\pi/6)}$$

$$= \frac{1 - \left(-\sqrt{3}/3\right)}{1 + \left(-\sqrt{3}/3\right)}$$

$$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}}\cdot\frac{3 + \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$$

17. $$\sin 105^\circ = \sin\left(60^\circ + 45^\circ\right)$$

$$= \sin 60^\circ\cos 45^\circ + \cos 60^\circ\sin 45^\circ$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos 105^\circ = \cos\left(60^\circ + 45^\circ\right)$$

$$= \cos 60^\circ\cos 45^\circ - \sin 60^\circ\sin 45^\circ$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan 105^\circ = \tan\left(60^\circ + 45^\circ\right)$$

$$= \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ\tan 45^\circ}$$

$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}}\cdot\frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

19. $$\begin{aligned}\sin(-195^\circ) &= \sin(30^\circ - 225^\circ)\\ &= \sin 30^\circ \cos 225^\circ - \cos 30^\circ \sin 225^\circ\\ &= \sin 30^\circ(-\cos 45^\circ) - \cos 30^\circ(-\sin 45^\circ)\\ &= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)\end{aligned}$$

$$\begin{aligned}\cos(-195^\circ) &= \cos(30^\circ - 225^\circ)\\ &= \cos 30^\circ \cos 225^\circ + \sin 30^\circ \sin 225^\circ\\ &= \cos 30^\circ(-\cos 45^\circ) + \sin 30^\circ(-45^\circ)\\ &= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)\end{aligned}$$

$$\begin{aligned}\tan(-195^\circ) &= \tan(30^\circ - 225^\circ)\\ &= \frac{\tan 30^\circ - \tan 225^\circ}{1 + \tan 30^\circ \tan 225^\circ}\\ &= \frac{\tan 30^\circ - \tan 45^\circ}{1 + \tan 30^\circ \tan 45^\circ}\\ &= \frac{\left(\frac{\sqrt{3}}{3}\right) - 1}{1 + \left(\frac{\sqrt{3}}{3}\right)} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}\\ &= \frac{-12 + 6\sqrt{3}}{6} = -2 + \sqrt{3}\end{aligned}$$

21. $$\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$$

$$\begin{aligned}\sin\frac{13\pi}{12} &= \sin\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)\\ &= \sin\frac{3\pi}{4}\cos\frac{\pi}{3} + \cos\frac{3\pi}{4}\sin\frac{\pi}{3}\\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)\end{aligned}$$

$$\begin{aligned}\cos\frac{13\pi}{12} &= \cos\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)\\ &= \cos\frac{3\pi}{4}\cos\frac{\pi}{3} - \sin\frac{3\pi}{4}\sin\frac{\pi}{3}\\ &= -\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)\end{aligned}$$

$$\begin{aligned}\tan\frac{13\pi}{12} &= \tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)\\ &= \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}\\ &= \frac{-1 + \sqrt{3}}{1 - (-1)\left(\sqrt{3}\right)}\\ &= -\frac{1 - \sqrt{3}}{1 + \sqrt{3}}\cdot\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\\ &= -\frac{4 - 2\sqrt{3}}{-2}\\ &= 2 - \sqrt{3}\end{aligned}$$

23. $$-\frac{5\pi}{12} = -\frac{\pi}{4} - \frac{\pi}{6}$$

$$\begin{aligned}\sin\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) &= \sin\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{6} - \cos\left(-\frac{\pi}{4}\right)\sin\frac{\pi}{6}\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)\end{aligned}$$

$$\begin{aligned}\cos\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{6} + \sin\left(-\frac{\pi}{4}\right)\sin\frac{\pi}{6}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)\end{aligned}$$

$$\begin{aligned}\tan\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) &= \frac{\tan\left(-\frac{\pi}{4}\right) - \tan\frac{\pi}{6}}{1 + \tan\left(-\frac{\pi}{4}\right)\tan\frac{\pi}{6}} = \frac{-1 - \frac{\sqrt{3}}{3}}{1 + (-1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}}\\ &= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}}\cdot\frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{-12 - 6\sqrt{3}}{6} = -2 - \sqrt{3}\end{aligned}$$

25. $285° = 225° + 60°$

$$\sin 285° = \sin(225° + 60°) = \sin 225° \cos 60° + \cos 225° \sin 60°$$

$$= -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos 285° = \cos(225° + 60°) = \cos 225° \cos 60° - \sin 225° \sin 60°$$

$$= -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\tan 285° = \tan(225° + 60°) = \frac{\tan 225° + \tan 60°}{1 - \tan 225° \tan 60°}$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3} = -\left(2 + \sqrt{3}\right)$$

27. $-165° = -(120° + 45°)$

$$\sin(-165°) = \sin\left[-(120° + 45°)\right] = -\sin(120° + 45°) = -\left[\sin 120° \cos 45° + \cos 120° \sin 45°\right]$$

$$= -\left[\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\cos(-165°) = \cos\left[-(120° + 45°)\right] = \cos(120° + 45°) = \cos 120° \cos 45° - \sin 120° \sin 45°$$

$$= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$$

$$\tan(-165°) = \tan\left[-(120° + 45°)\right] = -\tan(120° + \tan 45°) = -\frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$

$$= -\frac{-\sqrt{3} + 1}{1 - \left(-\sqrt{3}\right)(1)} = -\frac{1 - \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = -\frac{4 - 2\sqrt{3}}{-2} = 2 - \sqrt{3}$$

29. $\sin 3 \cos 1.2 - \cos 3 \sin 1.2 = \sin(3 - 1.2) = \sin 1.8$

31.
$$\cos 130° \cos 40° - \sin 130° \sin 40° = \cos(130° + 40°)$$
$$= \cos 170°$$

33.
$$\frac{\tan(\pi/15) + \tan(2\pi/5)}{1 - \tan(\pi/15)\tan(2\pi/5)} = \tan(\pi/15 + 2\pi/5)$$
$$= \tan(7\pi/15)$$

35. $\cos 3x \cos 2y + \sin 3x \sin 2y = \cos(3x - 2y)$

37.
$$\sin\frac{\pi}{12}\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\sin\frac{\pi}{4} = \sin\left(\frac{\pi}{12} + \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}$$

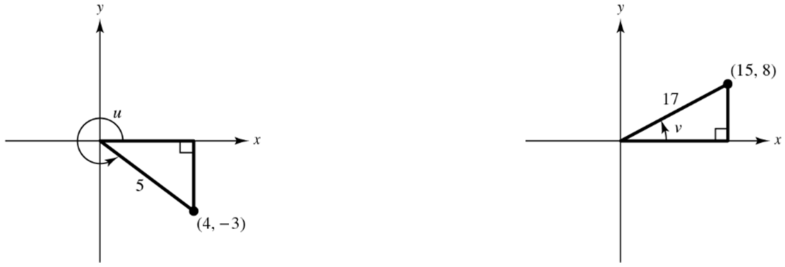
39.
$$\cos 130° \cos 10° + \sin 130° \sin 10° = \cos(130° - 10°)$$
$$= \cos 120°$$
$$= -\frac{1}{2}$$

41.
$$\frac{\tan(9\pi/8) - \tan(\pi/8)}{1 + \tan(9\pi/8)\tan(\pi/8)} = \tan\left(\frac{9\pi}{8} - \frac{\pi}{8}\right)$$
$$= \tan\pi$$
$$= 0$$

For Exercises 43–47, you have:

$\sin u = -\frac{3}{5}$, u in Quadrant IV $\Rightarrow$ $\cos u = \frac{4}{5}$, $\tan u = -\frac{4}{3}$

$\cos v = \frac{15}{17}$, v in Quadrant I $\Rightarrow$ $\sin v = \frac{8}{17}$, $\tan v = \frac{8}{15}$



Figures for Exercises 43–47

43. $$\begin{aligned}\sin(u+v) &= \sin u\cos v + \cos u \sin v\\ &= \left(-\frac{3}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{4}{5}\right)\left(\frac{8}{17}\right)\\ &= -\frac{13}{85}\end{aligned}$$

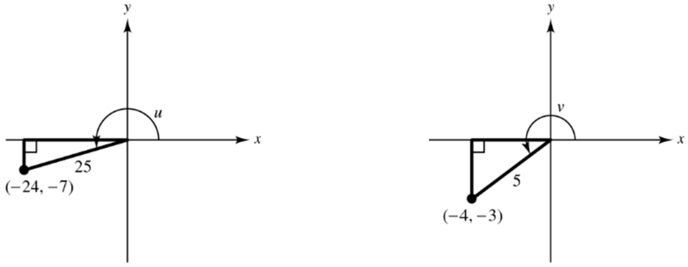
45. $$\begin{aligned}\tan(u+v) &= \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{-\frac{3}{4} + \left(\frac{8}{15}\right)}{1 - \left(-\frac{3}{4}\right)\left(\frac{8}{15}\right)}\\ &= \frac{-\frac{13}{60}}{1 + \frac{32}{60}} = \left(-\frac{13}{60}\right)\left(\frac{5}{7}\right) = -\frac{13}{84}\end{aligned}$$

47. $$\begin{aligned}\sec(v-u) &= \frac{1}{\cos(v-u)} = \frac{1}{\cos v\cos u + \sin v \sin u}\\ &= \frac{1}{\left(\frac{15}{17}\right)\left(\frac{4}{5}\right) + \left(\frac{8}{17}\right)\left(-\frac{3}{5}\right)} = \frac{1}{\left(\frac{60}{85}\right) + \left(-\frac{24}{85}\right)}\\ &= \frac{1}{\frac{36}{85}} = \frac{85}{36}\end{aligned}$$

For Exercises 49–53, you have:

$\sin u = -\frac{7}{25}$, u in Quadrant III $\Rightarrow$ $\cos u = -\frac{24}{25}$, $\tan u = \frac{7}{24}$

$\cos v = -\frac{4}{5}$, v in Quadrant III $\Rightarrow$ $\sin v = -\frac{3}{5}$, $\tan v = \frac{3}{4}$



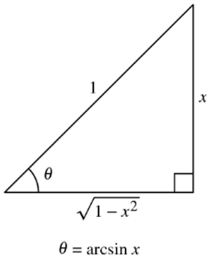
Figures for Exercises 49–53

49. $$\begin{aligned}\cos(u+v) &= \cos u\cos v - \sin u \sin v\\ &= \left(-\frac{24}{25}\right)\left(-\frac{4}{5}\right) - \left(-\frac{7}{25}\right)\left(-\frac{3}{5}\right)\\ &= \frac{3}{5}\end{aligned}$$

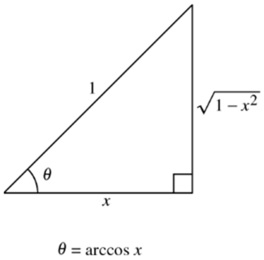
51. $$\begin{aligned}\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}\\ &= \frac{\frac{7}{24} - \frac{3}{4}}{1 + \left(\frac{7}{24}\right)\left(\frac{3}{4}\right)} = \frac{-\frac{11}{24}}{\frac{39}{32}} = -\frac{44}{117}\end{aligned}$$

53. $$\csc(u-v) = \frac{1}{\sin(u-v)} = \frac{1}{\sin u\cos v - \cos u \sin v}$$
$$= \frac{1}{\left(-\frac{7}{25}\right)\left(-\frac{4}{5}\right) - \left(-\frac{24}{25}\right)\left(-\frac{3}{5}\right)}$$
$$= \frac{1}{-\frac{44}{125}}$$
$$= -\frac{125}{44}$$

55. $$\sin(\arcsin x + \arccos x) = \sin(\arcsin x)\cos(\arccos x) + \sin(\arccos x)\cos(\arcsin x)$$
$$= x\cdot x + \sqrt{1-x^2}\cdot\sqrt{1-x^2}$$
$$= x^2 + 1 - x^2$$
$$= 1$$



θ = arcsin x



θ = arccos x

57. $$\cos(\arccos x + \arcsin x) = \cos(\arccos x)\cos(\arcsin x) - \sin(\arccos x)\sin(\arcsin x)$$
$$= x\cdot\sqrt{1-x^2} - \sqrt{1-x^2}\cdot x$$
$$= 0$$

(Use the triangles in Exercise 53.)

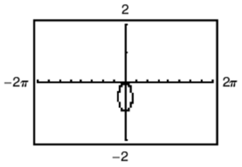
59. $$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$$
$$= (1)(\cos x) - (0)(\sin x)$$
$$= \cos x$$

61. $$\sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x$$
$$= \frac{1}{2}\left(\cos x + \sqrt{3}\sin x\right)$$

63. $$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\theta\tan\left(\frac{\pi}{4}\right)}$$
$$= \frac{\tan\theta - 1}{1 + \tan\theta}$$

65. $$\cos(\pi - \theta) + \sin\left(\frac{\pi}{2} + \theta\right) = \cos\pi\cos\theta + \sin\pi\sin\theta + \sin\frac{\pi}{2}\cos\theta + \cos\frac{\pi}{2}\sin\theta$$
$$= (-1)(\cos\theta) + (0)(\sin\theta) + (1)(\cos\theta) + (\sin\theta)(0)$$
$$= -\cos\theta + \cos\theta$$
$$= 0$$

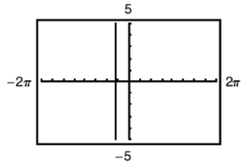
67. $$\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta$$
$$= (0)(\cos\theta) + (-1)(\sin\theta)$$
$$= -\sin\theta$$



The graphs appear to coincide, so $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta.$

69. $$\sin\left(\frac{3\pi}{2} + \theta\right) = \sin\frac{3\pi}{2}\cos\theta + \cos\frac{3\pi}{2}\sin\theta$$
$$= (-1)(\cos\theta) + (0)(\sin\theta)$$
$$= -\cos\theta$$

$$\csc\left(\frac{3\pi}{2} + \theta\right) = \frac{1}{\sin\left(\frac{3\pi}{2} + \theta\right)} = \frac{1}{-\cos\theta} = -\sec\theta$$



The graphs appear to coincide, so $\csc\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta.$

71. $$\cos(n\pi + \theta) = \cos n\pi\cos\theta - \sin n\pi\sin\theta$$
$$= (-1)^n(\cos\theta) - (0)(\sin\theta)$$
$$= (-1)^n(\cos\theta), \text{ where } n \text{ is an integer.}$$

73. $$C = \arctan\frac{b}{a} \Rightarrow \sin C = \frac{b}{\sqrt{a^2 + b^2}}, \cos C = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\sqrt{a^2 + b^2}\sin(B\theta + C) = \sqrt{a^2 + b^2}\left(\sin B\theta \cdot \frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}} \cdot \cos B\theta\right) = a\sin B\theta + b\cos B\theta$$

75. $\sin\theta + \cos\theta$

$a = 1, b = 1, B = 1$

(a) $C = \arctan\frac{b}{a} = \arctan 1 = \frac{\pi}{4}$

$$\sin\theta + \cos\theta = \sqrt{a^2 + b^2}\sin(B\theta + C)$$
$$= \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

(b) $C = \arctan\frac{a}{b} = \arctan 1 = \frac{\pi}{4}$

$$\sin\theta + \cos\theta = \sqrt{a^2 + b^2}\cos(B\theta - C)$$
$$= \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

77. $12\sin 3\theta + 5\cos 3\theta$

$a = 12, b = 5, B = 3$

(a) $C = \arctan\frac{b}{a} = \arctan\frac{5}{12} \approx 0.3948$

$$12\sin 3\theta + 5\cos 3\theta = \sqrt{a^2 + b^2}\sin(B\theta + C)$$
$$\approx 13\sin(3\theta + 0.3948)$$

(b) $C = \arctan\frac{a}{b} = \arctan\frac{12}{5} \approx 1.1760$

$$12\sin 3\theta + 5\cos 3\theta = \sqrt{a^2 + b^2}\cos(B\theta - C)$$
$$\approx 13\cos(3\theta - 1.1760)$$

79. $C = \arctan\frac{b}{a} = \frac{\pi}{4} \Rightarrow a = b, a > 0, b > 0$

$\sqrt{a^2 + b^2} = 2 \Rightarrow a = b = \sqrt{2}$

$B = 1$

$$2\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\sin\theta + \sqrt{2}\cos\theta$$

81. $y = \frac{1}{3}\sin 2t + \frac{1}{4}\cos 2t$

(a) $a = \frac{1}{3}, b = \frac{1}{4}, B = 2$

$$C = \arctan\frac{b}{a} = \arctan\frac{3}{4} \approx 0.6435$$

$$y \approx \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2}\sin(2t + 0.6435)$$

$$= \frac{5}{12}\sin(2t + 0.6435)$$

(b) Amplitude: $\frac{5}{12}$ feet

(c) Frequency: $\frac{1}{\text{period}} = \frac{B}{2\pi} = \frac{2}{2\pi}$

$$= \frac{1}{\pi}\text{ cycle per second}$$

83.

$$\begin{aligned}
\sin(x + \pi) - \sin x + 1 &= 0\\
\sin x\cos\pi + \cos x\sin\pi - \sin x + 1 &= 0\\
(\sin x)(-1) + (\cos x)(0) - \sin x + 1 &= 0\\
-2\sin x + 1 &= 0\\
\sin x &= \frac{1}{2}\\
x &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

85.

$$\begin{aligned}
\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) &= 1\\
\cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} - \left(\cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4}\right) &= 1\\
-2\sin x\left(\frac{\sqrt{2}}{2}\right) &= 1\\
-\sqrt{2}\sin x &= 1\\
\sin x &= -\frac{1}{\sqrt{2}}\\
\sin x &= -\frac{\sqrt{2}}{2}\\
x &= \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

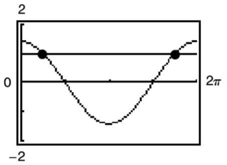
87.

$$\begin{aligned}
\tan(x + \pi) + 2\sin(x + \pi) &= 0\\
\frac{\tan x + \tan\pi}{1 - \tan x\tan\pi} + 2(\sin x\cos\pi + \cos x\sin\pi) &= 0\\
\frac{\tan x + 0}{1 - \tan x(0)} + 2\left[\sin x(-1) + \cos x(0)\right] &= 0\\
\frac{\tan x}{1} - 2\sin x &= 0\\
\frac{\sin x}{\cos x} &= 2\sin x\\
\sin x &= 2\sin x\cos x\\
\sin x(1 - 2\cos x) &= 0\\
\sin x = 0 \quad\text{or}\quad \cos x &= \frac{1}{2}\\
x = 0, \pi \qquad x &= \frac{\pi}{3}, \frac{5\pi}{3}
\end{aligned}$$

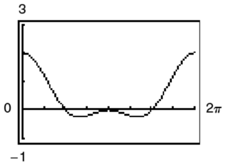
89. $\cos\left(x+\dfrac{\pi}{4}\right)+\cos\left(x-\dfrac{\pi}{4}\right)=1$

Graph $y_1=\cos\left(x+\dfrac{\pi}{4}\right)+\cos\left(x-\dfrac{\pi}{4}\right)$ and $y_2=1$.

$x=\dfrac{\pi}{4},\dfrac{7\pi}{4}$



91. $\sin\left(x+\dfrac{\pi}{2}\right)+\cos^2 x=0$



$x=\dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$

93. True.

$\sin(u+v)=\sin u\cos v+\cos u\sin v$

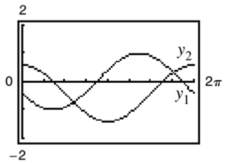
$\sin(u-v)=\sin u\cos v-\cos u\sin v$

So, $\sin(u\pm v)=\sin u\cos v\pm\cos u\sin v$.

95. The denominator should be

$1-\tan x\tan\left(\dfrac{\pi}{4}\right)=1-\tan x$.

97. $y_1=\cos(x+2)$, $y_2=\cos x+\cos 2$



No, $y_1\neq y_2$ because their graphs are different.

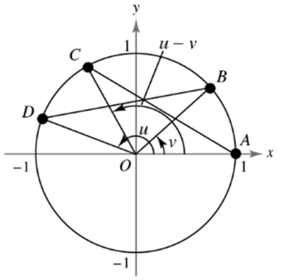
99. (a) To prove the identity for $\sin(u+v)$ you first need to prove the identity for $\cos(u-v)$.

Assume $0<v<u<2\pi$ and locate u, v, and $u-v$ on the unit circle.

The coordinates of the points on the circle are:

$A=(1,0)$, $B=(\cos v,\sin v)$, $C=(\cos(u-v),\sin(u-v))$, and $D=(\cos u,\sin u)$.

Because $\angle DOB=\angle COA$, chords AC and BD are equal. By the Distance Formula:



$$\sqrt{[\cos(u-v)-1]^2+[\sin(u-v)-0]^2}=\sqrt{(\cos u-\cos v)^2+(\sin u-\sin v)^2}$$

$$\cos^2(u-v)-2\cos(u-v)+1+\sin^2(u-v)=\cos^2 u-2\cos u\cos v+\cos^2 v+\sin^2 u-2\sin u\sin v+\sin^2 v$$

$$[\cos^2(u-v)+\sin^2(u-v)]+1-2\cos(u-v)=(\cos^2 u+\sin^2 u)+(\cos^2 v+\sin^2 v)-2\cos u\cos v-2\sin u\sin v$$

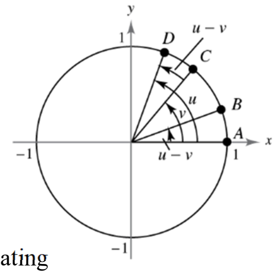
$$2-2\cos(u-v)=2-2\cos u\cos v-2\sin u\sin v$$

$$-2\cos(u-v)=-2(\cos u\cos v+\sin u\sin v)$$

$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$

Now, to prove the identity for $\sin(u+v)$, use cofunction identities.

$$\begin{aligned}\sin(u+v)&=\cos\left[\frac{\pi}{2}-(u+v)\right]=\cos\left[\left(\frac{\pi}{2}-u\right)-v\right]\\&=\cos\left(\frac{\pi}{2}-u\right)\cos v+\sin\left(\frac{\pi}{2}-u\right)\sin v\\&=\sin u\cos v+\cos u\sin v\end{aligned}$$



(b) First, prove $\cos(u - v) = \cos u \cos v + \sin u \sin v$ using the figure containing points

$A(1, 0)$
$B(\cos(u - v), \sin(u - v))$
$C(\cos v, \sin v)$
$D(\cos u, \sin u)$

on the unit circle.

Because chords AB and CD are each subtended by angle $u - v$, their lengths are equal. Equating $[d(A, B)]^2 = [d(C, D)]^2$ you have $(\cos(u - v) - 1)^2 + \sin^2(u - v) = (\cos u - \cos v)^2 + (\sin u - \sin v)^2$. Simplifying and solving for $\cos(u - v)$, you have $\cos(u - v) = \cos u \cos v + \sin u \sin v$.

Using $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

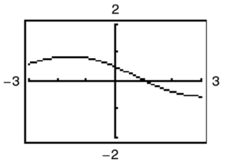
$$\begin{aligned}\sin(u - v) &= \cos\left[\frac{\pi}{2} - (u - v)\right] = \cos\left[\left(\frac{\pi}{2} - u\right) - (-v)\right] = \cos\left(\frac{\pi}{2} - u\right)\cos(-v) + \sin\left(\frac{\pi}{2} - u\right)\sin(-v)\\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

101. (a) The domains of f and g are the same, all real numbers h, except $h = 0$.

(b)

h	0.5	0.2	0.1	0.05	0.02	0.01
$f(h)$	0.267	0.410	0.456	0.478	0.491	0.496
$g(h)$	0.267	0.410	0.456	0.478	0.491	0.496

(c)



(d) As $h \to 0^*$, $f \to 0.5$ and $g \to 0.5$.

103. $\cos\theta = \frac{5}{13}$, θ in Quadrant IV

$$\sin\theta = -\sqrt{1 - \cos^2\theta} = -\sqrt{1 - \tfrac{25}{169}} = \tfrac{-12}{13}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \left(\frac{-12}{13}\right)\Big/\left(\frac{5}{13}\right) = \tfrac{-12}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{-13}{12}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{-5}{12}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{13}{5}$$

105. $\tan\theta = -\frac{1}{2}$, θ in Quadrant II

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\sec\theta = -\sqrt{1 + \tfrac{1}{4}} = \frac{-\sqrt{5}}{2}$$

$$\cos\theta = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$\sin\theta = \tan\theta \cdot \cos\theta = \left(\frac{-1}{2}\right)\left(\frac{-2\sqrt{5}}{5}\right) = \frac{\sqrt{5}}{5}$$

$$\csc\theta = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\cot\theta = -2$$

107. $2\cos x = 0$

$$\cos x = 0$$

$$x = \frac{\pi}{2} + 2n\pi, x = \frac{3\pi}{2} + 2n\pi$$

109. $1 + \sin x = 0$

$$\sin x = -1$$

$$x = \frac{3\pi}{2} + 2n\pi$$

111. $2\cos 3x = 0$

$$\cos 3x = 0$$

$$3x = \frac{\pi}{2}$$

$$x = \frac{\pi}{6} + \frac{2n\pi}{3}, x = \frac{\pi}{2} + \frac{2n\pi}{3}$$

Section 5.5 Multiple-Angle and Product-to-Sum Formulas

1. $\cos 2u$

3. *Sample answer:* Product-to-sum formulas are used in calculus to solve problems involving products of sines and cosines of different angles.

5.
$$\begin{aligned}\sin 2x - \sin x &= 0\\ 2\sin x\cos x - \sin x &= 0\\ \sin x(2\cos x - 1) &= 0\end{aligned}$$

$\sin x = 0$ or $2\cos x - 1 = 0$

$x = n\pi$ $\qquad \cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3} + 2n\pi, \dfrac{5\pi}{3} + 2n\pi$

7.
$$\begin{aligned}\cos 2x - \cos x &= 0\\ \cos 2x &= \cos x\\ \cos^2 x - \sin^2 x &= \cos x\\ \cos^2 x - (1 - \cos^2 x) - \cos x &= 0\\ 2\cos^2 x - \cos x - 1 &= 0\\ (2\cos x + 1)(\cos x - 1) &= 0\end{aligned}$$

$2\cos x + 1 = 0$ or $\cos x - 1 = 0$

$\cos x = -\dfrac{1}{2}$ $\qquad \cos x = 1$

$x = \dfrac{2n\pi}{3}$ $\qquad x = 0$

9.
$$\begin{aligned}\sin 4x &= -2\sin 2x\\ \sin 4x + 2\sin 2x &= 0\\ 2\sin 2x\cos 2x + 2\sin 2x &= 0\\ 2\sin 2x(\cos 2x + 1) &= 0\end{aligned}$$

$2\sin 2x = 0$ or $\cos 2x + 1 = 0$

$\sin 2x = 0$ $\qquad \cos 2x = -1$

$2x = n\pi$ $\qquad 2x = \pi + 2n\pi$

$x = \dfrac{n}{2}\pi$ $\qquad x = \dfrac{\pi}{2} + n\pi$

11.
$$\begin{aligned}\tan 2x - \cot x &= 0\\ \frac{2\tan x}{1 - \tan^2 x} &= \cot x\\ 2\tan x &= \cot x(1 - \tan^2 x)\\ 2\tan x &= \cot x - \cot x\tan^2 x\\ 2\tan x &= \cot x - \tan x\\ 3\tan x &= \cot x\\ 3\tan x - \cot x &= 0\\ 3\tan x - \frac{1}{\tan x} &= 0\\ \frac{3\tan^2 x - 1}{\tan x} &= 0\\ \frac{1}{\tan x}(3\tan^2 x - 1) &= 0\\ \cot x(3\tan^2 x - 1) &= 0\end{aligned}$$

$\cot x = 0$ or $3\tan^2 x - 1 = 0$

$x = \dfrac{\pi}{2} + n\pi$ $\qquad \tan^2 x = \dfrac{1}{3}$

$\tan x = \pm\dfrac{\sqrt{3}}{3}$

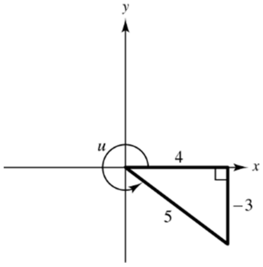
$x = \dfrac{\pi}{6} + n\pi, \dfrac{5\pi}{6} + n\pi$

13.
$$\begin{aligned}6\sin x\cos x &= 3(2\sin x\cos x)\\ &= 3\sin 2x\end{aligned}$$

15.
$$\begin{aligned}6\cos^2 x - 3 &= 3(2\cos^2 x - 1)\\ &= 3\cos 2x\end{aligned}$$

17.
$$\begin{aligned}4 - 8\sin^2 x &= 4(1 - 2\sin^2 x)\\ &= 4\cos 2x\end{aligned}$$

19. $\sin u = -\frac{3}{5}, \frac{3\pi}{2} < u < 2\pi$

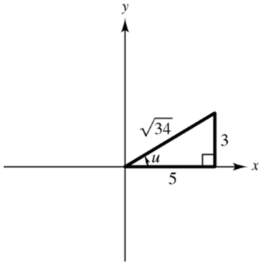


$$\sin 2u = 2\sin u\cos u = 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(-\frac{3}{4}\right)}{1 - \frac{9}{16}} = -\frac{3}{2}\left(\frac{16}{7}\right) = -\frac{24}{7}$$

21. $\tan u = \frac{3}{5}, 0 < u < \frac{\pi}{2}$



$$\sin 2u = 2\sin u\cos u = 2\left(\frac{3}{\sqrt{34}}\right)\left(\frac{5}{\sqrt{34}}\right) = \frac{15}{17}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \frac{25}{34} - \frac{9}{34} = \frac{8}{17}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(\frac{3}{5}\right)}{1 - \frac{9}{25}} = \frac{6}{5}\left(\frac{25}{16}\right) = \frac{15}{8}$$

23.
$$\begin{aligned}
\cos 4x &= \cos(2x + 2x) \\
&= \cos 2x\cos 2x - \sin 2x\sin 2x \\
&= \cos^2 2x - \sin^2 2x \\
&= \cos^2 2x - (1 - \cos^2 2x) \\
&= 2\cos^2 2x - 1 \\
&= 2(\cos 2x)^2 - 1 \\
&= 2(2\cos^2 x - 1)^2 - 1 \\
&= 2(4\cos^4 x - 4\cos x + 1) - 1 \\
&= 8\cos^4 x - 8\cos x + 1
\end{aligned}$$

25.
$$\begin{aligned}
\cos^4 x = (\cos^2 x)(\cos^2 x) = \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) &= \frac{1 + 2\cos 2x + \cos^2 2x}{4} \\
&= \frac{1 + 2\cos 2x + \frac{1 + \cos 4x}{2}}{4} \\
&= \frac{2 + 4\cos 2x + 1 + \cos 4x}{8} \\
&= \frac{3 + 4\cos 2x + \cos 4x}{8} \\
&= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)
\end{aligned}$$

27. $$\begin{aligned}\sin^4 2x &= \left(\sin^2 2x\right)^2\\ &= \left(\frac{1-\cos 4x}{2}\right)^2\\ &= \frac{1}{4}\left(1 - 2\cos 4x + \cos^2 4x\right)\\ &= \frac{1}{4}\left(1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right)\\ &= \frac{1}{4} - \frac{1}{2}\cos 4x + \frac{1}{8} + \frac{1}{8}\cos 8x\\ &= \frac{3}{8} - \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(3 - 4\cos 4x + \cos 8x)\end{aligned}$$

29. $$\begin{aligned}\tan^4 2x &= \left(\tan^2 2x\right)^2\\ &= \left(\frac{1-\cos 4x}{1+\cos 4x}\right)^2\\ &= \frac{1 - 2\cos 4x + \cos^2 4x}{1 + 2\cos 4x + \cos^2 4x}\\ &= \frac{1 - 2\cos 4x + \dfrac{1+\cos 8x}{2}}{1 + 2\cos 4x + \dfrac{1+\cos 8x}{2}}\\ &= \frac{\frac{1}{2}(2 - 4\cos 4x + 1 + \cos 8x)}{\frac{1}{2}(2 + 4\cos 4x + 1 + \cos 8x)}\\ &= \frac{3 - 4\cos 4x + \cos 8x}{3 + 4\cos 4x + \cos 8x}\end{aligned}$$

31. $$\begin{aligned}\sin^2 2x\cos^2 2x &= \left(\frac{1-\cos 4x}{2}\right)\left(\frac{1+\cos 4x}{2}\right)\\ &= \frac{1}{4}\left(1 - \cos^2 4x\right)\\ &= \frac{1}{4}\left(1 - \frac{1+\cos 8x}{2}\right)\\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(1 - \cos 8x)\end{aligned}$$

33. $$\begin{aligned}\sin 75^\circ &= \sin\left(\frac{1}{2}\cdot 150^\circ\right) = \sqrt{\frac{1-\cos 150^\circ}{2}} = \sqrt{\frac{1+\left(\sqrt{3}/2\right)}{2}}\\ &= \frac{1}{2}\sqrt{2+\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\cos 75^\circ &= \cos\left(\frac{1}{2}\cdot 150^\circ\right) = \sqrt{\frac{1+\cos 150^\circ}{2}} = \sqrt{\frac{1-\left(\sqrt{3}/2\right)}{2}}\\ &= \frac{1}{2}\sqrt{2-\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\tan 75^\circ &= \tan\left(\frac{1}{2}\cdot 150^\circ\right) = \frac{\sin 150^\circ}{1+\cos 150^\circ} = \frac{1/2}{1-\left(\sqrt{3}/2\right)}\\ &= \frac{1}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}\end{aligned}$$

35. $\sin 112^\circ\, 30' = \sin\left(\frac{1}{2}\cdot 225^\circ\right) = \sqrt{\frac{1-\cos 225^\circ}{2}} = \sqrt{\frac{1-\left(-\sqrt{2}/2\right)}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

$\cos 112^\circ\, 30' = \cos\left(\frac{1}{2}\cdot 225^\circ\right) = -\sqrt{\frac{1+\cos 225^\circ}{2}} = -\sqrt{\frac{1+\left(-\sqrt{2}/2\right)}{2}} = \frac{1}{2} - \sqrt{2-2}$

$\tan 112^\circ\, 30' = \tan\left(\frac{1}{2}\cdot 225^\circ\right) = \frac{\sin 225^\circ}{1+\cos 225^\circ} = \frac{-\sqrt{2}/2}{1+\left(-\sqrt{2}/2\right)} = \frac{-\sqrt{2}}{2-\sqrt{2}}\cdot\frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{-2\sqrt{2}-2}{2} = -1-\sqrt{2}$

37. $\sin\frac{\pi}{8} = \sin\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

$\cos\frac{\pi}{8} = \cos\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

$\tan\frac{\pi}{8} = \tan\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \sqrt{2}-1$

39. $\cos u = \frac{7}{25},\ 0 < u < \frac{\pi}{2}$

(a) Because u is in Quadrant I, $\frac{u}{2}$ is also in Quadrant I.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\cos\frac{u}{2} = \sqrt{\frac{1+\cos u}{2}} = \sqrt{\frac{1+\frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\frac{7}{25}}{\frac{24}{25}} = \frac{3}{4}$

41. $\tan u = -\frac{5}{12},\ \frac{3\pi}{2} < u < 2\pi$

(a) Because u is in Quadrant IV, $\frac{u}{2}$ is in Quadrant II.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\frac{12}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{\sqrt{26}}{26}$

$\cos\frac{u}{2} = -\sqrt{\frac{1+\cos u}{2}} = -\sqrt{\frac{1+\frac{12}{13}}{2}} = -\sqrt{\frac{25}{26}} = -\frac{5\sqrt{26}}{26}$

$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\frac{12}{13}}{\left(-\frac{5}{13}\right)} = -\frac{1}{5}$

43. $\sin\frac{x}{2} + \cos x = 0$

$$\pm\sqrt{\frac{1-\cos x}{2}} = -\cos x$$

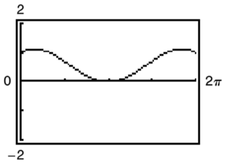
$$\frac{1-\cos x}{2} = \cos^2 x$$

$$0 = 2\cos^2 x + \cos x - 1$$

$$= (2\cos x - 1)(\cos x + 1)$$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$



By checking these values in the original equation, $x = \pi/3$ and $x = 5\pi/3$ are extraneous, and $x = \pi$ is the only solution.

45. $\cos\frac{x}{2} - \sin x = 0$

$$\pm\sqrt{\frac{1+\cos x}{2}} = \sin x$$

$$\frac{1+\cos x}{2} = \sin^2 x$$

$$1 + \cos x = 2\sin^2 x$$

$$1 + \cos x = 2 - 2\cos^2 x$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

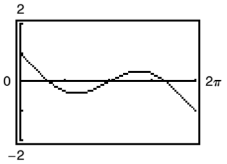
$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

$\pi/3, \pi,$ and $5\pi/3$ are all solutions to the equation.



47. $\sin 5\theta \sin 3\theta = \frac{1}{2}[\cos(5\theta - 3\theta) - \cos(5\theta + 3\theta)] = \frac{1}{2}(\cos 2\theta - \cos 8\theta)$

49. $\cos 2\theta \cos 4\theta = \frac{1}{2}[\cos(2\theta - 4\theta) + \cos(2\theta + 4\theta)] = \frac{1}{2}[\cos(-2\theta) + \cos 6\theta]$

51. $\sin 5\theta - \sin 3\theta = 2\cos\left(\frac{5\theta + 3\theta}{2}\right)\sin\left(\frac{5\theta - 3\theta}{2}\right) = 2\cos 4\theta \sin\theta$

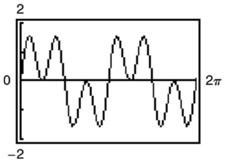
53. $\cos 6x + \cos 2x = 2\cos\left(\frac{6x + 2x}{2}\right)\cos\left(\frac{6x - 2x}{2}\right) = 2\cos 4x \cos 2x$

55. $\sin 75° + \sin 15° = 2\sin\left(\frac{75° + 15°}{2}\right)\cos\left(\frac{75° - 15°}{2}\right) = 2\sin 45° \cos 30° = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$

57. $\cos\frac{3\pi}{4} - \cos\frac{\pi}{4} = -2\sin\left(\frac{\frac{3\pi}{4} + \frac{\pi}{4}}{2}\right)\sin\left(\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right) = -2\sin\frac{\pi}{2}\sin\frac{\pi}{4}$

$\cos\frac{3\pi}{4} - \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

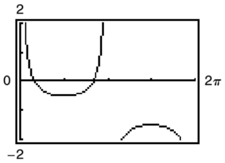
59.
$$\sin 6x + \sin 2x = 0$$
$$2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right) = 0$$
$$2(\sin 4x)\cos 2x = 0$$
$$\sin 4x = 0 \quad \text{or} \quad \cos 2x = 0$$
$$4x = n\pi \qquad 2x = \frac{\pi}{2} + n\pi$$
$$x = \frac{n\pi}{4} \qquad x = \frac{\pi}{4} + \frac{n\pi}{2}$$



In the interval $[0, 2\pi)$

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

61.
$$\frac{\cos 2x}{\sin 3x - \sin x} - 1 = 0$$
$$\frac{\cos 2x}{\sin 3x - \sin x} = 1$$
$$\frac{\cos 2x}{2\cos 2x \sin x} = 1$$
$$2\sin x = 1$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$



63.
$$\csc 2\theta = \frac{1}{\sin 2\theta}$$
$$= \frac{1}{2\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta}\cdot\frac{1}{2\cos\theta}$$
$$= \frac{\csc\theta}{2\cos\theta}$$

65.
$$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \frac{2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$
$$= \tan\left(\frac{x \pm y}{2}\right)$$

67. (a)
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{M}$$
$$\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 = \left(\frac{1}{M}\right)^2$$
$$\frac{1-\cos\theta}{2} = \frac{1}{M^2}$$
$$M^2(1-\cos\theta) = 2$$
$$1-\cos\theta = \frac{2}{M^2}$$
$$-\cos\theta = \frac{2}{M^2} - 1$$
$$\cos\theta = 1 - \frac{2}{M^2}$$
$$\cos\theta = \frac{M^2-2}{M^2}$$

(b) When $M = 2$, $\cos\theta = \frac{2^2-2}{2^2} = \frac{1}{2}$. So, $\theta = \frac{\pi}{3}$.

(c) When $M = 2$, $\frac{\text{speed of object}}{\text{speed of sound}} = M$

$$\frac{\text{speed of object}}{760 \text{ mph}} = 2$$
$$\text{speed of object} = 1520 \text{ mph.}$$

When $M = 4.5$, $\frac{\text{speed of object}}{\text{speed of sound}} = M$

$$\frac{\text{speed of object}}{760 \text{ mph}} = 4.5$$
$$\text{speed of object} = 3420 \text{ mph.}$$

69. Because ϕ and θ are complementary angles, $\sin\phi = \cos\theta$ and $\cos\phi = \sin\theta$.

(a) $$\begin{aligned}\sin(\phi - \theta) &= \sin\phi\cos\theta - \sin\theta\cos\phi\\ &= (\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta)\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos 2\theta\end{aligned}$$

(b) $$\begin{aligned}\cos(\phi - \theta) &= \cos\phi\cos\theta + \sin\phi\sin\theta\\ &= (\sin\theta)(\cos\theta) + (\cos\theta)(\sin\theta)\\ &= 2\sin\theta\cos\theta\\ &= \sin 2\theta\end{aligned}$$

71. True. Using the double angle formula and that sine is an odd function and cosine is an even function,

$$\begin{aligned}\sin(-2x) &= \sin[2(-x)]\\ &= 2\sin(-x)\cos(-x)\\ &= 2(-\sin x)\cos x\\ &= -2\sin x\cos x.\end{aligned}$$

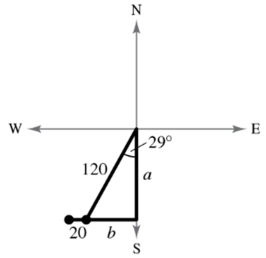
73. $B = 180° - A - C = 180° - 25° - 92° = 63°$

75. $$\begin{aligned}B + C &= 180° - A\\ B + B &= 180° - 90°\\ 2B &= 90°\\ B &= 45° = C\end{aligned}$$

77. $\cos 28° \approx 0.8829$

79. $\sec\left(\dfrac{11\pi}{15}\right) = \dfrac{1}{\cos(11\pi/15)} \approx -1.4945$

81.



(a) $\cos 29° = \dfrac{a}{120} \Rightarrow a \approx 104.95$ nautical miles south

$\sin 29° = \dfrac{b}{120} \Rightarrow b \approx 58.18$ nautical miles west

(b) $\tan\theta = \dfrac{20 + b}{a} \approx \dfrac{78.18}{104.95} \Rightarrow \theta \approx 36.7°$

Bearing: S 36.7° W

Distance: $d \approx \sqrt{104.95^2 + 78.18^2}$

≈ 130.9 nautical miles from port

Review Exercises for Chapter 5

1. $\cot x$

3. $\cos x$

5. $\cos\theta = -\dfrac{2}{5}$, $\tan\theta > 0$, θ is in Quadrant III.

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{2}$$

$$\sin\theta = -\sqrt{1 - \cos^2\theta} = -\sqrt{1 - \frac{4}{25}} = -\sqrt{\frac{21}{25}} = -\frac{\sqrt{21}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{\sqrt{21}} = -\frac{5\sqrt{21}}{21}$$

$$\tan\theta = \frac{\sin}{\cos\theta} = \frac{-\frac{\sqrt{21}}{5}}{-\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

7. $\frac{1}{\cot^2 x + 1} = \frac{1}{\csc^2 x} = \sin^2 x$

9. $$\begin{aligned}\tan^2 x(\csc^2 x - 1) &= \tan^2 x(\cot^2 x)\\ &= \tan^2 x\left(\frac{1}{\tan^2 x}\right)\\ &= 1\end{aligned}$$

11. $\frac{\cot\left(\frac{\pi}{2} - u\right)}{\cos u} = \frac{\tan u}{\cos u} = \tan u \sec u$

13. $$\begin{aligned}\cos^2 x + \cos^2 x \cot^2 x &= \cos^2 x(1 + \cot^2 x)\\ &= \cos^2 x(\csc^2 x)\\ &= \cos^2 x\left(\frac{1}{\sin^2 x}\right)\\ &= \frac{\cos^2 x}{\sin^2 x}\\ &= \cot^2 x\end{aligned}$$

15. $$\begin{aligned}\frac{1}{\csc\theta + 1} - \frac{1}{\csc\theta - 1} &= \frac{(\csc\theta - 1) - (\csc\theta + 1)}{(\csc\theta + 1)(\csc\theta - 1)}\\ &= \frac{-2}{\csc^2\theta - 1}\\ &= \frac{-2}{\cot^2\theta}\\ &= -2\tan^2\theta\end{aligned}$$

17. Let $x = 5\sin\theta$, then

$$\sqrt{25 - x^2} = \sqrt{25 - (5\sin\theta)^2} = \sqrt{25 - 25\sin^2\theta} = \sqrt{25(1 - \sin^2\theta)} = \sqrt{25\cos^2\theta} = 5\cos\theta.$$

19. $$\begin{aligned}\cos x(\tan^2 x + 1) &= \cos x \sec^2 x\\ &= \frac{1}{\sec x}\sec^2 x\\ &= \sec x\end{aligned}$$

21. $$\begin{aligned}\sin\left(\frac{\pi}{2} - \theta\right)\tan\theta &= \cos\theta\tan\theta\\ &= \cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \sin\theta\end{aligned}$$

23. $\frac{1}{\tan\theta\csc\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}} = \cos\theta$

25. $$\begin{aligned}\sin^5 x\cos^2 x &= \sin^4 x\cos^2 x\sin x\\ &= (1 - \cos^2 x)^2\cos^2 x\sin x\\ &= (1 - 2\cos^2 x + \cos^4 x)\cos^2 x\sin x\\ &= (\cos^2 x - 2\cos^4 x + \cos^6 x)\sin x\end{aligned}$$

27. $$\begin{aligned}\sin x &= \sqrt{3} - \sin x\\ \sin x &= \frac{\sqrt{3}}{2}\\ x &= \frac{\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n\end{aligned}$$

29. $$\begin{aligned}3\sqrt{3}\tan u &= 3\\ \tan u &= \frac{1}{\sqrt{3}}\\ u &= \frac{\pi}{6} + n\pi\end{aligned}$$

31. $$\begin{aligned}3\csc^2 x &= 4\\ \csc^2 x &= \frac{4}{3}\\ \sin x &= \pm\frac{\sqrt{3}}{2}\\ x &= \frac{\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n\end{aligned}$$

These can be combined as:

$$x = \frac{\pi}{3} + n\pi \quad\text{or}\quad x = \frac{2\pi}{3} + n\pi$$

33.
$$\begin{aligned} \sin^3 x &= \sin x \\ \sin^3 x - \sin x &= 0 \\ \sin x(\sin^2 x - 1) &= 0 \\ \sin x &= 0 \Rightarrow x = 0, \pi \\ \sin^2 x &= 1 \\ \sin x &= \pm 1 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

35.
$$\begin{aligned} \cos^2 x + \sin x &= 1 \\ 1 - \sin^2 x + \sin x - 1 &= 0 \\ -\sin x(\sin x - 1) &= 0 \end{aligned}$$

$$\sin x = 0 \qquad \sin x - 1 = 0$$
$$x = 0, \pi \qquad \sin x = 1$$
$$x = \frac{\pi}{2}$$

37.
$$\begin{aligned} 2 \sin 2x - \sqrt{2} &= 0 \\ \sin 2x &= \frac{\sqrt{2}}{2} \\ 2x &= \frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n \\ x &= \frac{\pi}{8} + \pi n, \frac{3\pi}{8} + \pi n \\ x &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8} \end{aligned}$$

45.
$$\begin{aligned} \tan^2 \theta + \tan \theta - 6 &= 0 \\ (\tan \theta + 3)(\tan \theta - 2) &= 0 \end{aligned}$$

$$\tan \theta + 3 = 0 \qquad \text{or} \qquad \tan \theta - 2 = 0$$
$$\tan \theta = -3 \qquad \tan \theta = 2$$
$$\theta = \arctan(-3) + n\pi \qquad \theta = \arctan 2 + n\pi$$

39.
$$\begin{aligned} 3 \tan^2\left(\frac{x}{3}\right) - 1 &= 0 \\ \tan^2\left(\frac{x}{3}\right) &= \frac{1}{3} \\ \tan \frac{x}{3} &= \pm\sqrt{\frac{1}{3}} \\ \tan \frac{x}{3} &= \pm\frac{\sqrt{3}}{3} \\ \frac{x}{3} &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \\ x &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \end{aligned}$$

$\frac{5\pi}{2}$ and $\frac{7\pi}{2}$ are greater than 2π, so they are not solutions. The solution is $x = \frac{\pi}{2}$.

41.
$$\cos 4x(\cos x - 1) = 0$$

$$\cos 4x = 0 \qquad \cos x - 1 = 0$$
$$4x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n \qquad \cos x = 1$$
$$x = \frac{\pi}{8} + \frac{\pi}{2}n, \frac{3\pi}{8} + \frac{\pi}{2}n \qquad x = 0$$
$$x = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

43.
$$\begin{aligned} \tan^2 x - 2 \tan x &= 0 \\ \tan x(\tan x - 2) &= 0 \end{aligned}$$

$$\tan x = 0 \qquad \text{or} \qquad \tan x - 2 = 0$$
$$x = n\pi \qquad \tan x = 2$$
$$x = \arctan 2 + n\pi$$

47. $\sin 75° = \sin(120° - 45°)$

$$= \sin 120° \cos 45° - \cos 120° \sin 45°$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$\cos 75° = \cos(120° - 45°)$

$$= \cos 120° \cos 45° + \sin 120° \sin 45°$$

$$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$\tan 75° = \tan(120° - 45°) = \dfrac{\tan 120° - \tan 45°}{1 + \tan 120° \tan 45°}$

$$= \frac{-\sqrt{3} - 1}{1 + \left(-\sqrt{3}\right)(1)} = \frac{-\sqrt{3} - 1}{1 - \sqrt{3}}$$

$$= \frac{-\sqrt{3} - 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{-4 - 2\sqrt{3}}{-2} = 2 + \sqrt{3}$$

49. $\sin \dfrac{25\pi}{12} = \sin\left(\dfrac{11\pi}{6} + \dfrac{\pi}{4}\right) = \sin \dfrac{11\pi}{6} \cos \dfrac{\pi}{4} + \cos \dfrac{11\pi}{6} \sin \dfrac{\pi}{4}$

$$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$\cos \dfrac{25\pi}{12} = \cos\left(\dfrac{11\pi}{6} + \dfrac{\pi}{4}\right) = \cos \dfrac{11\pi}{6} \cos \dfrac{\pi}{4} - \sin \dfrac{11\pi}{6} \sin \dfrac{\pi}{4}$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$\tan \dfrac{25\pi}{12} = \tan\left(\dfrac{11\pi}{6} + \dfrac{\pi}{4}\right) = \dfrac{\tan \dfrac{11\pi}{6} + \tan \dfrac{\pi}{4}}{1 - \tan \dfrac{11\pi}{6} \tan \dfrac{\pi}{4}}$

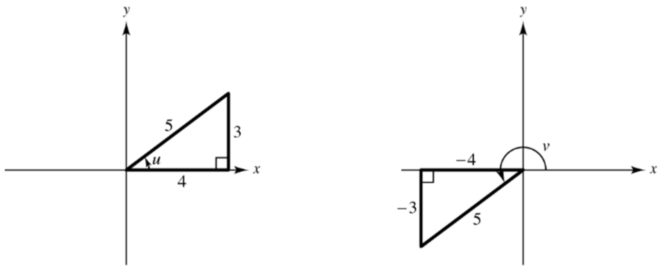
$$= \frac{\left(-\frac{\sqrt{3}}{3}\right) + 1}{1 - \left(-\frac{\sqrt{3}}{3}\right)(1)} = 2 - \sqrt{3}$$

51. $$\sin\left(-\frac{19\pi}{12}\right) = -\sin\left(\frac{19\pi}{12}\right)$$
$$= -\left[\sin\left(\frac{5\pi}{4} + \frac{\pi}{3}\right)\right]$$
$$= -\left[\sin\frac{5\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{5\pi}{4}\right]$$
$$= -\left[\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$= \frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$$

$$\cos\left(-\frac{19\pi}{4}\right) = \cos\left(\frac{19\pi}{4}\right)$$
$$= \cos\left(\frac{5\pi}{4} + \frac{\pi}{3}\right)$$
$$= \cos\frac{5\pi}{4}\cos\frac{\pi}{3} - \sin\frac{5\pi}{4}\sin\frac{\pi}{3}$$
$$= \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\tan\left(-\frac{19\pi}{4}\right) = \frac{\sin\left(-\frac{19\pi}{4}\right)}{\cos\left(-\frac{19\pi}{4}\right)}$$
$$= \frac{\sqrt{2}/4\left(1 + \sqrt{3}\right)}{\sqrt{2}/4\left(\sqrt{3} - 1\right)}$$
$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \cdot \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$
$$= \frac{4 + 2\sqrt{3}}{2}$$
$$= 2 + \sqrt{3}$$

53. $$\sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ = \sin\left(60^\circ - 45^\circ\right)$$
$$= \sin 15^\circ$$



Figures for Exercises 55 and 57

55. $$\sin(u + v) = \sin u \cos v + \cos u \sin v = \frac{3}{5}\left(-\frac{4}{5}\right) + \frac{4}{5}\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

57. $$\cos(u - v) = \cos u \cos v + \sin u \sin v = \frac{4}{5}\left(-\frac{4}{5}\right) + \frac{3}{5}\left(-\frac{3}{5}\right) = -1$$

59. $\cos\left(x + \dfrac{\pi}{2}\right) = \cos x \cos \dfrac{\pi}{2} - \sin x \sin \dfrac{\pi}{2} = \cos x(0) - \sin x(1) = -\sin x$

61. $\tan(\pi - x) = \dfrac{\tan \pi - \tan x}{1 - \tan \pi \tan x} = -\tan x$

63. $\sin\left(x + \dfrac{\pi}{4}\right) - \sin\left(x - \dfrac{\pi}{4}\right) = 1$

$$2 \cos x \sin \frac{\pi}{4} = 1$$

$$\cos x = \frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

65. $\sin u = -\dfrac{4}{5}, \pi < u < \dfrac{3\pi}{2}$

$$\cos u = -\sqrt{1 - \sin^2 u} = \frac{-3}{5}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{4}{3}$$

$$\sin 2u = 2 \sin u \cos u = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{24}{25}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = -\frac{7}{25}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u} = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = -\frac{24}{7}$$

67. $\sin 4x = \sin\left[2(2x)\right] = 2 \sin 2x \cos 2x = 2\left(2 \sin x \cos x\right)\left(\cos^2 x - \sin^2 x\right) = 4 \sin x \cos^3 x - 4 \sin^3 x \cos x$

Other forms possible. For example, $\sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x$.

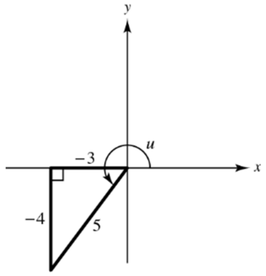
69. $\tan^2 3x = \dfrac{\sin^2 3x}{\cos^2 3x} = \dfrac{\frac{1 - \cos 6x}{2}}{\frac{1 + \cos 6x}{2}} = \dfrac{1 - \cos 6x}{1 + \cos 6x}$

71. $\sin(-75°) = -\sqrt{\dfrac{1 - \cos 150°}{2}} = -\sqrt{\dfrac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\dfrac{\sqrt{2 + \sqrt{3}}}{2} = -\dfrac{1}{2}\sqrt{2 + \sqrt{3}}$

$$\cos(-75°) = -\sqrt{\frac{1 + \cos 150°}{2}} = \sqrt{\frac{1 + \left(-\frac{\sqrt{3}}{2}\right)}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{1}{2}\sqrt{2 - \sqrt{3}}$$

$$\tan(-75°) = -\left(\frac{1 - \cos 150°}{\sin 150°}\right) = -\left(\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}\right) = -\left(2 + \sqrt{3}\right) = -2 - \sqrt{3}$$

73. $\tan u = \frac{4}{3}, \pi < u < \frac{3\pi}{2}$



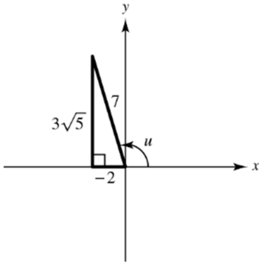
(a) Because u is in Quadrant III, $\frac{u}{2}$ is in Quadrant II.

(b) $$\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\frac{u}{2} = -\sqrt{\frac{1+\cos u}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = -\sqrt{\frac{1}{5}} = -\frac{\sqrt{5}}{5}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\left(-\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)} = -2$$

75. $\cos u = -\frac{2}{7}, \frac{\pi}{2} < u < \pi$



(a) Because u is in Quadrant II, $\frac{u}{2}$ is in Quadrant I.

(b) $$\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\left(-\frac{2}{7}\right)}{2}} = \sqrt{\frac{9}{14}} = \frac{3\sqrt{14}}{14}$$

$$\cos\frac{u}{2} = \sqrt{\frac{1+\cos u}{2}} = \sqrt{\frac{1+\left(-\frac{2}{7}\right)}{2}} = \sqrt{\frac{5}{14}} = \frac{\sqrt{70}}{14}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\left(-\frac{2}{7}\right)}{\frac{3\sqrt{5}}{7}} = \frac{3\sqrt{5}}{5}$$

77. $\cos 4\theta \sin 6\theta = \frac{1}{2}\left[\sin(4\theta + 6\theta) - \sin(4\theta - 6\theta)\right] = \frac{1}{2}\left[\sin 10\theta - \sin(-2\theta)\right]$

79. $\cos 6\theta + \cos 5\theta = 2\cos\left(\frac{6\theta + 5\theta}{2}\right)\cos\left(\frac{6\theta - 5\theta}{2}\right) = 2\cos\frac{11\theta}{2}\cos\frac{\theta}{2}$

81.
$$r = \frac{1}{32}v_0^2 \sin 2\theta$$
range = 100 feet

v_0 = 80 feet per second

$$r = \frac{1}{32}(80)^2 \sin 2\theta = 100$$
$$\sin 2\theta = 0.5$$
$$2\theta = 30^\circ$$
$$\theta = 15^\circ \text{ or } \frac{\pi}{12}$$

83. False. If $\frac{\pi}{2} < \theta < \pi$, then $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, and $\frac{\theta}{2}$ is in Quadrant I. $\cos\frac{\theta}{2} > 0$

85. False. $$\begin{aligned} 4\sin(-x)\cos(-x) &= -4\sin x\cos x \\ &= -2(2\sin x\cos x) \\ &= -2\sin 2x \end{aligned}$$

87. Yes. *Sample Answer*. When the domain is all real numbers, the solutions of $\sin x = \frac{1}{2}$ are $x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{5\pi}{6} + 2n\pi$, so there are infinitely many solutions.

Problem Solving for Chapter 5

1. $\sin\theta = \pm\sqrt{1-\cos^2\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} = \pm\frac{1}{\sqrt{1-\cos^2\theta}}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$$

You also have the following relationships:

$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$

$$\tan\theta = \frac{\cos\left[(\pi/2)-\theta\right]}{\cos\theta}$$

$$\csc\theta = \frac{1}{\cos\left[(\pi/2)-\theta\right]}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\cos\left[(\pi/2)-\theta\right]}$$

3. $\sin\left[\frac{(12n+1)\pi}{6}\right] = \sin\left[\frac{1}{6}(12n\pi+\pi)\right] = \sin\left(2n\pi+\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

So, $\sin\left[\frac{(12n+1)\pi}{6}\right] = \frac{1}{2}$ for all integers n.

5. From the figure, it appears that $u + v = w$. Assume that u, v, and w are all in Quadrant I.

From the figure:

$$\tan u = \frac{s}{3s} = \frac{1}{3}$$

$$\tan v = \frac{s}{2s} = \frac{1}{2}$$

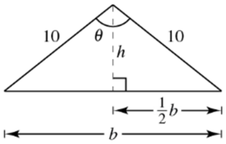
$$\tan w = \frac{s}{s} = 1$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v} = \frac{1/3 + 1/2}{1 - (1/3)(1/2)} = \frac{5/6}{1-(1/6)} = 1 = \tan w.$$

So, $\tan(u+v) = \tan w$. Because u, v, and w are all in Quadrant I, you have

$\arctan\left[\tan(u+v)\right] = \arctan[\tan w]u + v = w.$

7. (a)



$$\sin\frac{\theta}{2} = \frac{\frac{1}{2}b}{10} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10}$$

$$b = 20\sin\frac{\theta}{2} \qquad h = 10\cos\frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}\left(20\sin\frac{\theta}{2}\right)\left(10\cos\frac{\theta}{2}\right) \\ &= 100\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{aligned}$$

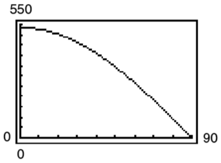
(b)
$$\begin{aligned} A &= 50\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\ &= 50\sin\left(2\left(\frac{\theta}{2}\right)\right) \\ &= 50\sin\theta \end{aligned}$$

Because $\sin\frac{\pi}{2} = 1$ is a maximum, $\theta = \frac{\pi}{2}$. So, the area is a maximum at $A = 50\sin\frac{\pi}{2} = 50$ square meters.

9. $F = \dfrac{0.6W \sin(\theta + 90^\circ)}{\sin 12^\circ}$

(a) $$F = \frac{0.6W(\sin\theta\cos 90^\circ + \cos\theta\sin 90^\circ)}{\sin 12^\circ}$$
$$= \frac{0.6W[(\sin\theta)(0) + (\cos\theta)(1)]}{\sin 12^\circ}$$
$$= \frac{0.6W\cos\theta}{\sin 12^\circ}$$

(b) Let $y_1 = \dfrac{0.6(185)\cos x}{\sin 12^\circ}$.



(c) The force is maximum (533.88 pounds) when $\theta = 0^\circ$.

The force is minimum (0 pounds) when $\theta = 90^\circ$.

11. $d = 35 - 28\cos\dfrac{\pi}{6.2}t$ when $t = 0$ corresponds to 12:00 A.M.

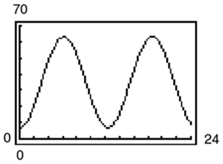
(a) The high tides occur when $\cos\dfrac{\pi}{6.2}t = -1$. Solving yields $t = 6.2$ or $t = 18.6$.

These t-values correspond to 6:12 A.M. and 6:36 P.M.

The low tide occurs when $\cos\dfrac{\pi}{6.2}t = 1$. Solving yields $t = 0$ and $t = 12.4$ which corresponds to 12:00 A.M. and 12:24 P.M.

(b) The water depth is never 3.5 feet. At low tide, the depth is $d = 35 - 28 = 7$ feet.

(c)



13. (a) $$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin\frac{\theta}{2}}$$
$$= \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$
$$= \cos\left(\frac{\alpha}{2}\right) + \cot\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)$$

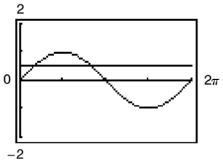
For $\alpha = 60^\circ$, $n = \cos 30^\circ + \cot\left(\dfrac{\theta}{2}\right)\sin 30^\circ$

$$n = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right).$$

(b) For glass, $n = 1.50$.

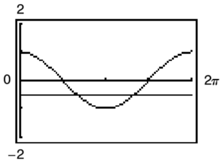
$$1.50 = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right)$$
$$2\left(1.50 - \frac{\sqrt{3}}{2}\right) = \cot\left(\frac{\theta}{2}\right)$$
$$\frac{1}{3 - \sqrt{3}} = \tan\left(\frac{\theta}{2}\right)$$
$$\theta = 2\tan^{-1}\left(\frac{1}{3 - \sqrt{3}}\right)$$
$$\theta \approx 76.5^\circ$$

15. (a) Let $y_1 = \sin x$ and $y_2 = 0.5$.



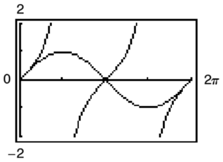
$\sin x \geq 0.5$ on the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

(b) Let $y_1 = \cos x$ and $y_2 = -0.5$.



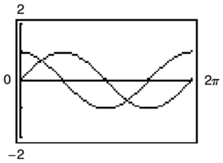
$\cos x \leq -0.5$ on the interval $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$.

(c) Let $y_1 = \tan x$ and $y_2 = \sin x$.



$\tan x < \sin x$ on the intervals $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(d) Let $y_1 = \cos x$ and $y_2 = \sin x$.



$\cos x \geq \sin x$ on the intervals $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right)$.

Practice Test for Chapter 5

1. Find the value of the other five trigonometric functions, given $\tan x = \frac{4}{11}$, $\sec x < 0$.

2. Simplify $\dfrac{\sec^2 x + \csc^2 x}{\csc^2 x(1 + \tan^2 x)}$.

3. Rewrite as a single logarithm and simplify $\ln|\tan\theta| - \ln|\cot\theta|$.

4. True or false:

   $\cos\left(\dfrac{\pi}{2} - x\right) = \dfrac{1}{\csc x}$

5. Factor and simplify: $\sin^4 x + (\sin^2 x)\cos^2 x$

6. Multiply and simplify: $(\csc x + 1)(\csc x - 1)$

7. Rationalize the denominator and simplify:

   $\dfrac{\cos^2 x}{1 - \sin x}$

8. Verify:

   $\dfrac{1 + \cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1 + \cos\theta} = 2\csc\theta$

9. Verify:

   $\tan^4 x + 2\tan^2 x + 1 = \sec^4 x$

10. Use the sum or difference formulas to determine:
    (a) $\sin 105°$
    (b) $\tan 15°$

11. Simplify: $(\sin 42°)\cos 38° - (\cos 42°)\sin 38°$

12. Verify $\tan\left(\theta + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

13. Write $\sin(\arcsin x - \arccos x)$ as an algebraic expression in x.

14. Use the double-angle formulas to determine:
    (a) $\cos 120°$
    (b) $\tan 300°$

15. Use the half-angle formulas to determine:
    (a) $\sin 22.5°$
    (b) $\tan\dfrac{\pi}{12}$

16. Given $\sin\theta = 4/5$, θ lies in Quadrant II, find $\cos(\theta/2)$.

17. Use the power-reducing identities to write $(\sin^2 x)\cos^2 x$ in terms of the first power of cosine.

18. Rewrite as a sum: $6(\sin 5\theta)\cos 2\theta$.

19. Rewrite as a product: $\sin(x+\pi)+\sin(x-\pi)$.

20. Verify $\dfrac{\sin 9x + \sin 5x}{\cos 9x - \cos 5x} = -\cot 2x$.

21. Verify:

$(\cos u)\sin v = \frac{1}{2}\left[\sin(u+v)-\sin(u-v)\right]$.

22. Find all solutions in the interval $[0, 2\pi)$:

$4\sin^2 x = 1$

23. Find all solutions in the interval $[0, 2\pi)$:

$\tan^2\theta + \left(\sqrt{3}-1\right)\tan\theta - \sqrt{3} = 0$

24. Find all solutions in the interval $[0, 2\pi)$:

$\sin 2x = \cos x$

25. Use the quadratic formula to find all solutions in the interval $[0, 2\pi)$:

$\tan^2 x - 6\tan x + 4 = 0$

CHAPTER 6

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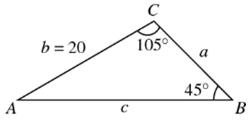
C H A P T E R 6
Additional Topics in Trigonometry

Section 6.1 Law of Sines

1. oblique

3. The two cases AAS (two angles and one side) or ASA (angle side angle) and SSA (two sides and an angle opposite) can be solved using the Law of Sines.

5.



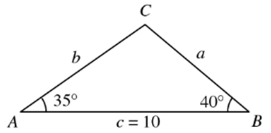
Given: $B = 45°, C = 105°, b = 20$

$A = 180° - B - C = 30°$

$a = \frac{b}{\sin B}(\sin A) = \frac{20 \sin 30°}{\sin 45°} = 10\sqrt{2} \approx 14.14$

$C = \frac{b}{\sin B}(\sin C) = \frac{20 \sin 105°}{\sin 45°} \approx 27.32$

7.



Given: $A = 35°, B = 40°, c = 10$

$C = 180° - A - B = 105°$

$a = \frac{c}{\sin C}(\sin A) = \frac{10 \sin 35°}{\sin 105°} \approx 5.94$

$b = \frac{c}{\sin C}(\sin B) = \frac{10 \sin 40°}{\sin 105°} \approx 6.65$

9. Given: $A = 102.4°, C = 16.7°, a = 21.6$

$B = 180° - A - C = 60.9°$

$b = \frac{a}{\sin A}(\sin B) = \frac{21.6}{\sin 102.4°}(\sin 60.9°) \approx 19.32$

$c = \frac{a}{\sin A}(\sin C) = \frac{21.6}{\sin 102.4°}(\sin 16.7°) \approx 6.36$

11. Given: $A = 83°20', C = 54.6°, c = 18.1$

$B = 180° - A - C = 180° - 83°20' - 54°36'$
$= 42°4'$

$a = \frac{c}{\sin C}(\sin A) = \frac{18.1}{\sin 54.6°}(\sin 83°20') \approx 22.05$

$b = \frac{c}{\sin C}(\sin B) = \frac{18.1}{\sin 54.6°}(\sin 42°4') \approx 14.88$

13. Given: $A = 35°, B = 65°, c = 10$

$C = 180° - A - B = 80°$

$a = \frac{c}{\sin C}(\sin A) = \frac{10 \sin 35°}{\sin 80°} \approx 5.82$

$b = \frac{c}{\sin C}(\sin B) = \frac{10 \sin 65°}{\sin 80°} \approx 9.20$

15. Given: $C = 55°, A = 42°, b = \frac{3}{4}$

$B = 180° - A - C = 83°$

$a = \frac{b}{\sin B}(\sin A) = \frac{0.75}{\sin 83°}(\sin 42°) \approx 0.51$

$c = \frac{b}{\sin B}(\sin C) = \frac{0.75}{\sin 83°}(\sin 55°) \approx 0.62$

17. Given: $A = 36°, a = 8, b = 5$

$\sin B = \frac{b \sin A}{a} = \frac{5 \sin 36°}{8} \approx 0.36737 \Rightarrow B \approx 21.55°$

$C = 180° - A - B \approx 180° - 36° - 21.55 = 122.45°$

$c = \frac{a}{\sin A}(\sin C) = \frac{8}{\sin 36°}(\sin 122.45°) \approx 11.49$

19. Given: $A = 145°, a = 14, b = 4$

$\sin B = \frac{b \sin A}{a} = \frac{4 \sin 145°}{14}$
$\approx 0.1639 \Rightarrow B \approx 9.43°$

$C = 180° - A - B \approx 25.57°$

$c = \frac{a}{\sin A}(\sin C) \approx \frac{14 \sin 25.57°}{\sin 145°} \approx 10.53$

21. Given: $B = 15°30', a = 4.5, b = 6.8$

$\sin A = \frac{a \sin B}{b} = \frac{4.5 \sin 15°30'}{6.8}$
$\approx 0.17685 \Rightarrow A \approx 10°11'$

$C = 180° - A - B \approx 180° - 10°11' - 15°30'$
$= 154°19'$

$c = \frac{b}{\sin B}(\sin C) = \frac{6.8}{\sin 15°30'}(\sin 154°19') \approx 11.03$

23. Given: $A = 110°, a = 125, b = 100$

$$\sin B = \frac{b \sin A}{a} = \frac{100 \sin 110°}{125} \approx 0.75175 \Rightarrow B \approx 48.74°$$

$$C = 180° - A - B \approx 21.26°$$

$$c = \frac{a \sin C}{\sin A} \approx \frac{125 \sin 21.26°}{\sin 110°} \approx 48.23$$

25. Given: $a = 18, b = 20, A = 76°$

$h = 20 \sin 76° \approx 19.41$

Because $a < h,$ no triangle is formed.

27. Given: $A = 58°, a = 11.4, c = 12.8$

$$\sin B = \frac{b \sin A}{a} = \frac{12.8 \sin 58°}{11.4} \approx 0.9522 \Rightarrow B \approx 72.21° \text{ or } B \approx 107.79°$$

Case 1

$B \approx 72.21°$

$C = 180° - A - B \approx 49.79°$

$$c = \frac{a}{\sin A}(\sin C) \approx \frac{11.4 \sin 49.79°}{\sin 58°} \approx 10.27$$

Case 2

$B \approx 107.79°$

$C = 180° - A - B \approx 14.21°$

$$c = \frac{a}{\sin A}(\sin C) \approx \frac{11.4 \sin 14.21°}{\sin 58°} \approx 3.30$$

29. Given: $A = 120°, a = b = 25$

No triangle is formed because A is obtuse and $a = b.$

31. Given: $A = 45°, a = b = 1$

Because $a = b = 1, B = 45°.$

$C = 180° - A - B = 90°$

$$c = \frac{a}{\sin A}(\sin C) = \frac{1 \sin 90°}{\sin 45°} = \sqrt{2} \approx 1.41$$

33. Given: $A = 36°, a = 5$

(a) One solution if $b \le 5$ or $b = \dfrac{5}{\sin 36°}.$

(b) Two solutions if $5 < b < \dfrac{5}{\sin 36°}.$

(c) No solution if $b > \dfrac{5}{\sin 36°}.$

35. Given: $A = 105°, a = 80$

(a) One solution if $b < 80.$

(b) Not possible for two solutions.

(c) No solution if $b \ge 80.$

37. $A = 125°, b = 9, c = 6$

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}(9)(6) \sin 125° \approx 22.1$$

39. $B = 39°, a = 25, c = 12$

$$\text{Area} = \frac{1}{2}ac \sin B = \frac{1}{2}(25)(12) \sin 39° = 94.4$$

41. $C = 103° 15', a = 16, b = 28$

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(16)(28) \sin 103° 15' \approx 218.0$$

43. $A = 67°, B = 43°, a = 8$

$$b = \frac{a}{\sin A}(\sin B) = \frac{8 \sin 43°}{\sin 67°} \approx 5.927$$

$C = 180° - A - B = 70°$

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(8)(5.927) \sin 70° \approx 22.3$$

45. Given: $A = 15°, B = 135°, c = 30$

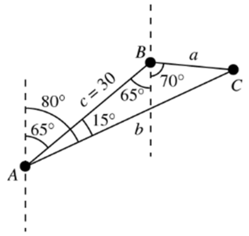
$C = 180° - A - B = 30°$

From Pine Knob:

$$b = \frac{c \sin B}{\sin C} = \frac{30 \sin 135°}{\sin 30°} \approx 42.4 \text{ kilometers}$$

From Colt Station:

$$a = \frac{c \sin A}{\sin C} = \frac{30 \sin 15°}{\sin 30°} \approx 15.5 \text{ kilometers}$$



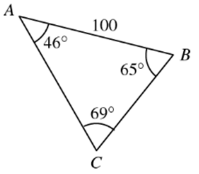
47. Given: $c = 100$

$A = 74° - 28° = 46°$,

$B = 180° - 41° - 74° = 65°$,

$C = 180° - 46° - 65° = 69°$

$$a = \frac{c}{\sin C}(\sin A) = \frac{100}{\sin 69°}(\sin 46°) \approx 77 \text{ meters}$$



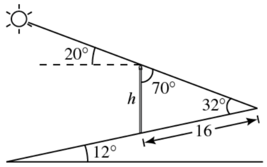
49. (a) $C = 180° - 94° - 30° = 56°$

$$\frac{h}{\sin 30°} = \frac{40}{\sin 56°}$$

$$h = \frac{40}{\sin 56°}(\sin 30°)$$

(b) $h = \dfrac{40}{\sin 56°}(\sin 30°) \approx 24.1$ meters

51. (a)



(b) $\dfrac{h}{\sin 32°} = \dfrac{16}{\sin 70°}$

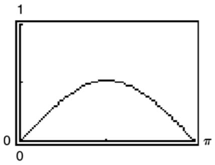
(c) $h = \dfrac{16 \sin 32°}{\sin 70°} \approx 9.0$ meters

53. (a) $\dfrac{\sin \alpha}{9} = \dfrac{\sin \beta}{18}$

$\sin \alpha = 0.5 \sin \beta$

$\alpha = \arcsin(0.5 \sin \beta)$

(b)



Domain: $0 < \beta < \pi$

Range: $0 < \alpha \le \dfrac{\pi}{6}$

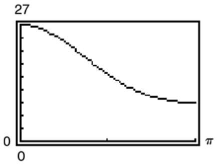
(c) $\gamma = \pi - \alpha - \beta = \pi - \beta - \arcsin(0.5 \sin \beta)$

$$\frac{c}{\sin \gamma} = \frac{18}{\sin \beta}$$

$$c = \frac{18 \sin \gamma}{\sin \beta}$$

$$= \frac{18 \sin\left[\pi - \beta - \arcsin(0.5 \sin \beta)\right]}{\sin \beta}$$

(d)



Domain: $0 < \beta < \pi$

Range: $9 < c < 27$

(e)

β	0.4	0.8	1.2	1.6	2.0	2.4	2.8
α	0.1960	0.3669	0.4848	0.5234	0.4720	0.3445	0.1683
c	25.95	23.07	19.19	15.33	12.29	10.31	9.27

As β increases from 0 to π, α increases and then decreases, and c decreases from 27 to 9.

55. True. If one angle of a triangle is obtuse, then there is less than 90° left for the other two angles, so it cannot contain a right angle. It must be oblique.

57. False. To solve an oblique triangle using the Law of Sines, you need to know two angles and any side, or two sides and an angle opposite one of them.

59. False. The sine of the angle included between the two given sides is needed.

61. $(1, -6), (5, 0)$

$$\text{Slope} = \frac{0 - (-6)}{5 - 1} = \frac{6}{4} = \frac{3}{2}$$

63. $(-5, 4), (-13, 4)$

$$\text{Slope} = \frac{4 - 4}{-5 - (-13)} = 0$$

65. $h(x) = x^2 - 10x + 25$

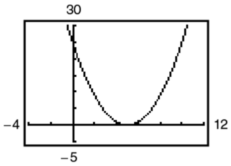
(a) $$\begin{aligned} x^2 - 10x + 25 &= 0 \\ (x-5)^2 &= 0 \\ x - 5 &= 0 \end{aligned}$$

Zero: $x = 5$

(b) $x = 5$ of multiplicity 2 (even multiplicity)

(c) Turning points: 1

(d)



67. $f(t) = -4t^4 - 10t^3 + 6t^2$

(a) $0 = -2t^2(2t^2 + 5t - 3)$

$0 = -2t^2(t + 3)(2t - 1)$

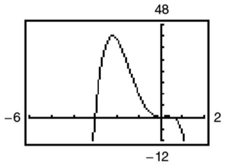
Zeros: $t = 0, t = -3, t = \frac{1}{2}$

(b) $x = -3, x = \frac{1}{2}$ of multiplicity 1 (odd multiplicity)

$x = 0$ of multiplicity 2 (even multiplicity)

(c) Turning points: 3

(d)



Section 6.2 Law of Cosines

1. alternative

3. Alternative Form: $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

5. Given: $a = 10, b = 12, c = 16$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{100 + 144 - 256}{2(10)(12)}$$

$$= -0.05 \Rightarrow C \approx 92.87°$$

$$\sin B = \frac{b \sin C}{c} \approx \frac{12 \sin 92.87°}{16}$$

$$\approx 0.749059 \Rightarrow B \approx 48.51°$$

$$A \approx 180° - 48.51° - 92.87° = 38.62°$$

7. Given: $a = 5, b = 8, c = 12$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 12^2}{2(5)(8)}$$

$$\approx \frac{-55}{80} = -0.6875 \Rightarrow C \approx 133.43°$$

$$\sin B = b\left(\frac{\sin C}{c}\right) = 8\left(\frac{\sin 133.43}{12}\right)$$

$$\approx 0.48412 \Rightarrow B \approx 28.96°$$

$$A \approx 180° - B - C = 17.61°$$

9. Given: $A = 30°, b = 15, c = 30$

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 225 + 900 - 2(15)(30)\cos 30° \approx 345.5771 \\ a &\approx 18.59 \end{aligned}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \approx \frac{(345.5771)^2 + 15^2 - 30^2}{2(18.59)(15)}$$

$$\approx -0.590681 \Rightarrow C \approx 126.21°$$

$$B \approx 180° - 30° - 126.21° = 23.79°$$

11. Given: $C = 108°, a = 10, b = 7$

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 10^2 + 7^2 - 2(10)(7)\cos 108° \\ &\approx 192.2624 \Rightarrow c \approx 13.87 \end{aligned}$$

$$\sin A = a\left(\frac{\sin C}{c}\right) \approx 10\left(\frac{\sin 108°}{13.8659}\right) \approx 0.685896$$

There are two angles between 0° and 180° whose sine is 0.685896, $A_1 \approx 43.31°$ and $A_2 \approx 180° - 43.31° \approx 136.69°$.

Because side c is the longest side of the triangle, C is the largest angle of triangle. So, $A \approx 43.31°$ and $B = 180° - A - C \approx 180° - 43.31° - 108° \approx 28.69°$.

13. Given: $a = 11, b = 15, c = 21$

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{121 + 225 - 441}{2(11)(15)}\\ &\approx -0.287879 \Rightarrow C \approx 106.73^\circ\\ \sin B &= \frac{b \sin C}{c} = \frac{15 \sin 106.73^\circ}{21}\\ &\approx 0.684051 \Rightarrow B \approx 43.16^\circ\\ A &\approx 180^\circ - 43.16^\circ - 106.73^\circ = 30.11^\circ\end{aligned}$$

15. Given: $a = 2.5, b = 1.8, c = 0.9$

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} = \frac{(1.8)^2 + (0.9)^2 - (2.5)^2}{2(1.8)(0.9)}\\ &= -0.679012 \Rightarrow A \approx 132.77^\circ\\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} = \frac{(2.5)^2 + (0.9)^2 - (1.8)^2}{2(2.5)(0.9)}\\ &\approx 0.848889 \Rightarrow B \approx 31.91^\circ\\ C &= 180^\circ - 132.77^\circ - 31.91^\circ = 15.32^\circ\end{aligned}$$

17. Given: $A = 120^\circ, b = 6, c = 7$

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos A\\ &= 36 + 49 - 2(6)(7) \cos 120^\circ\\ &= 127 \Rightarrow a \approx 11.27\\ \sin B &= \frac{b \sin A}{a} \approx \frac{6 \sin 120^\circ}{11.27}\\ &\approx 0.461061 \Rightarrow B \approx 27.46^\circ\\ C &\approx 180^\circ - 120^\circ - 27.46^\circ = 32.54^\circ\end{aligned}$$

19. Given: $B = 10^\circ\, 35', a = 40, c = 30$

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac \cos B\\ &= 1600 + 900 - 2(40)(30) \cos 10^\circ\, 35'\\ &\approx 140.8268 \Rightarrow b \approx 11.87\\ \sin C &= \frac{c \sin B}{b} = \frac{30 \sin 10^\circ\, 35'}{11.87}\\ &\approx 0.464192 \Rightarrow C \approx 27.66^\circ \approx 27^\circ\, 40'\\ A &\approx 180^\circ - 10^\circ\, 35' - 27^\circ\, 40' = 141^\circ\, 45'\end{aligned}$$

21. Given: $B = 125^\circ\, 40', a = 37, c = 37$

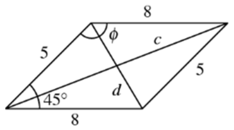
$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac \cos B\\ &= 1369 + 1369 - 2(37)(37) \cos 125^\circ\, 40'\\ &\approx 4334.4420 \Rightarrow b \approx 65.84\\ A &= C \Rightarrow 2A = 180 - 125^\circ\, 40'\\ &= 54^\circ\, 20' \Rightarrow A = C = 27^\circ\, 10'\end{aligned}$$

23. $C = 43^\circ, a = \frac{4}{9}, b = \frac{7}{9}$

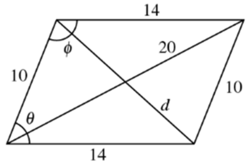
$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C\\ &= \left(\frac{4}{9}\right)^2 + \left(\frac{7}{9}\right)^2 - 2\left(\frac{4}{9}\right)\left(\frac{7}{9}\right) \cos 43^\circ\\ &\approx 0.296842 \Rightarrow c \approx 0.54\\ \sin A &= \frac{a \sin C}{c} = \frac{(4/9) \sin 43^\circ}{0.544832}\\ &\approx 0.556337 \Rightarrow A \approx 33.80^\circ\\ B &\approx 180^\circ - 43^\circ - 33.8^\circ = 103.20^\circ\end{aligned}$$

25.

$$\begin{aligned}d^2 &= 5^2 + 8^2 - 2(5)(8) \cos 45^\circ\\ &\approx 32.4315 \Rightarrow d \approx 5.69\\ 2\phi &= 360^\circ - 2(45^\circ) = 270^\circ \Rightarrow \phi = 135^\circ\\ c^2 &= 5^2 + 8^2 - 2(5)(8) \cos 135^\circ\\ &\approx 145.5685 \Rightarrow c \approx 12.07\end{aligned}$$



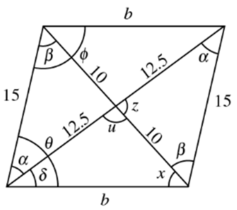
27.



$$.\ \cos \phi = \frac{10^2 + 14^2 - 20^2}{2(10)(14)} \qquad .$$

$$\begin{aligned}\phi &\approx 111.8^\circ\\ 2\theta &\approx 360^\circ - 2(111.8^\circ)\\ \theta &= 68.2^\circ\\ d^2 &= 10^2 + 14^2 - 2(10)(14) \cos 68.2^\circ\\ d &\approx 13.86\end{aligned}$$

29. $\cos\alpha = \dfrac{(12.5)^2 + (15)^2 - 10^2}{2(12.5)(15)} = 0.75 \Rightarrow \alpha \approx 41.41°$



$$\cos\beta = \frac{10^2 + 15^2 - (12.5)^2}{2(10)(15)} = 0.5625 \Rightarrow \beta \approx 55.77°$$

$$z = 180° - \alpha - \beta = 82.82°$$

$$u = 180° - z = 97.18°$$

$$b^2 = 12.5^2 + 10^2 - 2(12.5)(10)\cos 97.18° \approx 287.4967 \Rightarrow b \approx 16.96$$

$$\cos\delta = \frac{12.5^2 + 16.96^2 - 10^2}{2(12.5)(16.96)} \approx 0.8111 \Rightarrow \delta \approx 35.80°$$

$$\theta = \alpha + \delta = 41.41° + 35.80° = 77.2°$$

$$2\phi = 360° - 2\theta \Rightarrow \phi = \frac{360° - 2(77.21°)}{2} = 102.8°$$

31. Given: $a = 8, c = 5, B = 40°$

Given two sides and included angle, use the Law of Cosines.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 64 + 25 - 2(8)(5)\cos 40° \\ &\approx 27.7164 \Rightarrow b \approx 5.26 \end{aligned}$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{(5.26)^2 + 25 - 64}{2(5.26)(5)} \\ &\approx -0.2154 \Rightarrow A \approx 102.44° \end{aligned}$$

$$C \approx 180° - 102.44° - 40° = 37.56°$$

33. Given: $A = 24°, a = 4, b = 18$

Given two sides and an angle opposite one of them, use the Law of Sines.

$h = b\sin A = 18\sin 24° \approx 7.32$

Because $a < h$, no triangle is formed.

35. Given: $A = 42°, B = 35°, c = 1.2$

Given two angles and a side, use the Law of Sines.

$C = 180° - 42° - 35° = 103°$

$$a = \frac{c\sin A}{\sin C} = \frac{1.2\sin 42°}{\sin 103°} \approx 0.82$$

$$b = \frac{c\sin B}{\sin C} = \frac{1.2\sin 35°}{\sin 103°} \approx 0.71$$

37. $a = 6, b = 12, c = 17$

$$s = \frac{a + b + c}{2} = \frac{6 + 12 + 17}{2} = 17.5$$

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{17.5(11.5)(5.5)(0.5)} \approx 23.53 \end{aligned}$$

39. $a = 2.5, b = 10.2, c = 8$

$$s = \frac{a + b + c}{2} = \frac{2.5 + 10.2 + 8}{2} = 10.35$$

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{10.35(7.85)(0.15)(2.35)} \approx 5.35 \end{aligned}$$

41. Given: $a = 1, b = \dfrac{1}{2}, c = \dfrac{5}{4}$

$$s = \frac{a + b + c}{2} = \frac{1 + \frac{1}{2} + \frac{5}{4}}{2} = \frac{11}{8}$$

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\frac{11}{8}\left(\frac{3}{8}\right)\left(\frac{7}{8}\right)\left(\frac{1}{8}\right)} \approx 0.24 \end{aligned}$$

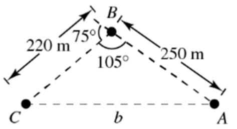
43. Given: $A = 80°, b = 75, c = 41$

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 75^2 + 41^2 - 2(75)(41)\cos 80° \\ &= 6238.0637 \Rightarrow a \approx 78.98 \end{aligned}$$

$$s = \frac{a + b + c}{2} = \frac{78.98 + 75 + 41}{2} = 97.49$$

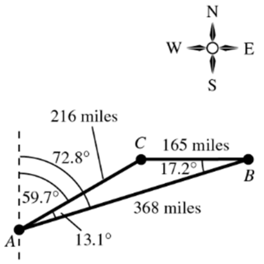
$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{97.49(18.51)(22.49)(56.49)} \\ &\approx 1514.14 \end{aligned}$$

45. $b^2 = 220^2 + 250^2 - 2(220)(250)\cos 105° \Rightarrow b \approx 373.3$ meters



47. $d = \sqrt{330^2 + 420^2 - 2(330)(420)\cos 8°} \approx 103.9$ feet

49.



$a = 165, b = 216, c = 368$

$$\cos B = \frac{165^2 + 368^2 - 216^2}{2(165)(368)} \approx 0.9551$$

$$B \approx 17.2°$$

$$\cos A = \frac{216^2 + 368^2 - 165^2}{2(216)(368)} \approx 0.9741$$

$$A \approx 13.1°$$

(a) Bearing of Minneapolis (C) from Phoenix (A)

N $(90° - 17.2° - 13.1°)$ E

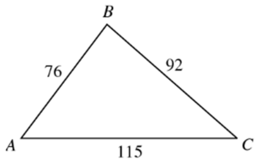
N 59.7° E

(b) Bearing of Albany (B) from Phoenix (A)

N $(90° - 17.2°)$ E

N 72.8° E

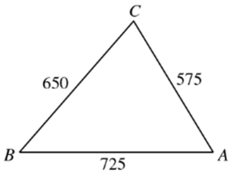
51.



$$\cos A = \frac{115^2 + 76^2 - 92^2}{2(115)(76)} \approx 0.6028 \Rightarrow A \approx 52.9°$$

$$\cos C = \frac{115^2 + 92^2 - 76^2}{2(115)(92)} \approx 0.75203 \Rightarrow c \approx 41.2°$$

53.



The largest angle is across from the largest side.

$$\cos C = \frac{650^2 + 575^2 - 725^2}{2(650)(575)}$$

$$C \approx 72.3°$$

55. $a = 200$

$b = 500$

$c = 600 \Rightarrow s = \dfrac{200 + 500 + 600}{2} = 650$

Area $= \sqrt{650(450)(150)(50)} \approx 46{,}837.5$ square feet

57. $s = \dfrac{510 + 840 + 1120}{2} = 1235$

Area $= \sqrt{1235(1235 - 510)(1235 - 840)(1235 - 1120)}$

$\approx 201{,}674$ square yards

Cost $\approx \left(\dfrac{201{,}674.02}{4840}\right)(2000) \approx \$83{,}336.37$

59. False. The average of the three sides of a triangle is $\dfrac{a + b + c}{3}$, not $\dfrac{a + b + c}{2} = s$.

61.
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= a^2 + b^2 - 2ab\cos 90° \\ &= a^2 + b^2 - 2ab(0) \\ &= a^2 + b^2 \end{aligned}$$

When $C = 90°$, you obtain the Pythagorean Theorem. The Pythagorean Theorem is a special case of the Law of Cosines.

63. (a) $$\begin{aligned}\frac{1}{2}bc(1+\cos A) &= \frac{1}{2}bc\left[1+\frac{b^2+c^2-a^2}{2bc}\right]\\ &= \frac{1}{2}bc\left[\frac{2bc+b^2+c^2-a^2}{2bc}\right]\\ &= \frac{1}{4}\left[(b+c)^2-a^2\right]\\ &= \frac{1}{4}\left[(b+c)+a\right]\left[(b+c)-a\right]\\ &= \frac{b+c+a}{2}\cdot\frac{b+c-a}{2}\\ &= \frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\end{aligned}$$

(b) $$\begin{aligned}\frac{1}{2}bc(1-\cos A) &= \frac{1}{2}bc\left[1+\frac{a^2-(b^2+c^2)}{2bc}\right]\\ &= \frac{1}{2}bc\left[\frac{2bc+a^2-b^2-c^2}{2bc}\right]\\ &= \frac{a^2-(b^2-2bc+c^2)}{4}\\ &= \frac{a^2-(b-c)^2}{4}\\ &= \left(\frac{a-(b-c)}{2}\right)\left(\frac{a+(b-c)}{2}\right)\\ &= \frac{a-b+c}{2}\cdot\frac{a+b-c}{2}\end{aligned}$$

65. $(1, 1), (5, 8)$

(a) $\text{Slope} = \frac{8-1}{5-1} = \frac{7}{4}$

(b) $$\begin{aligned}\text{distance} &= \sqrt{(5-1)^2+(8-1)^2} = \sqrt{4^2+7^2}\\ &= \sqrt{65}\end{aligned}$$

67. $(-7, -4), (-3, -5)$

(a) $\text{Slope} = \frac{-4-(-5)}{-7-(-3)} = \frac{1}{-4} = -\frac{1}{4}$

(b) $$\begin{aligned}\text{Distance} &= \sqrt{\left[-7-(-3)\right]^2+\left[-4-(-5)\right]^2}\\ &= \sqrt{(-4)^2+1^2}\\ &= \sqrt{17}\end{aligned}$$

69. $(1, 12), (-5, 12)$

(a) $\text{Slope} = \frac{12-12}{1-(-5)} = 0$

(b) $$\begin{aligned}\text{Distance} &= \sqrt{\left[1-(-5)\right]^2+(12-12)^2}\\ &= \sqrt{6^2+0^2}\\ &= 6\end{aligned}$$

71. $(12, 10), (-12, -10)$

(a) $\text{Slope} = \frac{10-(-10)}{12-(-12)} = \frac{20}{24} = \frac{5}{6}$

(b) $$\begin{aligned}\text{Distance} &= \sqrt{\left[12-(-12)\right]^2+\left[10-(-10)\right]^2}\\ &= \sqrt{576+400}\\ &= 4\sqrt{61}\end{aligned}$$

73. (a) $(x, y), (-3, 3)$

$$r = \sqrt{9+9} = 3\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{-3} = -1$$

$$\csc\theta = \frac{r}{y} = \frac{3\sqrt{2}}{3} = \sqrt{2}$$

$$\sec\theta = \frac{r}{x} = \frac{3\sqrt{2}}{-3} = -\sqrt{2}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{3} = -1$$

(b) $(x, y) = (3, -3)$

$$r = \sqrt{9+9} = 3\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1$$

$$\csc\theta = \frac{r}{y} = \frac{3\sqrt{2}}{-3} = -\sqrt{2}$$

$$\sec\theta = \frac{r}{x} = \frac{3\sqrt{2}}{3} = \sqrt{2}$$

$$\cot\theta = \frac{x}{y} = \frac{3}{-3} = -1$$

Section 6.3 Vectors in the Plane

1. directed line segment

3. multiplication; addition

5. Two vectors are equivalent if they have the same magnitude and direction.

7. $\|\mathbf{u}\| = \sqrt{(6-2)^2 + (5-4)^2} = \sqrt{17}$

$\|\mathbf{v}\| = \sqrt{(4-0)^2 + (1-0)^2} = \sqrt{17}$

$\text{slope}_{\mathbf{u}} = \dfrac{5-4}{6-2} = \dfrac{1}{4}$

$\text{slope}_{\mathbf{v}} = \dfrac{1-0}{4-0} = \dfrac{1}{4}$

u and **v** have the same magnitude and direction so they are equivalent.

9. $\|\mathbf{u}\| = \sqrt{(-1-2)^2 + (4-2)^2} = \sqrt{13}$

$\|\mathbf{v}\| = \sqrt{(-5-(-3))^2 + (2-(-1))^2} = \sqrt{13}$

$\text{slope}_{\mathbf{u}} = \dfrac{4-2}{-1-2} = -\dfrac{2}{3}$

$\text{slope}_{\mathbf{v}} = \dfrac{2-(-1)}{-5-(-3)} = -\dfrac{3}{2}$

u and **v** have the same magnitude but not the same direction so they are not equivalent.

11. $\|\mathbf{u}\| = \sqrt{(5-2)^2 + (-10-(-1))^2} = \sqrt{90} = 3\sqrt{10}$

$\|\mathbf{v}\| = \sqrt{(9-6)^2 + (-8-1)^2} = \sqrt{90} = 3\sqrt{10}$

$\text{slope}_{\mathbf{u}} = \dfrac{-10-(-1)}{5-2} = -3$

$\text{slope}_{\mathbf{v}} = \dfrac{-8-1}{9-6} = -3$

u and **v** have the same magnitude and direction so they are equivalent.

13. Initial point: $(0, 0)$

Terminal point: $(1, 3)$

$\mathbf{v} = \langle 1 - 0, 3 - 0 \rangle = \langle 1, 3 \rangle$

$\|\mathbf{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$

15. Initial point: $(3, -2)$

Terminal point: $(3, 3)$

$\mathbf{v} = \langle 3-3, 3-(-2) \rangle = \langle 0, 5 \rangle$

$\|\mathbf{v}\| = \sqrt{0^2 + 5^2} = \sqrt{25} = 5$

17. Initial point: $(-3, -5)$

Terminal point: $(-11, 1)$

$\mathbf{v} = \langle -11-(-3), 1-(-5) \rangle = \langle -8, 6 \rangle$

$\|\mathbf{v}\| = \sqrt{(-8)^2 + 6^2} = \sqrt{100} = 10$

19. Initial point: $(1, 3)$

Terminal point: $(-8, -9)$

$\mathbf{v} = \langle -8-1, -9-3 \rangle = \langle -9, -12 \rangle$

$\|\mathbf{v}\| = \sqrt{(-9)^2 + (-12)^2} = \sqrt{225} = 15$

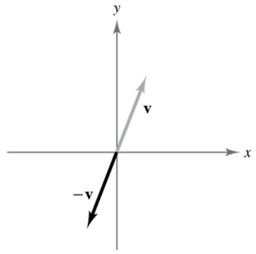
21. Initial point: $(-1, 5)$

Terminal point: $(15, -21)$

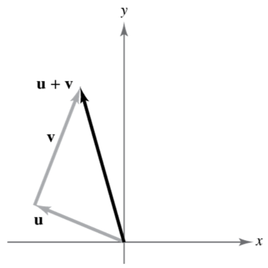
$\mathbf{v} = \langle 15-(-1), -21-5 \rangle = \langle 16, -26 \rangle$

$\|\mathbf{v}\| = \sqrt{(16)^2 + (-26)^2} = \sqrt{932} = 2\sqrt{233}$

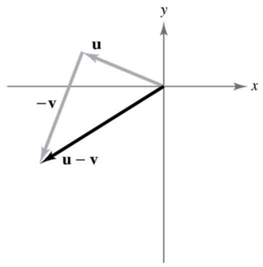
23. $-\mathbf{v}$



25. $\mathbf{u} + \mathbf{v}$

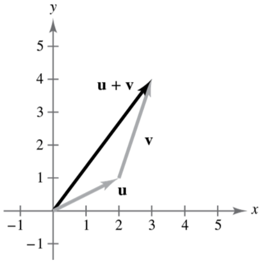


27. $\mathbf{u} - \mathbf{v}$

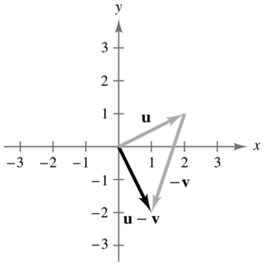


29. $\mathbf{u} = \langle 2, 1\rangle, \mathbf{v} = \langle 1, 3\rangle$

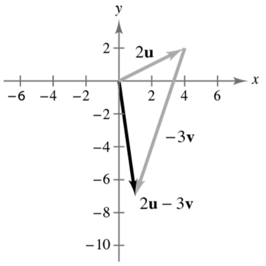
(a) $\mathbf{u} + \mathbf{v} = \langle 3, 4\rangle$



(b) $\mathbf{u} - \mathbf{v} = \langle 1, -2\rangle$

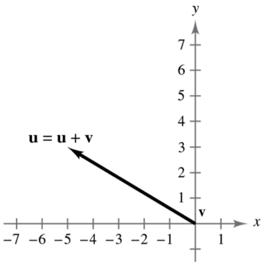


(c) $2\mathbf{u} - 3\mathbf{v} = \langle 4, 2\rangle - \langle 3, 9\rangle = \langle 1, -7\rangle$

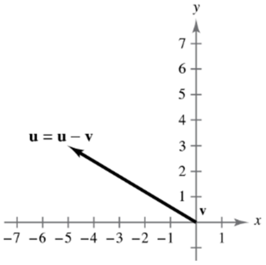


31. $\mathbf{u} = \langle -5, 3\rangle, \mathbf{v} = \langle 0, 0\rangle$

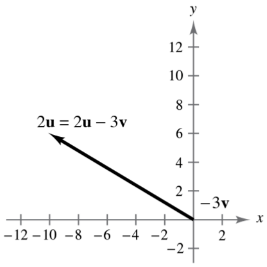
(a) $\mathbf{u} + \mathbf{v} = \langle -5, 3\rangle = \mathbf{u}$



(b) $\mathbf{u} - \mathbf{v} = \langle -5, 3\rangle = \mathbf{u}$



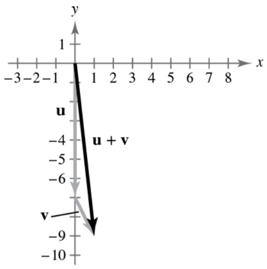
(c) $2\mathbf{u} - 3\mathbf{v} = \langle -10, 6\rangle = 2\mathbf{u}$



33. $\mathbf{u} = -7\mathbf{j}, \mathbf{v} = \mathbf{i} - 2\mathbf{j}$

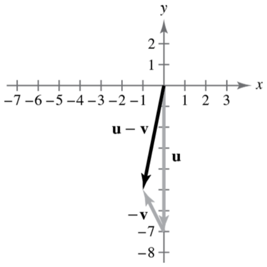
(a) $\mathbf{u} + \mathbf{v} = \mathbf{i} - 9\mathbf{j}$

$\langle 1, -9\rangle$



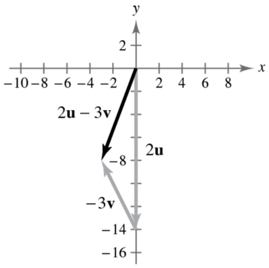
(b) $\mathbf{u} - \mathbf{v} = -\mathbf{i} - 5\mathbf{j}$

$\langle -1, -5\rangle$



(c) $2\mathbf{u} - 3\mathbf{v} = (-14\mathbf{j}) - (3\mathbf{i} - 6\mathbf{j})$
$= -3\mathbf{i} - 8\mathbf{j}$

$\langle -3, -8\rangle$



35.
$$\mathbf{u} = \langle 2, 0\rangle$$
$$5\mathbf{u} = \langle 10, 0\rangle$$
$$\|5\mathbf{u}\| = \sqrt{(10)^2 + 0^2} = 10$$

37. $\mathbf{v} = \langle -3, 6\rangle$

$-3\mathbf{v} = \langle 9, -18\rangle$

$\|-3\mathbf{v}\| = \sqrt{9^2 + (-18)^2} = \sqrt{405} = 9\sqrt{5}$

39. $\mathbf{v} = \langle 3, 0\rangle$

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{3^2 + 0^2}}\langle 3, 0\rangle = \frac{1}{3}\langle 3, 0\rangle = \langle 1, 0\rangle$$

$\|\mathbf{u}\| = \sqrt{1^2 + 0^2} = 1$

41. $\mathbf{v} = \langle -2, 2\rangle$

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(-2)^2 + 2^2}}\langle -2, 2\rangle = \frac{1}{2\sqrt{2}}\langle -2, 2\rangle$$

$$= \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$= \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$$\|\mathbf{u}\| = \sqrt{\left(\frac{-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

43. $\mathbf{v} = \langle 1, -6\rangle$

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{1^2 + (-6)^2}}\langle 1, -6\rangle = \frac{1}{\sqrt{37}}\langle 1, -6\rangle$$

$$= \frac{1}{\sqrt{37}}\langle 1, -6\rangle = \left\langle \frac{\sqrt{37}}{37}, -\frac{6\sqrt{37}}{37}\right\rangle$$

$$\|\mathbf{u}\| = \sqrt{\left(\frac{\sqrt{37}}{37}\right)^2 + \left(\frac{-6\sqrt{37}}{37}\right)^2} = 1$$

45. $$\mathbf{v} = 10\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) = 10\left(\frac{1}{\sqrt{(-3)^2 + 4^2}}\langle -3, 4\rangle\right)$$

$$= 2\langle -3, 4\rangle$$

$$= \langle -6, 8\rangle$$

47. $$9\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) = 9\left(\frac{1}{\sqrt{2^2 + 5^2}}\langle 2, 5\rangle\right) = \frac{9}{\sqrt{29}}\langle 2, 5\rangle$$

$$= \left\langle \frac{18}{\sqrt{29}}, \frac{45}{\sqrt{29}}\right\rangle = \left\langle \frac{18\sqrt{29}}{29}, \frac{45\sqrt{29}}{29}\right\rangle$$

49. $\mathbf{u} = \langle 3 - (-2), -2 - 1\rangle$

$= \langle 5, -3\rangle$

$= 5\mathbf{i} - 3\mathbf{j}$

51. $\mathbf{u} = \langle -6 - 0, 4 - 1\rangle$

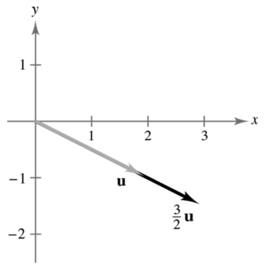
$\mathbf{u} = \langle -6, 3\rangle$

$\mathbf{u} = -6\mathbf{i} + 3\mathbf{j}$

53. $\mathbf{v} = \frac{3}{2}\mathbf{u}$

$= \frac{3}{2}(2\mathbf{i} - \mathbf{j})$

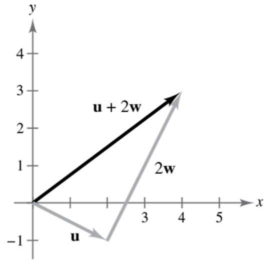
$= 3\mathbf{i} - \frac{3}{2}\mathbf{j} = \left\langle 3, -\frac{3}{2}\right\rangle$



55. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

$= (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j})$

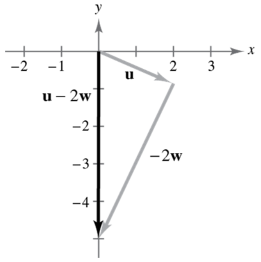
$= 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3\rangle$



57. $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

$= (2\mathbf{i} - \mathbf{j}) - 2(\mathbf{i} + 2\mathbf{j})$

$= -5\mathbf{j} = \langle 0, -5\rangle$



59. $\mathbf{v} = 6\mathbf{i} - 6\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{6^2 + (-6)^2} = \sqrt{72} = 6\sqrt{2}$

$\tan\theta = \frac{-6}{6} = -1$

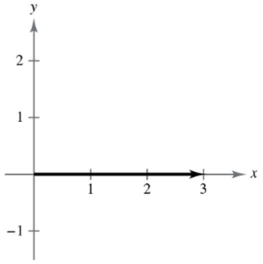
Since **v** lies in Quadrant IV, $\theta = 315°$.

61. $\mathbf{v} = 3(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

$\|\mathbf{v}\| = 3, \theta = 60°$

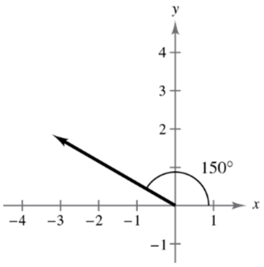
63. $\mathbf{v} = \langle 3\cos 0°, 3\sin 0°\rangle$

$= \langle 3, 0\rangle$



65. $\mathbf{v} = \left\langle \frac{7}{2}\cos 150°, \frac{7}{2}\sin 150°\right\rangle$

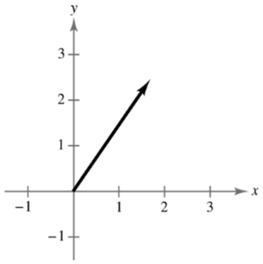
$= \left\langle -\frac{7\sqrt{3}}{4}, \frac{7}{4}\right\rangle$



67. $\mathbf{v} = 3\left(\frac{1}{\sqrt{3^2 + 4^2}}\right)(3\mathbf{i} + 4\mathbf{j})$

$= \frac{3}{5}(3\mathbf{i} + 4\mathbf{j})$

$= \frac{9}{5}\mathbf{i} + \frac{12}{5}\mathbf{j} = \left\langle \frac{9}{5}, \frac{12}{5}\right\rangle$



69. $\mathbf{u} = \langle 4\cos 60°, 4\sin 60°\rangle = \langle 2, 2\sqrt{3}\rangle$

$\mathbf{v} = \langle 4\cos 90°, 4\sin 90°\rangle = \langle 0, 4\rangle$

$\mathbf{u} + \mathbf{v} = \langle 2, 4 + 2\sqrt{3}\rangle$

71. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

$\mathbf{w} = 2\mathbf{i} - 2\mathbf{j}$

$\mathbf{u} = \mathbf{v} - \mathbf{w} = -\mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{2}$

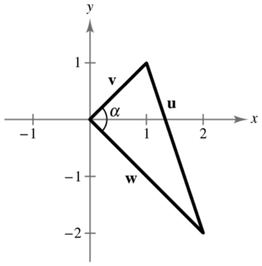
$\|\mathbf{w}\| = 2\sqrt{2}$

$\|\mathbf{v} - \mathbf{w}\| = \sqrt{10}$

$$\cos\alpha = \frac{\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2}{2\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

$$= \frac{2 + 8 - 10}{2\sqrt{2}\cdot 2\sqrt{2}} = 0$$

$\alpha = 90°$



73. Force One: $\mathbf{u} = 45\mathbf{i}$

Force Two: $\mathbf{v} = 60\cos\theta\mathbf{i} + 60\sin\theta\mathbf{j}$

Resultant Force: $\mathbf{u} + \mathbf{v} = (45 + 60\cos\theta)\mathbf{i} + 60\sin\theta\mathbf{j}$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(45 + 60\cos\theta)^2 + (60\sin\theta)^2} = 90$$

$$2025 + 5400\cos\theta + 3600 = 8100$$

$$5400\cos\theta = 2475$$

$$\cos\theta = \tfrac{2475}{5400} \approx 0.4583$$

$$\theta \approx 62.7°$$

75. Horizontal component of velocity: $1200 \cos 6^\circ \approx 1193.4$ ft/sec

Vertical component of velocity: $1200 \sin 6^\circ \approx 125.4$ ft/sec

77.

$$\begin{aligned} \mathbf{u} &= 300\mathbf{i} \\ \mathbf{v} &= (125 \cos 45^\circ)\mathbf{i} + (125 \sin 45^\circ)\mathbf{j} = \frac{125}{\sqrt{2}}\mathbf{i} + \frac{125}{\sqrt{2}}\mathbf{j} \\ \mathbf{u} + \mathbf{v} &= \left(300 + \frac{125}{\sqrt{2}}\right)\mathbf{i} + \frac{125}{\sqrt{2}}\mathbf{j} \\ \|\mathbf{u} + \mathbf{v}\| &= \sqrt{\left(300 + \frac{125}{\sqrt{2}}\right)^2 + \left(\frac{125}{\sqrt{2}}\right)^2} \approx 398.32 \text{ newtons} \\ \tan\theta &= \frac{\frac{125}{\sqrt{2}}}{300 + \left(\frac{125}{\sqrt{2}}\right)} \Rightarrow \theta \approx 12.8^\circ \end{aligned}$$

79.

$$\begin{aligned} \mathbf{u} &= (75 \cos 30^\circ)\mathbf{i} + (75 \sin 30^\circ)\mathbf{j} \approx 64.95\mathbf{i} + 37.5\mathbf{j} \\ \mathbf{v} &= (100 \cos 45^\circ)\mathbf{i} + (100 \sin 45^\circ)\mathbf{j} \approx 70.71\mathbf{i} + 70.71\mathbf{j} \\ \mathbf{w} &= (125 \cos 120^\circ)\mathbf{i} + (125 \sin 120^\circ)\mathbf{j} \approx -62.5\mathbf{i} + 108.3\mathbf{j} \\ \mathbf{u} + \mathbf{v} + \mathbf{w} &\approx 73.16\mathbf{i} + 216.5\mathbf{j} \\ \|\mathbf{u} + \mathbf{v} + \mathbf{w}\| &\approx 228.5 \text{ pounds} \\ \tan\theta &\approx \frac{216.5}{73.16} \approx 2.9593 \\ \theta &\approx 71.3^\circ \end{aligned}$$

81. Left crane: $\mathbf{u} = \|\mathbf{u}\|(\cos 155.7^\circ\mathbf{i} + \sin 155.7^\circ\mathbf{j})$

Right crane: $\mathbf{v} = \|\mathbf{v}\|(\cos 44.5^\circ\mathbf{i} + \sin 44.5^\circ\mathbf{j})$

Resultant: $\mathbf{u} + \mathbf{v} = -20{,}240\mathbf{j}$

System of equations:

$\|\mathbf{u}\| \cos 155.7^\circ + \|\mathbf{v}\| \cos 44.5^\circ = 0$

$\|\mathbf{u}\| \sin 155.7^\circ + \|\mathbf{v}\| \sin 44.5^\circ = 20{,}240$

Solving this system of equations yields the following:

Left crane $= \|\mathbf{u}\| \approx 15{,}484$ pounds

Right crane $= \|\mathbf{v}\| \approx 19{,}786$ pounds

83. Horizontal force: $\mathbf{u} = \|\mathbf{u}\|\mathbf{i}$

Weight: $\mathbf{w} = -\mathbf{j}$

Rope: $\mathbf{t} = \|\mathbf{t}\|(\cos 135^\circ\mathbf{i} + \sin 135^\circ\mathbf{j})$

$$\begin{aligned} \mathbf{u} + \mathbf{w} + \mathbf{t} = \mathbf{0} \Rightarrow \|\mathbf{u}\| + \|\mathbf{t}\| \cos 135^\circ &= 0 \\ -1 + \|\mathbf{t}\| \sin 135^\circ &= 0 \end{aligned}$$

$\|\mathbf{t}\| \approx \sqrt{2}$ pounds

$\|\mathbf{u}\| \approx 1$ pound

85. $W = 100, \theta = 12^\circ$

$$\sin\theta = \frac{F}{W}$$

$$F = W \sin\theta = 100 \sin 12^\circ \approx 20.8 \text{ pounds}$$

87. $F = 5000, W = 15{,}000$

$$\begin{aligned} \sin\theta &= \frac{F}{W} \\ \theta &= \frac{5000}{15{,}000} \\ \theta &= \sin^{-1}\frac{1}{3} \approx 19.5^\circ \end{aligned}$$

89. Airspeed: $\mathbf{u} = (875\cos 58°)\mathbf{i} - (875\sin 58°)\mathbf{j}$

Groundspeed: $\mathbf{v} = (800\cos 50°)\mathbf{i} - (800\sin 50°)\mathbf{j}$

Wind: $\mathbf{w} = \mathbf{v} - \mathbf{u} = (800\cos 50° - 875\cos 58°)\mathbf{i} + (-800\sin 50° + 875\sin 58°)\mathbf{j}$

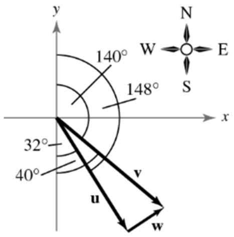
$\approx 50.5507\mathbf{i} + 129.2065\mathbf{j}$

Wind speed: $\|\mathbf{w}\| \approx \sqrt{(50.5507)^2 + (129.2065)^2} \approx 138.7$ kilometers per hour

Wind direction: $\tan\theta \approx \dfrac{129.2065}{50.5507}$

$\theta \approx 68.6°;\ 90° - \theta = 21.4°$

Bearing: N 21.4° E



91. True. Two directed line segments that have the same magnitude and direction are equivalent (see Example 1).

93. True. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = 0$ is the zero vector, then $a = b = 0$. So, $a = -b$.

95. The order of subtraction should be switched.

$\mathbf{u} = \langle 6 - (-3), -1 - 4\rangle = \langle 9, -5\rangle$

97. Let $\mathbf{v} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.

$\|\mathbf{v}\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

So, $\mathbf{v}$ is a unit vector for any value of θ.

99.

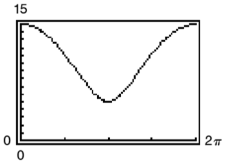
$$\begin{aligned}\mathbf{u} &= \langle 5 - 1, 2 - 6\rangle = \langle 4, -4\rangle\\ \mathbf{v} &= \langle 9 - 4, 4 - 5\rangle = \langle 5, -1\rangle\\ \mathbf{u} - \mathbf{v} &= \langle -1, -3\rangle \text{ or } \mathbf{v} - \mathbf{u} = \langle 1, 3\rangle\end{aligned}$$

101. $\mathbf{F}_1 = \langle 10, 0\rangle,\ \mathbf{F}_2 = 5\langle\cos\theta, \sin\theta\rangle$

(a) $\mathbf{F}_1 + \mathbf{F}_2 = \langle 10 + 5\cos\theta, 5\sin\theta\rangle$

$$\begin{aligned}\|\mathbf{F}_1 + \mathbf{F}_2\| &= \sqrt{(10 + 5\cos\theta)^2 + (5\sin\theta)^2}\\ &= \sqrt{100 + 100\cos\theta + 25\cos^2\theta + 25\sin^2\theta}\\ &= 5\sqrt{4 + 4\cos\theta + \cos^2\theta + \sin^2\theta}\\ &= 5\sqrt{4 + 4\cos\theta + 1}\\ &= 5\sqrt{5 + 4\cos\theta}\end{aligned}$$

(b)



(c) Range: $[5, 15]$

Maximum is 15 when $\theta = 0$.

Minimum is 5 when $\theta = \pi$.

(d) The magnitude of the resultant is never 0 because the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ are not the same.

103. $(2, 8), (8, 2)$

$$\begin{aligned}\text{Distance} &= \sqrt{(2 - 8)^2 + (8 - 2)^2}\\ &= \sqrt{36 + 36} = 6\sqrt{2} \approx 8.49\end{aligned}$$

105. $(-15, 16), (-14, 19)$

$$\begin{aligned}\text{Distance} &= \sqrt{[-14 - (-15)]^2 + (19 - 16)^2}\\ &= \sqrt{1 + 9} = \sqrt{10} \approx 3.16\end{aligned}$$

107. $(5, -20), (11, -19)$

$$\begin{aligned}\text{Distance} &= \sqrt{(11 - 5)^2 + [-19 - (-20)]^2}\\ &= \sqrt{36 + 1}\\ &= \sqrt{37} \approx 6.08\end{aligned}$$

109. $(-3, 0), (13, 6)$

$$\begin{aligned}\text{Distance} &= \sqrt{[13 - (-3)]^2 + (6 - 0)^2}\\ &= \sqrt{16^2 + 6^2}\\ &= \sqrt{292} \approx 17.09\end{aligned}$$

111. $\cos x = \dfrac{-3(3) + 1(9)}{\sqrt{10(90)}}$

$$\cos x = \frac{-9 + 9}{\sqrt{900}}$$

$$\cos x = 0$$

$$x = \frac{\pi}{2} + n\pi$$

113. $\cos \dfrac{\pi}{4} = \dfrac{4x + 21}{\sqrt{1250}}$

$$\frac{\sqrt{2}}{2} = \frac{4x + 21}{25\sqrt{2}}$$

$$25 = 4x + 21$$

$$4 = 4x$$

$$1 = x$$

115. Given: $a = 9, b = 8, c = 16$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9^2 + 8^2 - 16^2}{2(9)(8)} = -0.7708\overline{3}$$

$$\Rightarrow C \approx 140.43°$$

$$\sin B = \sin C\left(\frac{b}{c}\right) = \sin\left(140.43\right)\left(\frac{8}{16}\right) \approx 0.3185$$

$$\Rightarrow B \approx 18.57°$$

$$A \approx 180° - B - C = 21.00°$$

117. Given: $B = 59°, C = 84°, c = 20$

$$A = 180° - B - C = 180° - 59° - 84° = 37°$$

$$a = \sin A\left(\frac{c}{\sin C}\right) = \sin 37°\frac{20}{\sin 84°} \approx 12.10$$

$$b = \sin B\left(\frac{c}{\sin C}\right) = \sin 59°\frac{20}{\sin 84°} \approx 17.24$$

Section 6.4 Vectors and Dot Products

1. *Sample answer:*

$$\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

3. $\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

5. $\mathbf{u} = \langle 7, 1\rangle, \mathbf{v} = \langle -3, 2\rangle$

$$\mathbf{u} \cdot \mathbf{v} = 7(-3) + 1(2) = -19$$

7. $\mathbf{u} = \langle -6, 2\rangle, \mathbf{v} = \langle 1, 3\rangle$

$$\mathbf{u} \cdot \mathbf{v} = -6(1) + 2(3) = 0$$

9. $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j}, \mathbf{v} = \mathbf{i} - \mathbf{j}$

$$\mathbf{u} \cdot \mathbf{v} = 4(1) + (-2)(-1) = 6$$

11. $\mathbf{u} = \langle 3, 3\rangle$

$$\mathbf{u} \cdot \mathbf{u} = 3(3) + 3(3) = 18$$

The result is a scalar.

13. $\mathbf{u} = \langle 3, 3\rangle, \mathbf{v} = \langle -4, 2\rangle$

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \left[3(-4) + 3(2)\right]\langle -4, 2\rangle$$

$$= -6\langle -4, 2\rangle$$

$$= \langle 24, -12\rangle$$

The result is a vector.

15. $\mathbf{u} = \langle 3, 3\rangle, \mathbf{v} = \langle -4, 2\rangle, \mathbf{w} = \langle 3, -1\rangle$

$$(\mathbf{v} \cdot \mathbf{0})\mathbf{w} = 0\langle 3, -1\rangle = \langle 0, 0\rangle = \mathbf{0}$$

The result is a vector.

17. $\mathbf{v} = \langle -4, 2\rangle$

$$3\|\mathbf{v}\|^2 = 3\left(\sqrt{(-4)^2 + (2)^2}\right)^2$$

$$= 3(20)$$

$$= 60$$

The result is a scalar.

19. $\mathbf{w} = \langle 3, -1\rangle$

$$\|\mathbf{w}\| - 1 = \sqrt{3^2 + (-1)^2} - 1 = \sqrt{10} - 1$$

The result is a scalar.

21. $\mathbf{u} = \langle 3, 3\rangle, \mathbf{v} = \langle -4, 2\rangle, \mathbf{w} = \langle 3, -1\rangle$

$$\begin{aligned}(\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{w}) &= \left[3(-4) + 3(2)\right] - \left[3(3) + 3(-1)\right]\\ &= -6 - 6\\ &= -12\end{aligned}$$

The result is a scalar.

23. $\mathbf{u} = \langle -8, 15\rangle$

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(-8)(-8) + 15(15)} = \sqrt{289} = 17$$

25. $\mathbf{u} = 20\mathbf{i} + 25\mathbf{j}$

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(20)^2 + (25)^2} = \sqrt{1025} = 5\sqrt{41}$$

27. $\mathbf{u} = 6\mathbf{j}$

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(0)^2 + (6)^2} = \sqrt{36} = 6$$

29. $\mathbf{u} = \langle 1, 0\rangle, \mathbf{v} = \langle 0, -2\rangle$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{(1)(2)} = 0$$

$$\theta = \frac{\pi}{2}\text{ radians}$$

31. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = -\frac{8}{(5)(2)}$$

$$\theta = \arccos\left(-\frac{4}{5}\right)$$

$$\theta \approx 2.50\text{ radians}$$

33. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}, \mathbf{v} = 6\mathbf{i} - 3\mathbf{j}$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\\ &= \frac{2(6) + (-1)(-3)}{\sqrt{2^2 + (-1)^2}\sqrt{6^2 + (-3)^2}}\\ &= \frac{15}{\sqrt{225}} = 1\end{aligned}$$

$$\theta = 0$$

35. $\mathbf{u} = -6\mathbf{i} - 3\mathbf{j}, \mathbf{v} = -8\mathbf{i} + 4\mathbf{j}$

$$\cos\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-6(-8) + (-3)(4)}{\sqrt{45}\sqrt{80}} = \frac{36}{60} = 0.6$$

$$\theta \approx 0.93\text{ radian}$$

37. $\mathbf{u} = \left(\cos\dfrac{\pi}{3}\right)\mathbf{i} + \left(\sin\dfrac{\pi}{3}\right)\mathbf{j} = \dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}$

$$\mathbf{v} = \left(\cos\frac{3\pi}{4}\right)\mathbf{i} + \left(\sin\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1$$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \mathbf{u}\cdot\mathbf{v}\\ &= \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

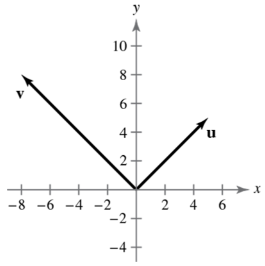
$$\theta = \arccos\left(\frac{-\sqrt{2} + \sqrt{6}}{4}\right) = \frac{5\pi}{12}$$

39. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$

$\mathbf{v} = -7\mathbf{i} + 5\mathbf{j}$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\\ &= \frac{3(-7) + 4(5)}{3\sqrt{74}}\\ &= \frac{-1}{5\sqrt{74}} \approx -0.0232\end{aligned}$$

$$\theta \approx 91.33°$$



41. $\mathbf{u} = -5\mathbf{i} - 5\mathbf{j}$

$\mathbf{v} = -8\mathbf{i} + 8\mathbf{j}$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\\ &= \frac{-5(-8) + (-5)(8)}{\sqrt{50}\sqrt{128}}\\ &= 0\end{aligned}$$

$$\theta = 90°$$

43. $P = (1, 2), Q = (3, 4), R = (2, 5)$

$\overline{PQ} = \langle 2, 2\rangle, \overline{PR} = \langle 1, 3\rangle, \overline{QR} = \langle -1, 1\rangle$

$$\cos\alpha = \frac{\overline{PQ}\cdot\overline{PR}}{\|\overline{PQ}\|\|\overline{PR}\|} = \frac{8}{\left(2\sqrt{2}\right)\left(\sqrt{10}\right)} \Rightarrow \alpha$$

$$= \arccos\frac{2}{\sqrt{5}} \approx 26.57^\circ$$

$$\cos\beta = \frac{\overline{PQ}\cdot\overline{QR}}{\|\overline{PQ}\|\|\overline{QR}\|} = 0 \Rightarrow \beta = 90^\circ$$

$$\gamma = 180^\circ - 26.57^\circ - 90^\circ = 63.43^\circ$$

45. $P = (-3, 0), Q = (2, 2), R = (0, 6)$

$\overline{QP} = \langle -5, -2\rangle, \overline{PR} = \langle 3, 6\rangle, \overline{QR} = \langle -2, 4\rangle,$

$\overline{PQ} = \langle 5, 2\rangle$

$$\cos\alpha = \frac{\overline{PQ}\cdot\overline{PR}}{\|\overline{PQ}\|\|\overline{PR}\|} = \frac{27}{\sqrt{29}\sqrt{45}} \Rightarrow \alpha \approx 41.63^\circ$$

$$\cos\beta = \frac{\overline{QP}\cdot\overline{QR}}{\|\overline{QP}\|\|\overline{PR}\|} = \frac{2}{\sqrt{29}\sqrt{20}} \Rightarrow \beta \approx 85.24^\circ$$

$$\delta = 180^\circ - 41.63^\circ - 85.24^\circ = 53.13^\circ$$

47. $\|\mathbf{u}\| = 4, \|\mathbf{v}\| = 10, \theta = \dfrac{2\pi}{3}$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ &= (4)(10)\cos\frac{2\pi}{3}\\ &= 40\left(-\frac{1}{2}\right)\\ &= -20\end{aligned}$$

49. $\|\mathbf{u}\| = 100, \|\mathbf{v}\| = 250, \theta = \dfrac{\pi}{6}$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ &= (100)(250)\cos\frac{\pi}{6}\\ &= 25{,}000\cdot\frac{\sqrt{3}}{2}\\ &= 12{,}500\sqrt{3}\end{aligned}$$

51. $\mathbf{u} = \langle 3, 15\rangle, \mathbf{v} = \langle -1, 5\rangle$

$\mathbf{u} \neq k\mathbf{v} \Rightarrow$ Not parallel

$\mathbf{u}\cdot\mathbf{v} \neq 0 \Rightarrow$ Not orthogonal

Neither

53. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}, \mathbf{v} = -\mathbf{i} - \mathbf{j}$

$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ **u** and **v** are orthogonal.

55. $\mathbf{u} = 1, \mathbf{v} = -2\mathbf{i} + 2\mathbf{j}$

$\mathbf{u} \neq k\mathbf{v} \Rightarrow$ Not parallel

$\mathbf{u}\cdot\mathbf{v} \neq 0 \Rightarrow$ Not orthogonal

Neither

57. $\mathbf{u} = \langle 2, 2\rangle, \mathbf{v} = \langle 6, 1\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \frac{14}{37}\langle 6, 1\rangle = \frac{1}{37}\langle 84, 14\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 2\rangle - \frac{14}{37}\langle 6, 1\rangle = \left\langle -\frac{10}{37}, \frac{60}{37}\right\rangle$$

$$= \frac{10}{37}\langle -1, 6\rangle = \frac{1}{37}\langle -10, 60\rangle$$

$$\mathbf{u} = \frac{1}{37}\langle 84, 14\rangle + \frac{1}{37}\langle -10, 60\rangle = \langle 2, 2\rangle$$

59. $\mathbf{u} = \langle 4, 2\rangle, \mathbf{v} = \langle 1, -2\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = 0\langle 1, -2\rangle = \langle 0, 0\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 4, 2\rangle - \langle 0, 0\rangle = \langle 4, 2\rangle$$

$$\mathbf{u} = \langle 4, 2\rangle + \langle 0, 0\rangle = \langle 4, 2\rangle$$

61. $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$ because **u** and **v** are parallel.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{3(6) + 2(4)}{\left(\sqrt{6^2 + 4^2}\right)^2}\langle 6, 4\rangle = \frac{1}{2}\langle 6, 4\rangle$$

$$= \langle 3, 2\rangle = \mathbf{u}$$

63. Because **u** and **v** are orthogonal, $\mathbf{u}\cdot\mathbf{v} = 0$ and $\text{proj}_{\mathbf{v}}\mathbf{u} = 0$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = 0, \text{ because } \mathbf{u}\cdot\mathbf{v} = 0.$$

65. $\mathbf{u} = \langle 3, 5\rangle$

For **v** to be orthogonal to **u**, $\mathbf{u}\cdot\mathbf{v}$ must equal 0.

Two possibilities: $\langle -5, 3\rangle$ and $\langle 5, -3\rangle$

67. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$

For **u** and **v** to be orthogonal, $\mathbf{u}\cdot\mathbf{v}$ must equal 0.

Two possibilities: $\mathbf{v} = \frac{2}{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$ and $\mathbf{v} = -\frac{2}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

69. Work $= \left\|\text{proj}_{\overline{PQ}}\mathbf{v}\right\| \left\|\overline{PQ}\right\|$ where $\overline{PQ} = \langle 4, 7\rangle$ and $\mathbf{v} = \langle 1, 4\rangle$.

$$\text{proj}_{\overline{PQ}}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\overline{PQ}}{\left\|\overline{PQ}\right\|^2}\right)\overline{PQ} = \left(\frac{32}{65}\right)\langle 4, 7\rangle$$

$$\text{Work} = \left\|\text{proj}_{\overline{PQ}}\mathbf{v}\right\| \left\|\overline{PQ}\right\| = \left(\frac{32\sqrt{65}}{65}\right)\left(\sqrt{65}\right) = 32$$

71. (a) $\mathbf{u}\cdot\mathbf{v} = 1225(12.20) + 2445(8.50)$
$= 35{,}727.5$

The total amount paid to the employees is \$35,727.50.

(b) To increase wages by 2%, use scalar multiplication to multiply 1.02 by **v**.

73. (a) Force due to gravity:

$\mathbf{F} = -30{,}000\mathbf{j}$

Unit vector along hill:

$\mathbf{v} = (\cos d)\mathbf{i} + (\sin d)\mathbf{j}$

Projection of **F** onto **v**:

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = -30{,}000 \sin d\mathbf{v}$$

The magnitude of the force is $30{,}000 \sin d$.

(b)

d	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
Force	0	523.6	1047.0	1570.1	2092.7	2614.7	3135.9	3656.1	4175.2	4693.0	5209.4

(c) Force perpendicular to the hill when $d = 5°$:

$$\text{Force} = \sqrt{(30{,}000)^2 - (2614.7)^2} \approx 29{,}885.8 \text{ pounds}$$

75. Work $= (245)(3) = 735$ newton-meters

77. Work $= (\cos 30°)(45)(20) \approx 779.4$ foot-pounds

79. Work $= (\cos\theta)\|\mathbf{F}\|\left\|\overline{PQ}\right\|$
$= (\cos 20°)(25 \text{ pounds})(50 \text{ feet})$
≈ 1174.62 foot-pounds

81. False. Work is represented by a scalar.

83. $\mathbf{u}\cdot\mathbf{v} = \langle 8, 4\rangle\cdot\langle 2, -k\rangle = 16 - 4k = 0$

$16 - 4k = 0$
$-4k = -16$
$k = 4$

85. The dot product is a scalar, not a vector.

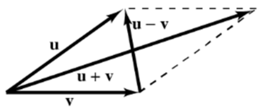
$\langle 5, 8\rangle\cdot\langle -2, 7\rangle = (5)(-2) + (8)(7)$
$= -10 + 56$
$= 46$

87. $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$
$= 1^2 = 1$

89. In a rhombus, $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0\end{aligned}$$

So, the diagonals are orthogonal.



91. (0, 9), (3, 13)

(a) Distance $= \sqrt{(3-0)^2 + (13-9)^2}$
$= \sqrt{9 + 16}$
$= \sqrt{25} = 5$

(b) Midpoint $= \left(\dfrac{0+3}{2}, \dfrac{9+13}{2}\right) = \left(\dfrac{3}{2}, 11\right)$

93. $(-2.5, 3.5), (2.5, -3.5)$

(a) $\text{Distance} = \sqrt{\left[2.5 - (-2.5)\right]^2 + \left[3.5 - (-3.5)\right]^2}$
$= \sqrt{5^2 + 7^2}$
$= \sqrt{74}$

(b) $\text{Midpoint} = \left(\dfrac{2.5 - 2.5}{2}, \dfrac{-3.5 + 3.5}{2}\right) = (0, 0)$

95. $(6 + 11i) + (-18 + 3i) = -12 + 14i$

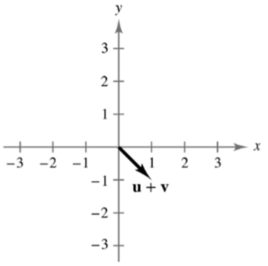
97. $(4 - 9i) - (10 - 5i) = -6 + 4i$

99. The complex conjugate of $1 + 3i$ is $1 - 3i$.

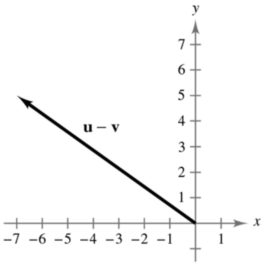
101. The complex conjugate of 4 is 4.

103. $\mathbf{u} = \langle -3, 2\rangle, \mathbf{v} = \langle 4, -3\rangle$

(a) $\mathbf{u} + \mathbf{v} = \langle -3, 2\rangle + \langle 4, -3\rangle = \langle 1, -1\rangle$



(b) $\mathbf{u} - \mathbf{v} = \langle -3, 2\rangle - \langle 4, -3\rangle = \langle -7, 5\rangle$



Section 6.5 The Complex Plane

1. real; imaginary

3. To find the complex conjugate of the complex number $a + bi$, reverse the sign of the imaginary part $b : a - bi$.

5. $2 = 2 + 0i$ matches (b)

6. $3i = 0 + 3i$ matches (d)

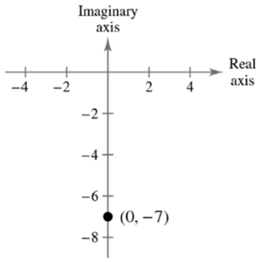
7. $1 + 2i$ matches (f)

8. $3 - i$ matches (a)

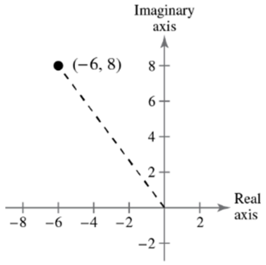
9. $-3 + i$ matches (e)

10. $-2 - i$ matches (c)

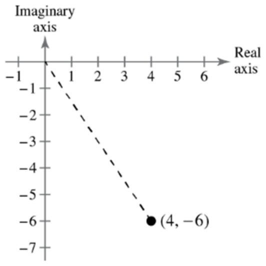
11. $|-7i| = \sqrt{0^2 + (-7)^2}$
$= \sqrt{49} = 7$



13. $|-6 + 8i| = \sqrt{(-6)^2 + 8^2}$
$= \sqrt{100} = 10$



15. $|4 - 6i| = \sqrt{4^2 + (-6)^2}$
$= \sqrt{52} = 2\sqrt{13}$



17. $(3 + i) + (2 + 5i) = 5 + 6i$

19. $(8 - 2i) + (2 + 6i) = 10 + 4i$

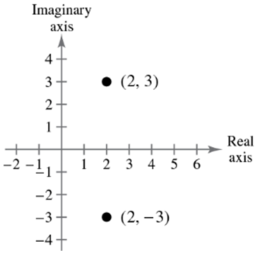
21. $(-1 + 3i) + (2 + 4i) = 1 + 7i$

23. $(4 + 2i) - (6 + 4i) = -2 - 2i$

25. $(5 - i) - (-5 + 2i) = 10 - 3i$

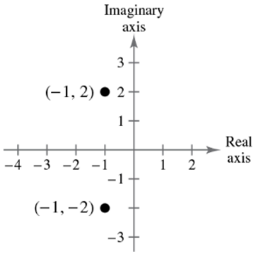
27. $2 - (2 + 6i) = -6i$

29.



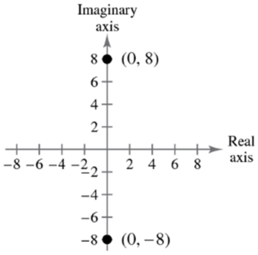
The complex conjugate of $2 + 3i$ is $2 - 3i$

31.



The complex conjugate of $-1 - 2i$ is $-1 + 2i$.

33.



The complex conjugate of $8i$ is $-8i$.

35. $d = \sqrt{(-1 - 1)^2 + (4 - 2)^2}$

$= \sqrt{8} = 2\sqrt{2} \approx 2.83$

37. $d = \sqrt{(3 - 0)^2 + (-4 - 6)^2}$

$= \sqrt{109} \approx 10.44$

39. Midpoint $= \left(\dfrac{2 + 6}{2}, \dfrac{1 + 5}{2}\right)$

$= 4 + 3i = (4, 3)$

41. Midpoint $= \left(\dfrac{0 + 9}{2}, \dfrac{7 - 10}{2}\right)$

$= \dfrac{9}{2} - \dfrac{3}{2}i = \left(\dfrac{9}{2}, -\dfrac{3}{2}\right)$

43. (a) Ship A: $3 + 4i$

Ship B: $-5 + 2i$

(b) To find the distance between the two ships using complex numbers, you can find the modulus of the difference of the two complex numbers.

$d = \sqrt{(-5 - 3)^2 + (2 - 4)^2}$

$= \sqrt{68}$

$\approx$ 8.25 miles.

45. False. The modulus of a complex number is always real.

47. False. The modulus of the sum of two complex numbers is not equal to the sum of their moduli.

$|1 + i| + |1 - i| = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$\neq |(1 + i) + (1 - i)| = |2| = 2$

49. The set of all points with the same modulus represent a circle in the complex plane. The modulus represents the distance from the origin, that is the radius of the circle.

51. $f(x) = x^3 + 2x^2 + x + 2$

Because i is a zero, so is i.

$f(x) = x^2(x + 2) + x + 2$

$= (x + 2)(x^2 + 1)$

zeros: $x = -2, \pm i$

53. $f(x) = x^3 - 9x^2 + 28x - 40$

Because $2 - 2i$ is a zero, so is $2 + 2i$, and

$$\begin{aligned}\left[x - (2 - 2i)\right]\left[x - (2 + 2i)\right] &= \left[(x - 2) + 2i\right]\left[(x - 2) - 2i\right]\\ &= (x - 2)^2 - (2i)^2\\ &= x^2 - 4x + 4 - 4i^2\\ &= x^2 - 4x + 8\end{aligned}$$

is a factor of $f(x)$. By long division you have:

$$\begin{array}{r} x - 5 \\ x^2 - 4x + 8\overline{\smash{\big)}\,x^3 - 9x^2 + 28x - 40} \\ \underline{x^3 - 4x^2 + 8x\phantom{{}- 40}} \\ -5x^2 + 20x - 40 \\ \underline{-5x^2 + 20x - 40} \\ 0 \end{array}$$

$f(x) = (x^2 - 4x + 8)(x - 5)$

Zeros: $x = 2 \pm 2i, 5$

55. $\tan\theta < 0$ and $\cos\theta = -\frac{3}{5} \Rightarrow \theta$ is in Quadrant II $\Rightarrow x < 0$ and $y > 0$.

$\cos\theta = \frac{x}{r} = \frac{-3}{5} \Rightarrow y^2 = r^2 - x^2 = 25 - 9 = 16 \Rightarrow y = 4$

$\sin\theta = \frac{y}{r} = \frac{4}{5}$

$\tan\theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$

$\csc\theta = \frac{r}{y} = \frac{5}{4}$

$\sec\theta = \frac{r}{x} = \frac{5}{-3} = -\frac{5}{3}$

$\cot\theta = \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4}$

57. $\tan\theta = 5$ and $\sec\theta < 0 \Rightarrow \theta$ is in Quadrant III $\Rightarrow x < 0, y < 0$

$\tan\theta = \frac{y}{x} = \frac{-5}{-1} \Rightarrow x^2 + y^2 = r^2 = 1 + 25 = 26 \Rightarrow r = \sqrt{26}$

$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{26}} = -\frac{5\sqrt{26}}{26}$

$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{26}} = -\frac{\sqrt{26}}{26}$

$\csc\theta = \frac{r}{y} = -\frac{\sqrt{26}}{5}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{26}}{-1} = -\sqrt{26}$

$\cot\theta = \frac{x}{y} = \frac{-1}{-5} = \frac{1}{5}$

59. $1 + 2\sin\theta = 0$

$$\sin\theta = \frac{-1}{2}$$

$$\theta = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$$

61. $2\cot^2\theta - 1 = 5$

$$\cot^2\theta = 3$$

$$\cot\theta = \pm\sqrt{3}$$

$$\theta = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$$

Section 6.6 Trigonometric Form of a Complex Number

1. DeMoivre's

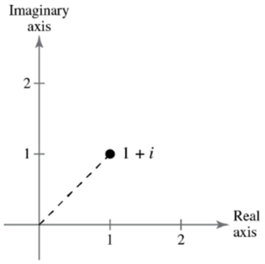
3. In the equation $z = v(\cos\theta + i\sin\theta)$, r is the modulus and θ is an argument of z.

5. $z = 1 + i$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\tan\theta = 1$, θ is in Quadrant I $\Rightarrow \theta = \frac{\pi}{4}$.

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

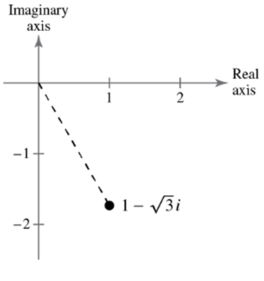


7. $z = 1 - \sqrt{3}i$

$$r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$\tan\theta = -\sqrt{3}$, θ is in Quadrant IV $\Rightarrow \theta = \frac{5\pi}{3}$.

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

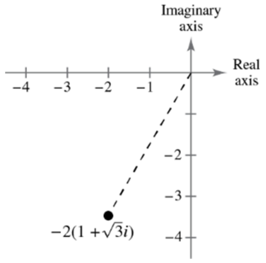


9. $z = -2\left(1 + \sqrt{3}i\right)$

$$r = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$, θ is in Quadrant III $\Rightarrow \theta = \frac{4\pi}{3}$.

$$z = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

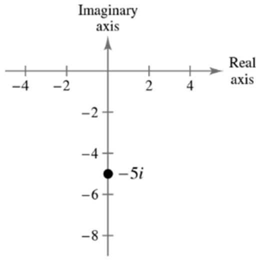


11. $z = -5i$

$$r = \sqrt{0^2 + (-5)^2} = \sqrt{25} = 5$$

$\tan\theta = \frac{-5}{0}$, undefined $\Rightarrow \theta = \frac{3\pi}{2}$

$$z = 5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

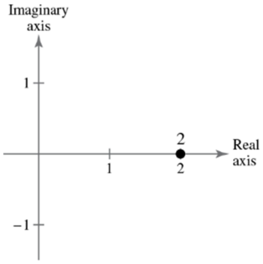


13. $z = 2$

$r = \sqrt{2^2 + 0^2} = \sqrt{4} = 2$

$\tan\theta = 0 \Rightarrow \theta = 0$

$z = 2(\cos 0 + i\sin 0)$

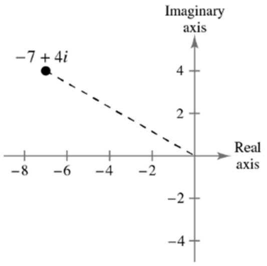


15. $z = -7 + 4i$

$r = \sqrt{(-7)^2 + (4)^2} = \sqrt{65}$

$\tan\theta = \dfrac{4}{-7}$, θ is in Quadrant II $\Rightarrow \theta \approx 2.62$.

$z \approx \sqrt{65}(\cos 2.62 + i\sin 2.62)$

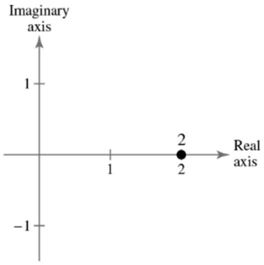


17. $z = 3 + \sqrt{3}i$

$r = \sqrt{(3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$\tan\theta = \dfrac{\sqrt{3}}{3} \Rightarrow \theta = \dfrac{\pi}{6}$

$z = 2\sqrt{3}\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$



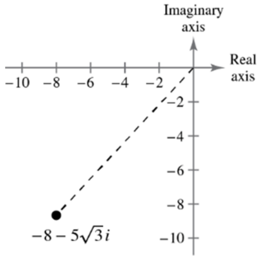
19. $z = -8 - 5\sqrt{3}i$

$r = \sqrt{(-8)^2 + (-5\sqrt{3})^2} = \sqrt{139}$

$\tan\theta = \dfrac{5\sqrt{3}}{8}$

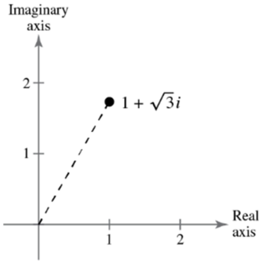
$\theta \approx 3.97$

$z \approx \sqrt{139}(\cos 3.97 + i\sin 3.97)$



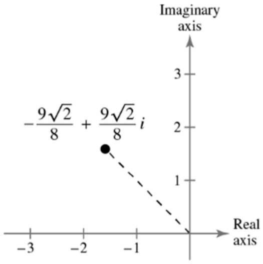
21. $2(\cos 60° + i\sin 60°) = 2\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

$= 1 + \sqrt{3}i$

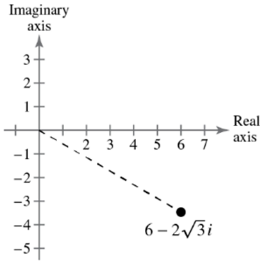


23. $\dfrac{9}{4}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right) = \dfrac{9}{4}\left(-\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)$

$= -\dfrac{9\sqrt{2}}{8} + \dfrac{9\sqrt{2}}{8}i$



25. $\sqrt{48}\left[\cos(-30^\circ) + i\sin(-30^\circ)\right] = 4\sqrt{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 6 - 2\sqrt{3}i$



27. $5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right) \approx 4.6985 + 1.7101i$

29. $2(\cos 155^\circ + i\sin 155^\circ) \approx -1.8126 + 0.8452i$

31. $$\left[2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\left[6\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right] = (2)(6)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)\right]$$
$$= 12\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

33. $$\left[\tfrac{5}{3}(\cos 120^\circ + i\sin 120^\circ)\right]\left[\tfrac{2}{3}(\cos 30^\circ + i\sin 30^\circ)\right] = \tfrac{5}{3}\left(\tfrac{2}{3}\right)\left[\cos(120^\circ + 30^\circ) + i\sin(120^\circ + 30^\circ)\right]$$
$$= \tfrac{10}{9}(\cos 150^\circ + i\sin 150^\circ)$$

35. $$\frac{3(\cos 50^\circ + i\sin 50^\circ)}{9(\cos 20^\circ + i\sin 20^\circ)} = \frac{1}{3}\left[\cos(50^\circ - 20^\circ) + i\sin(50^\circ - 20^\circ)\right] = \frac{1}{3}(\cos 30^\circ + i\sin 30^\circ)$$

37. $$\frac{\cos\pi + i\sin\pi}{\cos(\pi/3) + i\sin(\pi/3)} = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

39. $$\left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]\left[\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right] = (2)\left(\frac{1}{2}\right)\left[\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)\right]$$
$$= (\cos\pi + i\sin\pi)$$
$$= -1 + 0i$$
$$= -1$$

41. $$\left[5(\cos 20^\circ + i\sin 20^\circ)\right]^3 = 5^3(\cos 60^\circ + i\sin 60^\circ)$$
$$= \frac{125}{2} + \frac{125\sqrt{3}}{2}i$$

43. $$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12} = \cos\frac{12\pi}{4} + i\sin\frac{12\pi}{4}$$
$$= \cos 3\pi + i\sin 3\pi$$
$$= -1$$

45. $$\left[5(\cos 3.2 + i\sin 3.2)\right]^4 = 5^4(\cos 12.8 + i\sin 12.8)$$
$$\approx 608.0 + 144.7i$$

47. $$\left[3(\cos 15^\circ + i\sin 15^\circ)\right]^4 = 81(\cos 60^\circ + i\sin 60^\circ)$$
$$= \frac{81}{2} + \frac{81\sqrt{3}}{2}i$$

49. $$(1 + i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^5$$
$$= \left(\sqrt{2}\right)^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$= 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$
$$= -4 - 4i$$

51. $$\begin{aligned}(-1+i)^6 &= \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^6\\ &= \left(\sqrt{2}\right)^6\left(\cos\frac{18\pi}{4} + i\sin\frac{18\pi}{4}\right)\\ &= 8\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)\\ &= 8(0+i)\\ &= 8i\end{aligned}$$

53. $$\begin{aligned}2\left(\sqrt{3}+i\right)^{10} &= 2\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{16}\\ &= 2\left[2^{10}\left(\cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}\right)\right]\\ &= 2048\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\\ &= 2048\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\\ &= 1024 - 1024\sqrt{3}i\end{aligned}$$

55. (a) Square roots of $5(\cos 120^\circ + i\sin 120^\circ)$:

$$\sqrt{5}\left[\cos\left(\frac{120^\circ + 360^\circ k}{2}\right) + i\sin\left(\frac{120^\circ + 360^\circ k}{2}\right)\right],$$

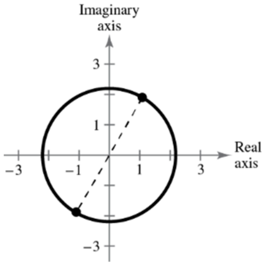
$k = 0, 1$

$k = 0$: $\sqrt{5}(\cos 60^\circ + i\sin 60^\circ)$

$k = 1$: $\sqrt{5}(\cos 240^\circ + i\sin 240^\circ)$

(b) $\dfrac{\sqrt{5}}{2} + \dfrac{\sqrt{15}}{2}i, -\dfrac{\sqrt{5}}{2} - \dfrac{\sqrt{15}}{2}i$

(c)



57. (a) Cube roots of $8\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$:

$$\sqrt[3]{8}\left[\cos\left(\frac{(2\pi/3) + 2\pi k}{3}\right) + i\sin\left(\frac{(2\pi/3) + 2\pi k}{3}\right)\right],$$

$k = 0, 1, 2$

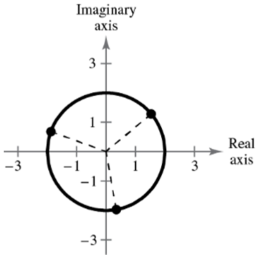
$k = 0$: $2\left(\cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9}\right)$

$k = 1$: $2\left(\cos\dfrac{8\pi}{9} + i\sin\dfrac{8\pi}{9}\right)$

$k = 2$: $2\left(\cos\dfrac{14\pi}{9} + i\sin\dfrac{14\pi}{9}\right)$

(b) $1.5321 + 1.2856i, -1.8794 + 0.6840i,$
$0.3473 - 1.9696i$

(c)



59. (a) Cube roots of $1000 = 1000(\cos 0 + i\sin 0)$:

$$\sqrt[3]{1000}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$$

$k = 0, 1, 2$

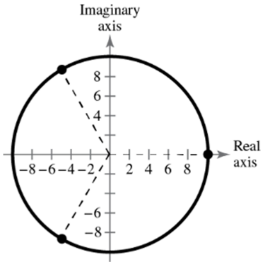
$k = 0$: $10(\cos 0 + i\sin 0)$

$k = 1$: $10\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$k = 2$: $10\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

(b) $10, -5 + 5\sqrt{3}i, -5 - 5\sqrt{3}i$

(c)



61. (a) Fourth roots of $-4 = 4(\cos \pi + i \sin \pi)$:

$$\sqrt[4]{4}\left[\cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right)\right]$$

$k = 0, 1, 2, 3$

$k = 0$: $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

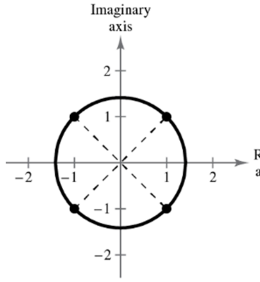
$k = 1$: $\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

$k = 2$: $\sqrt{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

$k = 3$: $\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$

(b) $1 + i, -1 + i, -1 - i, 1 - i$

(c)



63. (a) Fourth roots of $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$:

$$\sqrt[4]{1}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right)\right],$$

$k = 0, 1, 2, 3$

$k = 0$: $\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$

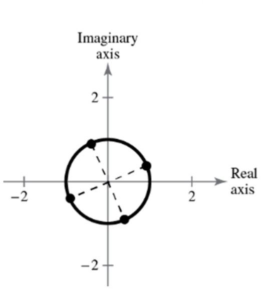
$k = 1$: $\cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8}$

$k = 2$: $\cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}$

$k = 3$: $\cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8}$

(b) $0.9239 + 0.3827i, -0.3827 + 0.9239i,$
$-0.9239 - 0.3827i, 0.3827 - 0.9239i$

(c)



65. (a) Square roots of $-25i = 25\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$:

$$\sqrt{25}\left[\cos\left(\frac{\frac{3\pi}{2} + 2k\pi}{2}\right) + i \sin\left(\frac{\frac{3\pi}{2} + 2k\pi}{2}\right)\right],$$

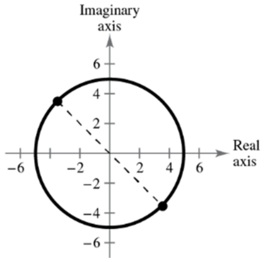
$k = 0, 1$

$k = 0$: $5\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

$k = 1$: $5\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$

(b) $-\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i, \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

(c)



67. (a) Fifth roots of

$4(1 - i) = 4\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$:

$$\sqrt[5]{4\sqrt{2}}\left[\cos\left(\frac{\frac{7\pi}{4} + 2\pi k}{5}\right) + i \sin\left(\frac{\frac{7\pi}{4} + 2\pi k}{5}\right)\right],$$

$k = 0, 1, 2, 3, 4$

$k = 0$: $\sqrt{2}\left(\cos \frac{7\pi}{20} + i \sin \frac{7\pi}{20}\right)$

$k = 1$: $\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

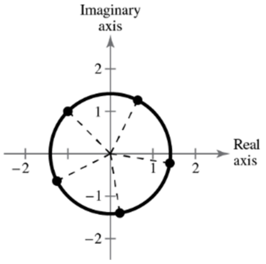
$k = 2$: $\sqrt{2}\left(\cos \frac{23\pi}{20} + i \sin \frac{23\pi}{20}\right)$

$k = 3$: $\sqrt{2}\left(\cos \frac{31\pi}{20} + i \sin \frac{31\pi}{20}\right)$

$k = 4$: $\sqrt{2}\left(\cos \frac{39\pi}{20} + i \sin \frac{39\pi}{20}\right)$

(b) $0.6420 + 1.2601i, -1 + 1i, -1.2601 - 0.6420i,$
$0.2212 - 1.3968i, 1.3968 - 0.2212i$

(c)



69. (a) Cube roots of $-\frac{125}{2}\left(1 + \sqrt{3}i\right) = 125\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$:

$$\sqrt[3]{125}\left[\cos\left(\frac{\frac{4\pi}{3} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{4\pi}{3} + 2k\pi}{3}\right)\right], k = 0, 1, 2$$

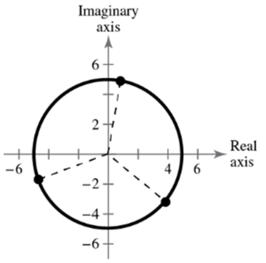
$$k = 0\text{: } 5\left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$$

$$k = 1\text{: } 5\left(\cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9}\right)$$

$$k = 2\text{: } 5\left(\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right)$$

(b) $0.8682 + 4.9240i, -4.6985 - 1.7101i, 3.8302 - 3.2140i$

(c)



71. $x^4 + i = 0$

$x^4 = -i$

The solutions are the fourth roots of

$i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$:

$$\sqrt[4]{1}\left[\cos\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right)\right],$$

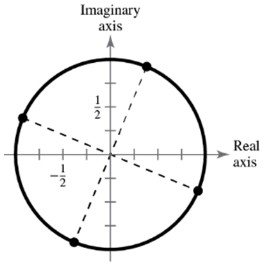
$k = 0, 1, 2, 3$

$$k = 0\text{: } \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8} \approx 0.3827 + 0.9239i$$

$$k = 1\text{: } \cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8} \approx -0.9239 + 0.3827i$$

$$k = 2\text{: } \cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8} \approx -0.3827 - 0.9239i$$

$$k = 3\text{: } \cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8} \approx 0.9239 - 0.3827i$$



73. $x^5 + 243 = 0$

$x^5 = -243$

The solutions are the fifth roots of

$-243 = 243(\cos\pi + i\sin\pi)$:

$$\sqrt[5]{243}\left[\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right],$$

$k = 0, 1, 2, 3, 4$

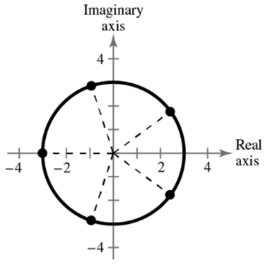
$$k = 0\text{: } 3\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) \approx 2.4271 + 1.7634i$$

$$k = 1\text{: } 3\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) \approx -0.9271 + 2.8532i$$

$$k = 2\text{: } 3(\cos\pi + i\sin\pi) = -3$$

$$k = 3\text{: } 3\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right) \approx -0.9271 - 2.8532i$$

$$k = 4\text{: } 3\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right) \approx 2.4271 - 1.7634i$$



75. $x^4 + 16i = 0$

$$x^4 = -16i$$

The solutions are the fourth roots of $-16i = 16\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)$:

$$\sqrt[4]{16}\left[\cos\frac{\frac{3\pi}{2} + 2\pi k}{4} + i\sin\frac{\frac{3\pi}{2} + 2\pi k}{4}\right],$$

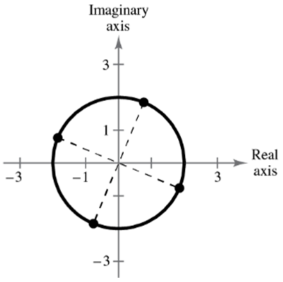
$k = 0, 1, 2, 3$

$$k = 0:\ 2\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right) \approx 0.7654 + 1.8478i$$

$$k = 1:\ 2\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right) \approx -1.8478 + 0.7654i$$

$$k = 2:\ 2\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right) \approx -0.7654 - 1.8478i$$

$$k = 3:\ 2\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right) \approx 1.8478 - 0.7654i$$



77. $x^3 - (1 - i) = 0$

$$x^3 = 1 - i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

The solutions are the cube roots of $1 - i$:

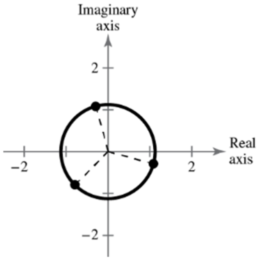
$$\sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{(7\pi/4) + 2\pi k}{3}\right) + i\sin\left(\frac{(7\pi/4) + 2\pi k}{3}\right)\right],$$

$k = 0, 1, 2$

$$k = 0:\ \sqrt[6]{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \approx -0.2905 + 1.0842i$$

$$k = 1:\ \sqrt[6]{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \approx -0.7937 - 0.7937i$$

$$k = 2:\ \sqrt[6]{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) \approx 1.0842 - 0.2905i$$



79. $E = IZ = \left[6(\cos 41^\circ + i\sin 41^\circ)\right]\left(4\left[\cos(-11^\circ) + i\sin(-11^\circ)\right]\right) = 24(\cos 30^\circ + i\sin 30^\circ)$ volts

$$E = 24\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 12\sqrt{3} + 12i \text{ volts}$$

81. False. They are equally spaced along the circle centered at the origin with radius $\sqrt[n]{r}$.

83.
$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

85. The quotient is $4\left[\cos(90° - 30°) + i\sin(90° - 30°)\right]$

$$= 4(\cos 30° + i\sin 30°).$$

Because $\sin 30° = \frac{1}{2}$, not $-\frac{1}{2}$, the standard form should be $2\sqrt{3} + 2i$.

87.

$$\begin{aligned} 2x + y &= 5 \\ 2(2) + 1 &\overset{?}{=} 5 \\ 4 + 1 &= 5 \end{aligned}$$

Yes, $(2,1)$ satisfies the equation.

89.

$$\begin{aligned} 3y - 2x &= -8 \\ 3(1) - 2(-2) &\overset{?}{=} -8 \\ 3 + 4 &\neq -8 \end{aligned}$$

No, $(-2, 1)$ does not satisfy the equation.

91.

$$\begin{aligned} x + y &= 4 \\ y &= 4 - x \end{aligned}$$

93.

$$\begin{aligned} 5x - 3y &= 6 \\ -3y &= 6 - 5x \\ y &= \frac{5}{3}x - 2 \end{aligned}$$

95.

$$\begin{aligned} 3x^2 + 4x - (2x + 1) &= 7 \\ 3x^2 + 2x - 8 &= 0 \\ (x + 2)(3x - 4) &= 0 \\ x &= -2, \tfrac{4}{3} \end{aligned}$$

97.

$$\begin{aligned} x^2 + x + 4 &= 3 \\ x^2 + x + 1 &= 0 \\ x &= \frac{-1 \pm \sqrt{1 - 4}}{2} \\ x &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

Review Exercises for Chapter 6

1. Given: $A = 38°, B = 70°, a = 8$

$$C = 180° - 38° - 70° = 72°$$

$$b = \frac{a\sin B}{\sin A} = \frac{8\sin 70°}{\sin 38°} \approx 12.21$$

$$c = \frac{a\sin C}{\sin A} = \frac{8\sin 72°}{\sin 38°} \approx 12.36$$

3. Given: $B = 72°, C = 82°, b = 54$

$$A = 180° - 72° - 82° = 26°$$

$$a = \frac{b\sin A}{\sin B} = \frac{54\sin 26°}{\sin 72°} \approx 24.89$$

$$c = \frac{b\sin C}{\sin B} = \frac{54\sin 82°}{\sin 72°} \approx 56.23$$

5. Given: $A = 16°, B = 98°, c = 8.4$

$$C = 180° - 16° - 98° = 66°$$

$$a = \frac{c\sin A}{\sin C} = \frac{8.4\sin 16°}{\sin 66°} \approx 2.53$$

$$b = \frac{c\sin B}{\sin C} = \frac{8.4\sin 98°}{\sin 66°} \approx 9.11$$

7. Given: $A = 24°, C = 48°, b = 27.5$

$$B = 180° - 24° - 48° = 108°$$

$$a = \frac{b\sin A}{\sin B} = \frac{27.5\sin 24°}{\sin 108°} \approx 11.76$$

$$c = \frac{b\sin C}{\sin B} = \frac{27.5\sin 48°}{\sin 108°} \approx 21.49$$

9. Given: $B = 150°, b = 30, c = 10$

$$\sin C = \frac{c\sin B}{b} = \frac{10\sin 150°}{30} \approx 0.1667$$

$$\Rightarrow C \approx 9.59°$$

$$A \approx 180° - 150° - 9.59° = 20.41°$$

$$a = \frac{b\sin A}{\sin B} = \frac{30\sin 20.41°}{\sin 150°} \approx 20.92$$

11. $A = 75°, a = 51.2, b = 33.7$

$$\sin B = \frac{b\sin A}{a} = \frac{33.7\sin 75°}{51.2} \approx 0.6358$$

$$\Rightarrow B \approx 39.48°$$

$$C \approx 180° - 75° - 39.48° = 65.52°$$

$$c = \frac{a\sin C}{\sin A} = \frac{51.2\sin 65.52°}{\sin 75°} \approx 48.24$$

13. $A = 33°, b = 7, c = 10$

$$\text{Area} = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}(7)(10)\sin 33° \approx 19.1$$

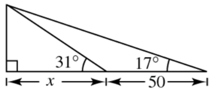
15. $C = 119°, a = 18, b = 6$

$$\text{Area} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}(18)(6)\sin 119° \approx 47.2$$

17. $\tan 17° = \dfrac{h}{x + 50} \Rightarrow h = (x + 50)\tan 17°$

$$h = x\tan 17° + 50\tan 17°$$

$\tan 31° = \dfrac{h}{x} \Rightarrow h = x\tan 31°$



$x\tan 17° + 50\tan 17° = x\tan 31°$

$$50\tan 17° = x(\tan 31° - \tan 17°)$$

$$\frac{50\tan 17°}{\tan 31° - \tan 17°} = x$$

$$x \approx 51.7959$$

$h = x\tan 31° \approx 51.7959\tan 31° \approx 31.1$ meters

The height of the building is approximately 31.1 meters.

19. Given: $a = 6, b = 9, c = 14$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{36 + 81 - 196}{2(6)(9)} \approx -0.7315 \Rightarrow C \approx 137.01°$$

$$\sin B = \frac{b\sin C}{c} \approx \frac{9\sin 137.01°}{14} \approx 0.4383 \Rightarrow B \approx 26.00°$$

$$A \approx 180° - 26.00° - 137.01° = 16.99°$$

21. Given: $a = 2.5, b = 5.0, c = 4.5$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = 0.0667 \Rightarrow B \approx 86.18°$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = 0.44 \Rightarrow C \approx 63.90°$$

$$A = 180° - B - C \approx 29.92°$$

23. Given: $B = 108°, a = 11, c = 11$

$$b^2 = a^2 + c^2 - 2ac\cos B = 11^2 + 11^2 - 2(11)(11)\cos 108° \Rightarrow b \approx 17.80$$

$$A = C = \tfrac{1}{2}(180° - 108°) = 36°$$

25. Given: $C = 43°, a = 22.5, b = 31.4$

$$c = \sqrt{a^2 + b^2 - 2ab\cos C} \approx 21.42$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \approx -0.02169 \Rightarrow B \approx 91.24°$$

$$A = 180° - B - C \approx 45.76°$$

27. Given: $B = 38°, a = 15, b = 6$

Given two sides and angle opposite one of them, use the Law of Sines.

$$\sin A = \frac{a\sin B}{b} = \frac{15\sin 38°}{6} = 1.54 > 1$$

There is no solution.

29. Given: $a = 13, b = 15, c = 24$

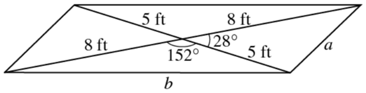
Given three sides, the Law of Cosines can be used.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{169 + 225 - 576}{2(13)(15)} \approx -0.46667 \Rightarrow C \approx 117.82°$$

$$\sin A = \frac{a\sin C}{c} \approx \frac{13\sin 117.82°}{24} \approx 0.47906 \Rightarrow A \approx 28.62°$$

$$B \approx 180° - 28.62° - 117.82° = 33.56°$$

31.



$a^2 = 5^2 + 8^2 - 2(5)(8)\cos 28° \approx 18.364$

$a \approx 4.3$ feet

$b^2 = 8^2 + 5^2 - 2(8)(5)\cos 152° \approx 159.636$

$b \approx 12.6$ feet

33. $a = 3, b = 6, c = 8$

$$s = \frac{a+b+c}{2} = \frac{3+6+8}{2} = 8.5$$

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{8.5(5.5)(2.5)(0.5)}\\ &\approx 7.64\end{aligned}$$

35. $a = 12.3, b = 15.8, c = 3.7$

$$s = \frac{a+b+c}{2} = \frac{12.3+15.8+3.7}{2} = 15.9$$

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{15.9(3.6)(0.1)(12.2)} = 8.36\end{aligned}$$

37. $\|\mathbf{u}\| = \sqrt{\left[4-(-2)\right]^2 + (6-1)^2} = \sqrt{61}$

$\|\mathbf{v}\| = \sqrt{(5-0)^2 + \left[4-(-2)\right]^2} = \sqrt{61}$

$$\text{Slope } \mathbf{u} = \frac{6-1}{4-(-2)} = \frac{5}{6}$$

$$\text{Slope } \mathbf{v} = \frac{4-(-2)}{5-0} = \frac{6}{5}$$

u and **v** do not have the same direction, so they are not equivalent.

39. Initial point: $(0, 10)$

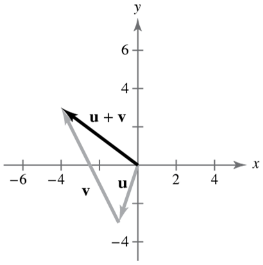
Terminal point: $(7, 3)$

$\mathbf{v} = \langle 7-0, 3-10\rangle = \langle 7, -7\rangle$

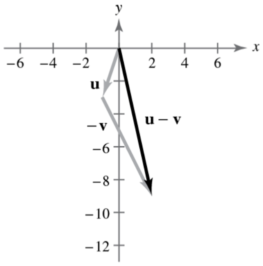
$\|\mathbf{v}\| = \sqrt{7^2 + (-7)^2} = \sqrt{98} = 7\sqrt{2}$

41. $\mathbf{u} = \langle -1, -3\rangle, \mathbf{v} = \langle -3, 6\rangle$

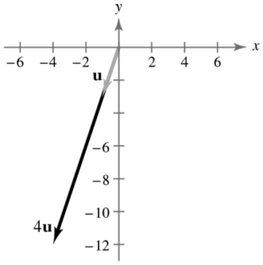
(a) $\mathbf{u} + \mathbf{v} = \langle -1, -3\rangle + \langle -3, 6\rangle = \langle -4, 3\rangle$



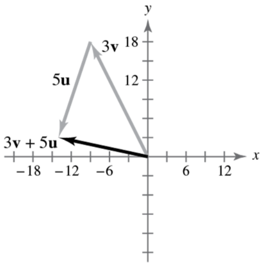
(b) $\mathbf{u} - \mathbf{v} = \langle -1, -3\rangle - \langle -3, 6\rangle = \langle 2, -9\rangle$



(c) $4\mathbf{u} = 4\langle -1, -3\rangle = \langle -4, -12\rangle$

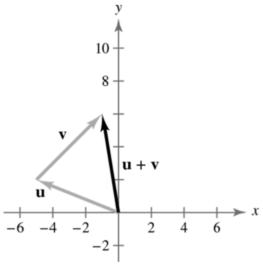


(d) $3\mathbf{v} + 5\mathbf{u} = 3\langle -3, 6\rangle + 5\langle -1, -3\rangle = \langle -9, 18\rangle + \langle -5, -15\rangle = \langle -14, 3\rangle$

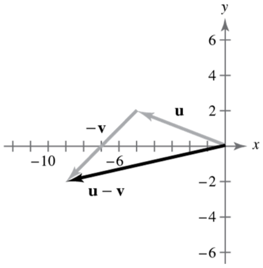


43. $\mathbf{u} = \langle -5, 2\rangle, \mathbf{v} = \langle 4, 4\rangle$

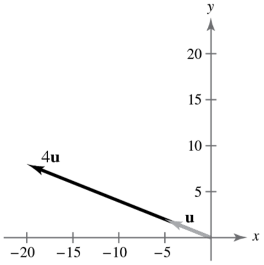
(a) $\mathbf{u} + \mathbf{v} = \langle -5, 2\rangle + \langle 4, 4\rangle = \langle -1, 6\rangle$



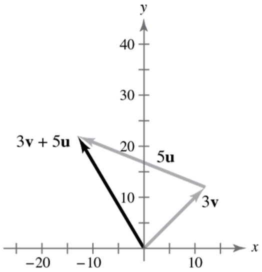
(b) $\mathbf{u} - \mathbf{v} = \langle -5, 2\rangle - \langle 4, 4\rangle = \langle -9, -2\rangle$



(c) $4\mathbf{u} = 4\langle -5, 2\rangle = \langle -20, 8\rangle$

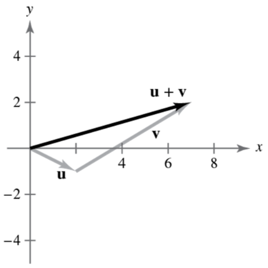


(d) $3\mathbf{v} + 5\mathbf{u} = 3\langle 4, 4\rangle + 5\langle -5, 2\rangle = \langle 12, 12\rangle + \langle -25, 10\rangle = \langle -13, 22\rangle$

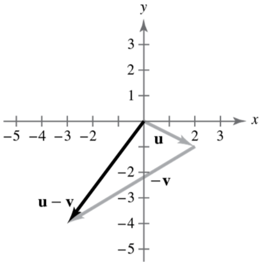


45. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}, \mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$

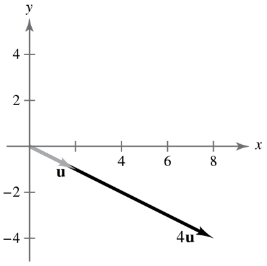
(a) $\mathbf{u} + \mathbf{v} = (2\mathbf{i} - \mathbf{j}) + (5\mathbf{i} + 3\mathbf{j}) = 7\mathbf{i} + 2\mathbf{j}$



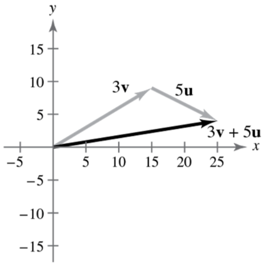
(b) $\mathbf{u} - \mathbf{v} = (2\mathbf{i} - \mathbf{j}) - (5\mathbf{i} + 3\mathbf{j}) = -3\mathbf{i} - 4\mathbf{j}$



(c) $4\mathbf{u} = 4(2\mathbf{i} - \mathbf{j}) = 8\mathbf{i} - 4\mathbf{j}$

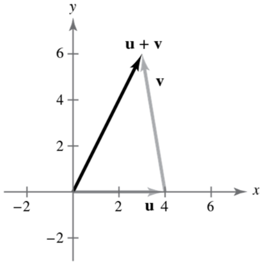


(d) $3\mathbf{v} + 5\mathbf{u} = 3(5\mathbf{i} + 3\mathbf{j}) + 5(2\mathbf{i} - \mathbf{j}) = 15\mathbf{i} + 9\mathbf{j} + 10\mathbf{i} - 5\mathbf{j}$
$= 25\mathbf{i} + 4\mathbf{j}$

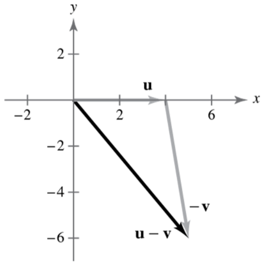


47. $\mathbf{u} = 4\mathbf{i}, \mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

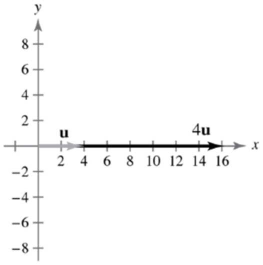
(a) $\mathbf{u} + \mathbf{v} = 4\mathbf{i} + (-\mathbf{i} + 6\mathbf{j}) = 3\mathbf{i} + 6\mathbf{j}$



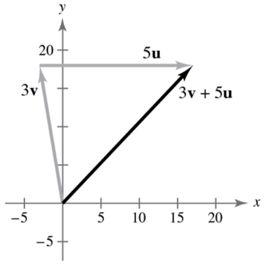
(b) $\mathbf{u} - \mathbf{v} = 4\mathbf{i} - (-\mathbf{i} + 6\mathbf{j}) = 5\mathbf{i} - 6\mathbf{j}$



(c) $4\mathbf{u} = 4(4\mathbf{i}) = 16\mathbf{i}$



(d) $3\mathbf{v} + 5\mathbf{u} = 3(-\mathbf{i} + 6\mathbf{j}) + 5(4\mathbf{i}) = -3\mathbf{i} + 18\mathbf{j} + 20\mathbf{i} = 17\mathbf{i} + 18\mathbf{j}$



49. $P = (2, 3), Q = (1, 8)$

$\overrightarrow{PQ} = \mathbf{v} = \langle 1 - 2, 8 - 3 \rangle$

$\mathbf{v} = \langle -1, 5 \rangle$

$\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$

51. $P = (3, 4), Q = (9, 8)$

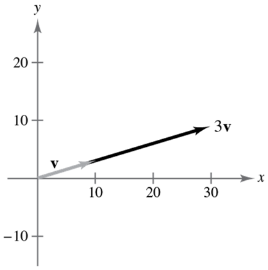
$\overrightarrow{PQ} = \mathbf{v} = \langle 9 - 3, 8 - 4 \rangle$

$\mathbf{v} = \langle 6, 4 \rangle$

$\mathbf{v} = 6\mathbf{i} + 4\mathbf{j}$

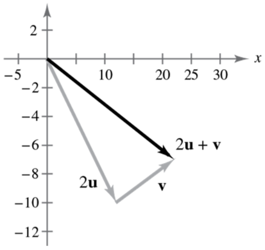
53. $\mathbf{v} = 10\mathbf{i} + 3\mathbf{j}$

$$\begin{aligned} 3\mathbf{v} &= 3(10\mathbf{i} + 3\mathbf{j}) \\ &= 30\mathbf{i} + 9\mathbf{j} \\ &= \langle 30, 9 \rangle \end{aligned}$$



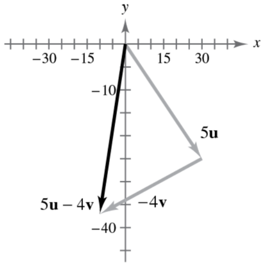
55. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j}, \mathbf{v} = 10\mathbf{i} + 3\mathbf{j}$

$$\begin{aligned} 2\mathbf{u} + \mathbf{v} &= 2(6\mathbf{i} - 5\mathbf{j}) + (10\mathbf{i} + 3\mathbf{j}) \\ &= 22\mathbf{i} - 7\mathbf{j} \\ &= \langle 22, -7 \rangle \end{aligned}$$



57. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j}, \mathbf{v} = 10\mathbf{i} + 3\mathbf{j}$

$$\begin{aligned} 5\mathbf{u} - 4\mathbf{v} &= 5(6\mathbf{i} - 5\mathbf{j}) - 4(10\mathbf{i} + 3\mathbf{j}) \\ &= 30\mathbf{i} - 25\mathbf{j} - 40\mathbf{i} - 12\mathbf{j} \\ &= -10\mathbf{i} - 37\mathbf{j} \\ &= \langle -10, -37 \rangle \end{aligned}$$



59. $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{5^2 + 4^2} = \sqrt{41}$

$\tan\theta = \frac{4}{5} \Rightarrow \theta \approx 38.7°$

61. $\mathbf{v} = -3\mathbf{i} - 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$

$\tan\theta = \dfrac{-3}{-3} = 1 \Rightarrow \theta = 225°$

63. $\mathbf{v} = 7(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

$\|\mathbf{v}\| = 7$

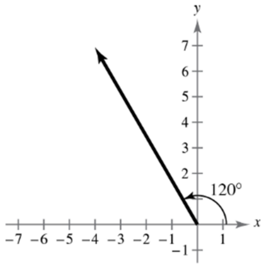
$\theta = 60°$

65. $\mathbf{v} = 8(\cos 120° + i \sin 120°)$

$= 8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -4 + 4\sqrt{3}i$

$= \left\langle -4, 4\sqrt{3} \right\rangle$



67. Force One:

$\mathbf{u} = 85(\cos 45°\mathbf{i} + \sin 45°\mathbf{j})$

$= 85\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right)$

$= \frac{85\sqrt{2}}{2}\mathbf{i} + \frac{85\sqrt{2}}{2}\mathbf{j}$

Force Two:

$\mathbf{v} = 50(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

$= 50\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$

$= 25\mathbf{i} + 25\sqrt{3}\mathbf{j}$

Resultant Force:

$$\mathbf{u} + \mathbf{v} = \left(\frac{85\sqrt{2}}{2} + 25\right)\mathbf{i} + \left(\frac{85\sqrt{2}}{2} + 25\sqrt{3}\right)\mathbf{j}$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{\left(\frac{85\sqrt{2}}{2} + 25\right)^2 + \left(\frac{85\sqrt{2}}{2} + 25\sqrt{3}\right)^2}$$

≈ 133.92 pounds

$$\tan\theta = \frac{\frac{85\sqrt{2}}{2} + 25\sqrt{3}}{\frac{85\sqrt{2}}{2} + 25}$$

$\theta = 50.5°$

69. $\mathbf{u} = \langle 6, 7\rangle, \mathbf{v} = \langle -3, 9\rangle$

$\mathbf{u} \cdot \mathbf{v} = 6(-3) + 7(9) = 45$

71. $\mathbf{u} = 3\mathbf{i} + 7\mathbf{j}, \mathbf{v} = 11\mathbf{i} - 5\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 3(11) + 7(-5) = -2$

73. $\mathbf{u} = \langle -4, 2\rangle$

$2\mathbf{u} = \langle -8, 4\rangle$

$2\mathbf{u} \cdot \mathbf{u} = -8(-4) + 4(2) = 40$

The result is a scalar.

75. $\mathbf{u} = \langle -4, 2\rangle$

$4 - \|\mathbf{u}\| = 4 - \sqrt{(-4)^2 + 2^2} = 4 - \sqrt{20}$

$= 4 - 2\sqrt{5}$

The result is a scalar.

77. $\mathbf{u} = \langle -4, 2\rangle, \mathbf{v} = \langle 5, 1\rangle$

$\mathbf{u}(\mathbf{u} \cdot \mathbf{v}) = \langle -4, 2\rangle[-4(5) + 2(1)]$

$= -18\langle -4, 2\rangle$

$= \langle 72, -36\rangle$

The result is a vector.

79. $\mathbf{u} = \langle -4, 2\rangle, \mathbf{v} = \langle 5, 1\rangle$

$(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v}) = [-4(-4) + 2(2)] - [-4(5) + 2(1)]$

$= 20 - (-18)$

$= 38$

The result is a scalar.

81. $\mathbf{u} = \left\langle 2\sqrt{2}, -4\right\rangle, \mathbf{v} = \left\langle -\sqrt{2}, 1\right\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-8}{\left(\sqrt{24}\right)\left(\sqrt{3}\right)} \Rightarrow \theta \approx 160.5°$$

83. $\mathbf{u} = \cos\frac{7\pi}{4}\mathbf{i} + \sin\frac{7\pi}{4}\mathbf{j} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

$\mathbf{v} = \cos\frac{5\pi}{6}\mathbf{i} + \sin\frac{5\pi}{6}\mathbf{j} = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-\sqrt{3} - 1}{2\sqrt{2}} \Rightarrow \theta = 165°$$

85. $\mathbf{u} = \langle -3, 8\rangle$

$\mathbf{v} = \langle 8, 3\rangle$

$\mathbf{u} \cdot \mathbf{v} = -3(8) + 8(3) = 0$

u and **v** are orthogonal.

87. $\mathbf{u} = -\mathbf{i}$

$\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} \neq 0 \Rightarrow$ Not orthogonal

$\mathbf{v} \neq k\mathbf{u} \Rightarrow$ Not parallel

Neither

89. $\mathbf{u} = \langle -4, 3\rangle, \mathbf{v} = \langle -8, -2\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{26}{68}\right)\langle -8, -2\rangle = -\frac{13}{17}\langle 4, 1\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle -4, 3\rangle - \left(-\frac{13}{17}\right)\langle 4, 1\rangle = \frac{16}{17}\langle -1, 4\rangle$$

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = -\frac{13}{17}\langle 4, 1\rangle + \frac{16}{17}\langle -1, 4\rangle$$

91. $\mathbf{u} = \langle 2, 7\rangle, \mathbf{v} = \langle 1, -1\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = -\frac{5}{2}\langle 1, -1\rangle = \frac{5}{2}\langle -1, 1\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 7\rangle - \left(\frac{5}{2}\right)\langle -1, 1\rangle = \frac{9}{2}\langle 1, 1\rangle$$

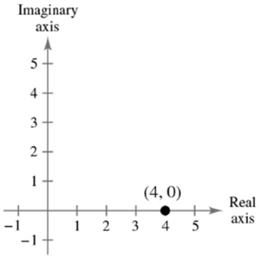
$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = \frac{5}{2}\langle -1, 1\rangle + \frac{9}{2}\langle 1, 1\rangle$$

93. $P = (5, 3), Q = (8, 9) \Rightarrow \overline{PQ} = \langle 3, 6\rangle$

Work $= \mathbf{v}\cdot\overline{PQ} = \langle 2, 7\rangle\cdot\langle 3, 6\rangle = 48$

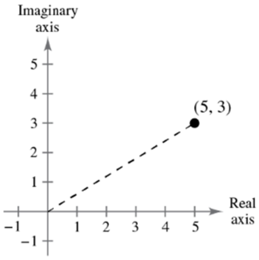
95. Work $= (18{,}000)\left(\frac{48}{12}\right) = 72{,}000$ foot-pounds

97. $|4| = 4$



99. $|5 + 3i| = \sqrt{5^2 + 3^2}$

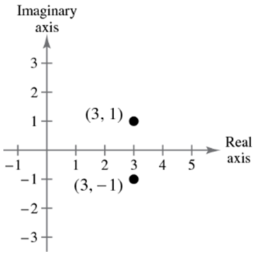
$= \sqrt{34}$



101. $(2 + 3i) + (1 - 2i) = 3 + i$

103. $(1 + 2i) - (3 + i) = -2 + i$

105. The complex conjugate of $3 + i$ is $3 - i$



107. $d = \sqrt{(2 - 3)^2 + (-1 - 2)^2} = \sqrt{10}$

109. Midpoint $= \left(\frac{1 + 4}{2}, \frac{1 + 3}{2}i\right)$

$= \frac{5}{2} + 2i$

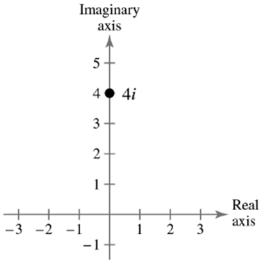
$= \left(\frac{5}{2}, 2\right)$

111. $z = 4i$

$r = \sqrt{0^2 + 4^2} = \sqrt{16} = 4$

$\tan\theta = \frac{4}{0}$, undefined $\Rightarrow \theta = \frac{\pi}{2}$

$z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

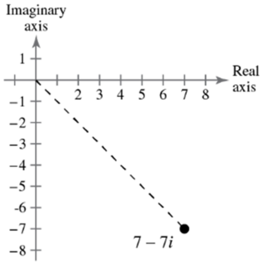


113. $z = 7 - 7i$

$r = \sqrt{(7)^2 + (-7)^2} = \sqrt{98} = 7\sqrt{2}$

$\tan\theta = \frac{-7}{7} = -1 \Rightarrow \theta = \frac{7\pi}{4}$ because the complex number lies in Quadrant IV.

$7 - 7i = 7\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

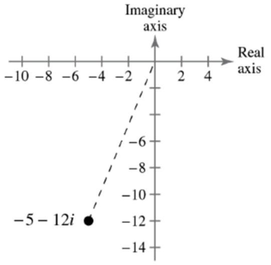


115. $z = -5 - 12i$

$$r = \sqrt{(-5)^2 + (-12)^2} = \sqrt{169} = 13$$

$$\tan\theta = \frac{12}{5}, \theta \text{ is in Quadrant III} \Rightarrow \theta \approx 4.32$$

$$z = 13(\cos 4.32 + i\sin 4.32)$$



117. $$\left[2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right] = (2)(2)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right] = 4\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

119. $$\frac{2[\cos 60^\circ + i\sin 60^\circ]}{3[\cos 15^\circ + i\sin 15^\circ]} = \frac{2}{3}(\cos(60^\circ - 15^\circ) + i\sin(60^\circ - 15^\circ)) = \frac{2}{3}(\cos 45^\circ + i\sin 45^\circ)$$

121. $$\left[5\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]^4 = 5^4\left(\cos\frac{4\pi}{12} + i\sin\frac{4\pi}{12}\right)$$
$$= 625\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 625\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= \frac{625}{2} + \frac{625\sqrt{3}}{2}i$$

123. $$(2 + 3i)^6 \approx \left[\sqrt{13}(\cos 56.3^\circ + i\sin 56.3^\circ)\right]^6$$
$$= 13^3(\cos 337.9^\circ + i\sin 337.9^\circ)$$
$$\approx 13^3(0.9263 - 0.3769i)$$
$$\approx 2035 - 828i$$

125. Sixth roots of $-729i = 729\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$:

(a) $$\sqrt[6]{729}\left[\cos\left(\frac{\frac{3\pi}{2} + 2k\pi}{6}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2k\pi}{6}\right)\right],$$

$k = 0, 1, 2, 3, 4, 5$

$k = 0$: $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$k = 1$: $3\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$k = 2$: $3\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

$k = 3$: $3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$k = 4$: $3\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

$k = 5$: $3\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$

(b) $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$-0.776 + 2.898i$

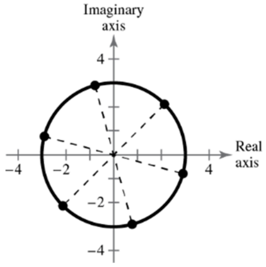
$-2.898 + 0.776i$

$\frac{-3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

$0.776 - 2.898i$

$2.898 - 0.776i$

(c)



127. Cube roots of $8 = 8(\cos 0 + i\sin 0)$, $k = 0, 1, 2$

(a) $\sqrt[3]{8}\left[\cos\left(\frac{0+2\pi k}{3}\right) + i\sin\left(\frac{0+2\pi k}{3}\right)\right]$

$k = 0\text{: } 2(\cos 0 + i\sin 0)$

$k = 1\text{: } 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

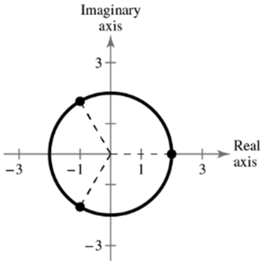
$k = 2\text{: } 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

(b) 2

$-1 + \sqrt{3i}$

$-1 - \sqrt{3i}$

(c)



129. $x^4 + 81 = 0$

$x^4 = -81$ Solve by finding the fourth roots of -81.

$-81 = 81(\cos\pi + i\sin\pi)$

$\sqrt[4]{-81} = \sqrt[4]{81}\left[\cos\left(\frac{\pi+2\pi k}{4}\right) + i\sin\left(\frac{\pi+2\pi k}{4}\right)\right],$

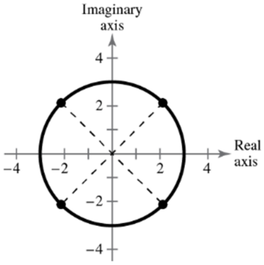
$k = 0, 1, 2, 3$

$k = 0\text{: } 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$k = 1\text{: } 3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$k = 2\text{: } 3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

$k = 3\text{: } 3\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$



131. $x^3 + 8i = 0$

$x^3 = -8i$ Solve by finding the cube roots of $-8i$.

$-8i = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

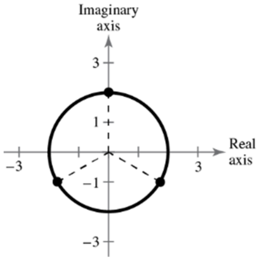
$\sqrt[3]{-8i} = \sqrt[3]{8}\left[\cos\left(\frac{\frac{3\pi}{2}+2\pi k}{3}\right) + i\sin\left(\frac{\frac{3\pi}{2}+2\pi k}{3}\right)\right],$

$k = 0, 1, 2$

$k = 0\text{: } 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$

$k = 1\text{: } 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$

$k = 2\text{: } 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$



133. True. sin 90° is defined in the Law of Sines.

135. A vector in the plane has both a magnitude and a direction.

Problem Solving for Chapter 6

1. $\left(\overline{PQ}\right)^2 = 4.7^2 + 6^2 - 2(4.7)(6)\cos 25°A$

$\overline{PQ} \approx 2.6409$ feet

$\dfrac{\sin\alpha}{4.7} = \dfrac{\sin 25°}{2.6409} \Rightarrow \alpha \approx 48.78°$

$\theta + \beta = 180° - 25° - 48.78° = 106.22°$

$(\theta + \beta) + \theta = 180° \Rightarrow \theta = 180° - 106.22° = 73.78°$

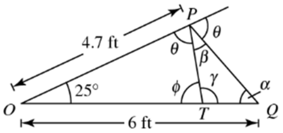
$\beta = 106.22° - 73.78° = 32.44°$

$\gamma = 180° - \alpha - \beta = 180° - 48.78° - 32.44° = 98.78°$

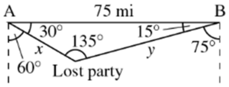
$\phi = 180° - \gamma = 180° - 98.78° = 81.22°$

$\dfrac{\overline{PT}}{\sin 25°} = \dfrac{4.7}{\sin 81.22°}$

$\overline{PT} \approx 2.01$ feet



3. (a)



(b) $\dfrac{x}{\sin 15°} = \dfrac{75}{\sin 135°}$

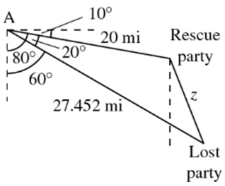
$y \approx 27.45$ miles

and

$\dfrac{y}{\sin 30°} = \dfrac{75}{\sin 135°}$

$y \approx 53.03$ miles

(c)



$z^2 = (27.45)^2 + (20)^2 - 2(27.45)(20)\cos 20°$

$z \approx 11.03$ miles

$\dfrac{\sin\theta}{27.45} = \dfrac{\sin 20°}{11.03}$

$\sin\theta \approx 0.8511$

$\theta = 180° - \sin^{-1}(0.8511)$

$\theta \approx 121.7°$

To find the bearing, we have $\theta - 10° - 90° \approx 21.7°$.

Bearing: S 21.7° E

5. If $\mathbf{u} \neq 0$, $\mathbf{v} \neq 0$, and $\mathbf{u} + \mathbf{v} \neq 0$, then $\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = \left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$ because all of these are magnitudes of unit vectors.

(a) $\mathbf{u} = \langle 1, -1\rangle, \quad \mathbf{v} = \langle -1, 2\rangle, \quad \mathbf{u} + \mathbf{v} = \langle 0, 1\rangle$

(i) $\|\mathbf{u}\| = \sqrt{2}$ (ii) $\|\mathbf{v}\| = \sqrt{5}$ (iii) $\|\mathbf{u} + \mathbf{v}\| = 1$ (iv) $\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$ (v) $\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$ (vi) $\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

(b) $\mathbf{u} = \langle 0, 1\rangle, \quad \mathbf{v} = \langle 3, -3\rangle, \quad \mathbf{u} + \mathbf{v} = \langle 3, -2\rangle$

(i) $\|\mathbf{u}\| = 1$ (ii) $\|\mathbf{v}\| = \sqrt{18} = 3\sqrt{2}$ (iii) $\|\mathbf{u} + \mathbf{v}\| = \sqrt{13}$ (iv) $\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$ (v) $\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$ (vi) $\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

(c) $\mathbf{u} = \left\langle 1, \dfrac{1}{2}\right\rangle, \mathbf{v} = \langle 2, 3\rangle, \mathbf{u} + \mathbf{v} = \left\langle 3, \dfrac{7}{2}\right\rangle$

(i) $\|\mathbf{u}\| = \dfrac{\sqrt{5}}{2}$ (ii) $\|\mathbf{v}\| = \sqrt{13}$ (iii) $\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \dfrac{49}{4}} = \dfrac{\sqrt{85}}{2}$ (iv) $\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$ (v) $\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$ (vi) $\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

(d) $\mathbf{u} = \langle 2, -4 \rangle, \quad \mathbf{v} = \langle 5, 5 \rangle, \quad \mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$

(i) $\|\mathbf{u}\| = \sqrt{20} = 2\sqrt{5}$ (ii) $\|\mathbf{v}\| = \sqrt{50} = 5\sqrt{2}$ (iii) $\|\mathbf{u} + \mathbf{v}\| = \sqrt{50} = 5\sqrt{2}$ (iv) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$ (v) $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$

(vi) $\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\| = 1$

7. Let $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$.

Then,
$$\begin{aligned} \mathbf{u} \cdot (c\mathbf{v} + d\mathbf{w}) &= \mathbf{u} \cdot c\mathbf{v} + \mathbf{u} \cdot d\mathbf{w} \\ &= c(\mathbf{u} \cdot \mathbf{v}) + d(\mathbf{u} \cdot \mathbf{w}) \\ &= c(0) + d(0) \\ &= 0. \end{aligned}$$

So for all scalars c and d, $\mathbf{u}$ is orthogonal to $c\mathbf{v} + d\mathbf{w}$.

9. (a)
$$\begin{aligned} z_1 &= 2(\cos 30° + i \sin 30°) \\ z_2 &= 2(\cos 150° + i \sin 150°) \\ z_3 &= 2(\cos 270° + i \sin 270°) \end{aligned}$$

(b)
$$\begin{aligned} z_1 &= 3(\cos 45° + i \sin 45°) \\ z_2 &= 3(\cos 135° + i \sin 135°) \\ z_3 &= 2(\cos 225° + i \sin 225°) \\ z_4 &= 2(\cos 315° + i \sin 315°) \end{aligned}$$

11. $\|\mathbf{u} + \mathbf{v}\|$ is larger in figure (a) because the angle between **u** and **v** is acute rather than obtuse as in figure (b). As the angle between the two vectors becomes more acute the magnitude becomes greater.

Practice Test for Chapter 6

For Exercises 1 and 2, use the Law of Sines to find the remaining sides and angles of the triangle.

1. $A = 40°, B = 12°, b = 100$

2. $C = 150°, a = 5, c = 20$

3. Find the area of the triangle: $a = 3, b = 6, C = 130°$.

4. Determine the number of solutions to the triangle: $a = 10, b = 35, A = 22.5°$.

For Exercises 5 and 6, use the Law of Cosines to find the remaining sides and angles of the triangle.

5. $a = 49, b = 53, c = 38$

6. $C = 29°, a = 100, c = 300$

7. Use Heron's Formula to find the area of the triangle: $a = 4.1, b = 6.8, c = 5.5$.

8. A ship travels 40 miles due east, then adjusts its course 12° southward. After traveling 70 miles in that direction, how far is the ship from its point of departure?

9. $\mathbf{w} = 4\mathbf{u} - 7\mathbf{v}$ where $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$. Find **w**.

10. Find a unit vector in the direction of $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$.

11. Find the dot product and the angle between $\mathbf{u} = 6\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

12. **v** is a vector of magnitude 4 making an angle of 30° with the positive x-axis. Find **v** in component form.

13. Find the projection of **u** onto **v** given $\mathbf{u} = \langle 3, -1 \rangle$ and $\mathbf{v} = \langle -2, 4 \rangle$.

14. Give the trigonometric form of $z = 5 - 5i$.

15. Give the standard form of $z = 6(\cos 225° + i \sin 225°)$.

16. Multiply $[7(\cos 23° + i \sin 23°)][4(\cos 7° + i \sin 7°)]$.

17. Divide $\dfrac{9\left(\cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4}\right)}{3(\cos \pi + i \sin \pi)}$.

18. Find $(2 + 2i)^8$.

19. Find the cube roots of $8\left(\cos \dfrac{\pi}{3} + i \sin \dfrac{\pi}{3}\right)$.

20. Find all the solutions to $x^4 + i = 0$.

CHAPTER 7
Systems of Equations and Inequalities

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CHAPTER 7

Systems of Equations and Inequalities

Section 7.1 Linear and Nonlinear Systems of Equations

1. system, equations

3. substitution

5. break-even point

7. $\begin{cases} 2x - y = 4 \\ 8x + y = -9 \end{cases}$

(a) $(0, -4)$

$8(0) - 4 \neq -9$

$(0, -4)$ *is not* a solution.

(b) $(3, -1)$

$2(3) - (-1) \neq 4$

$(3, -1)$ *is not* a solution.

(c) $\left(\frac{3}{2}, -1\right)$

$8\left(\frac{3}{2}\right) - 1 \neq -9$

$\left(\frac{3}{2}, -1\right)$ *is not* a solution.

(d) $\left(-\frac{1}{2}, -5\right)$

$$\begin{aligned} 2\left(-\tfrac{1}{2}\right) + 5 &\overset{?}{=} 4 \\ -1 + 5 &= 4 \end{aligned}$$

$$\begin{aligned} 8\left(-\tfrac{1}{2}\right) - 5 &\overset{?}{=} -9 \\ -4 - 5 &= -9 \end{aligned}$$

$\left(-\frac{1}{2}, -5\right)$ *is* a solution.

9. $\begin{cases} 2x + y = 6 & \text{Equation 1} \\ -x + y = 0 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = 6 - 2x$

Substitute for y in Equation 2: $-x + (6 - 2x) = 0$

Solve for x: $-3x + 6 = 0 \Rightarrow x = 2$

Back-substitute $x = 2$: $y = 6 - 2(2) = 2$

Solution: $(2, 2)$

11. $\begin{cases} x - y = -4 & \text{Equation 1} \\ x^2 - y = -2 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = x + 4$

Substitute for y in Equation 2: $x^2 - (x + 4) = -2$

Solve for x:
$x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x = -1, 2$

Back-substitute $x = -1$: $y = -1 + 4 = 3$

Back-substitute $x = 2$: $y = 2 + 4 = 6$

Solutions: $(-1, 3), (2, 6)$

13. $\begin{cases} x^2 + y = 0 & \text{Equation 1} \\ x^2 - 4x - y = 0 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = -x^2$

Substitute for y in Equation 2: $x^2 - 4x - (-x^2) = 0$

Solve for x:
$2x^2 - 4x = 0 \Rightarrow 2x(x - 2) = 0 \Rightarrow x = 0, 2$

Back-substitute $x = 0$: $y = -0^2 = 0$

Back-substitute $x = 2$: $y = -2^2 = -4$

Solutions: $(0, 0), (2, -4)$

15. $\begin{cases} y = x^3 - 3x^2 + 1 & \text{Equation 1} \\ y = x^2 - 3x + 1 & \text{Equation 2} \end{cases}$

Substitute for y in Equation 2:

$$\begin{aligned} x^3 - 3x^2 + 1 &= x^2 - 3x + 1 \\ x^3 - 4x^2 + 3x &= 0 \\ x(x - 1)(x - 3) &= 0 \Rightarrow x = 0, 1, 3 \end{aligned}$$

Back-substitute $x = 0$: $y = 0^3 - 3(0)^2 + 1 = 1$

Back-substitute $x = 1$: $y = 1^3 - 3(1)^2 + 1 = -1$

Back-substitute $x = 3$: $y = 3^3 - 3(3)^2 + 1 = 1$

Solutions: $(0, 1), (1, -1), (3, 1)$

17. $\begin{cases} x - y = 2 & \text{Equation 1} \\ 6x - 5y = 16 & \text{Equation 2} \end{cases}$

Solve for *x* in Equation 1: $x = y + 2$

Substitute for *x* in Equation 2: $6(y + 2) - 5y = 16 \Rightarrow 6y + 12 - 5y = 16 \Rightarrow y = 4$

Back-substitute $y = 4$: $x - 4 = 2 \Rightarrow x = 6$

Solution: $(6, 4)$

19. $\begin{cases} x + 4y = 3 & \text{Equation 1} \\ 2x - 7y = -24 & \text{Equation 2} \end{cases}$

Solve for *x* in Equation 1: $x = 3 - 4y$

Substitute for *x* in Equation 2: $2(3 - 4y) - 7y = -24 \Rightarrow 6 - 8y - 7y = -24 \Rightarrow y = 2$

Back-substitute $y = 2$: $x + 4(2) = 3 \Rightarrow x = -5$

Solution: $(-5, 2)$

21. $\begin{cases} 2x - y + 2 = 0 & \text{Equation 1} \\ 4x + y - 5 = 0 & \text{Equation 2} \end{cases}$

Solve for *y* in Equation 1: $y = 2x + 2$

Substitute for *y* in Equation 2: $4x + (2x + 2) - 5 = 0$

Solve for *x*: $6x - 3 = 0 \Rightarrow x = \frac{1}{2}$

Back-substitute $x = \frac{1}{2}$: $y = 2x + 2 = 2\left(\frac{1}{2}\right) + 2 = 3$

Solution: $\left(\frac{1}{2}, 3\right)$

23. $\begin{cases} 1.5x + 0.8y = 2.3 & \text{Equation 1} \\ 0.3x - 0.2y = 0.1 & \text{Equation 2} \end{cases}$

Multiply the equations by 10.

$$\begin{aligned} 15x + 8y &= 23 \quad \text{Revised Equation 1} \\ 3x - 2y &= 1 \quad \text{Revised Equation 2} \end{aligned}$$

Solve for *y* in revised Equation 2: $y = \frac{3}{2}x - \frac{1}{2}$

Substitute for *y* in revised Equation 1: $15x + 8\left(\frac{3}{2}x - \frac{1}{2}\right) = 23$

Solve for *x*: $15x + 12x - 4 = 23 \Rightarrow 27x = 27 \Rightarrow x = 1$

Back-substitute $x = 1$: $y = \frac{3}{2}(1) - \frac{1}{2} = 1$

Solution: $(1, 1)$

25. $\begin{cases} 0.5x + 3.2y = 9.0 & \text{Equation 1} \\ 0.2x - 1.6y = -3.6 & \text{Equation 2} \end{cases}$

Multiply the equations by 10.

$5x + 32y = 90$ Revised Equation 1

$2x - 16y = -36$ Revised Equation 2

Solve for x in revised Equation 2: $x = 8y - 18$

Substitute for x in revised Equation 1: $5(8y - 18) + 32y = 90$

Solve for y: $40y - 90 + 32y = 90 \Rightarrow 72y = 180 \Rightarrow y = \frac{5}{2}$

Back-substitute $y = \frac{5}{2}$: $x = 8\left(\frac{5}{2}\right) - 18 = 2$

Solution: $\left(2, \frac{5}{2}\right)$

27. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 & \text{Equation 1} \\ x + y = 20 & \text{Equation 2} \end{cases}$

Solve for x in Equation 2: $x = 20 - y$

Substitute for x in Equation 1: $\frac{1}{5}(20 - y) + \frac{1}{2}y = 8$

Solve for y: $4 + \frac{3}{10}y = 8 \Rightarrow y = \frac{40}{3}$

Back-substitute $y = \frac{40}{3}$: $x = 20 - y = 20 - \frac{40}{3} = \frac{20}{3}$

Solution: $\left(\frac{20}{3}, \frac{40}{3}\right)$

29. $\begin{cases} 6x + 5y = -3 & \text{Equation 1} \\ -x - \frac{5}{6}y = -7 & \text{Equation 2} \end{cases}$

Solve for x in Equation 2: $x = 7 - \frac{5}{6}y$

Substitute for x in Equation 1: $6\left(7 - \frac{5}{6}y\right) + 5y = -3$

Solve for y: $42 - 5y + 5y = -3 \Rightarrow 42 = -3$ (False)

No solution

31. $\begin{cases} \frac{1}{2}x + y = 5 & \text{Equation 1} \\ x - \frac{1}{2}y = 3 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = 5 - \frac{1}{2}x$

Substitute for y in Equation 2: $x - \frac{1}{2}\left(5 - \frac{1}{2}x\right) = 3 \Rightarrow \frac{5}{4}x = \frac{11}{2} \Rightarrow x = \frac{22}{5}$

Back-substitute $x = \frac{22}{5}$: $y = 5 - \frac{1}{2}\left(\frac{22}{5}\right) = \frac{14}{5}$

Solution: $\left(\frac{22}{5}, \frac{14}{5}\right)$

33. $\begin{cases} x + y = 12{,}000 \\ 0.02x + 0.06y = 500 \end{cases}$

Solve for y in Equation 1: $y = 12{,}000 - x$

Substitute for y in Equation 2: $0.02x + 0.06(12{,}000 - x) = 500$

Solve for x: $0.02x + 720 - 0.06x = 500$

$$-0.04x = -220$$

$$x = 5500$$

Back-substitute $x = 5500$: $y = 12{,}000 - 5500 = 6500$

So, \$5500 is invested at 2% and \$6500 is invested at 6%.

35. $\begin{cases} x + y = 12{,}000 \\ 0.028x + 0.038y = 396 \end{cases}$

Solve for y in Equation 1: $y = 12{,}000 - x$

Substitute for y in Equation 2: $0.028x + 0.038(12{,}000 - x) = 396$

Solve for x: $0.028x + 456 - 0.038x = 396$

$$-0.01x = -60$$

$$x = 6000$$

Back-substitute $x = 6000$: $y = 12{,}000 - 6000 = 6000$

So, \$6000 is invested at 2.8% and \$6000 is invested at 3.8%.

37. $\begin{cases} x^2 - y = 0 & \text{Equation 1} \\ 2x + y = 0 & \text{Equation 2} \end{cases}$

Solve for y in Equation 2: $y = -2x$

Substitute for y in Equation 1: $x^2 - (-2x) = 0$

Solve for x:

$$x^2 + 2x = 0 \Rightarrow x(x + 2) = 0 \Rightarrow x = 0, -2$$

Back-substitute $x = 0$: $y = -2(0) = 0$

Back-substitute $x = -2$: $y = -2(-2) = 4$

Solutions: $(0, 0), (-2, 4)$

39. $\begin{cases} x - y = -1 & \text{Equation 1} \\ x^2 - y = -4 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = x + 1$

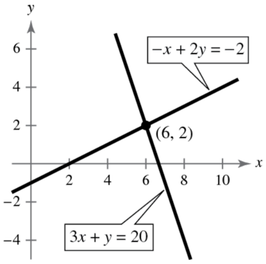
Substitute for y in Equation 2: $x^2 - (x + 1) = -4$

Solve for x: $x^2 - x - 1 = -4 \Rightarrow x^2 - x + 3 = 0$

The Quadratic Formula yields no real solutions.

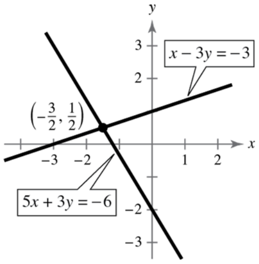
41. $\begin{cases} -x + 2y = -2 \\ 3x + y = 20 \end{cases}$

Point of intersection: $(6, 2)$



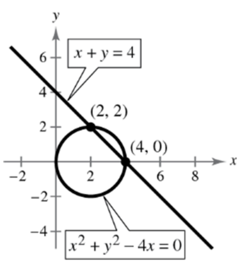
43. $\begin{cases} x - 3y = -3 \\ 5x + 3y = -6 \end{cases}$

Point of intersection: $\left(-\frac{3}{2}, \frac{1}{2}\right)$

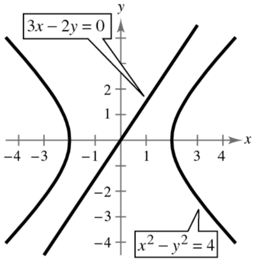


45. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

Points of intersection: $(2, 2), (4, 0)$

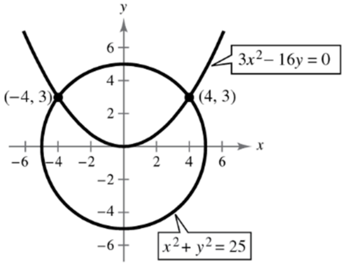


47. $\begin{cases} 3x - 2y = 0 \\ x^2 - y^2 = 4 \end{cases}$



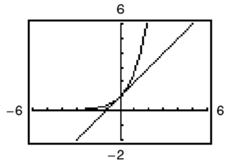
No points of intersection ⇒ No solution

49. $\begin{cases} x^2 + y^2 = 25 \\ 3x^2 - 16y = 0 \end{cases}$



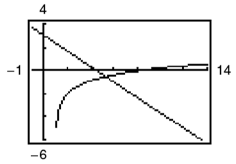
Points of intersection: $(-4, 3), (4, 3)$

51. $\begin{cases} y = e^x \\ x - y + 1 = 0 \Rightarrow y = x + 1 \end{cases}$



Point of intersection: $(0, 1)$

53. $\begin{cases} y = -2 + \ln(x - 1) \\ 3y + 2x = 9 \Rightarrow y = -\frac{2}{3}x + 3 \end{cases}$



Point of intersection: $(5.31, -0.54)$

55. $\begin{cases} y = 2x & \text{Equation 1} \\ y = x^2 + 1 & \text{Equation 2} \end{cases}$

Substitute for y in Equation 2: $2x = x^2 + 1$

Solve for x: $x^2 - 2x + 1 = (x - 1)^2 = 0 \Rightarrow x = 1$

Back-substitute $x = 1$ in Equation 1: $y = 2x = 2$

57. $\begin{cases} x - 2y = 4 & \text{Equation 1} \\ x^2 - y = 0 & \text{Equation 2} \end{cases}$

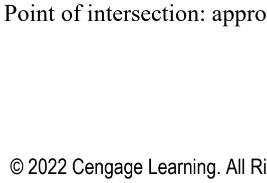
Solve for y in Equation 2: $y = x^2$

Substitute for y in Equation 1: $x - 2x^2 = 4$

Solve for x: $0 = 2x^2 - x + 4 \Rightarrow x = \dfrac{1 \pm \sqrt{1 - 4(2)(4)}}{2(2)} \Rightarrow x = \dfrac{1 \pm \sqrt{-31}}{4}$

The discriminant in the Quadratic Formula is negative.
No real solution

59. $\begin{cases} y - e^{-x} = 1 \Rightarrow y = e^{-x} + 1 \\ y - \ln x = 3 \Rightarrow y = \ln x + 3 \end{cases}$



Point of intersection: approximately $(0.287), (1.751)$

61. $\begin{cases} xy - 1 = 0 & \text{Equation 1} \\ 2x - 4y + 7 = 0 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = \dfrac{1}{x}$

Substitute for y in Equation 2: $2x - 4\left(\dfrac{1}{x}\right) + 7 = 0$

Solve for x: $2x^2 - 4 + 7x = 0 \Rightarrow (2x - 1)(x + 4) = 0 \Rightarrow x = \dfrac{1}{2}, -4$

Back-substitute $x = \dfrac{1}{2}$: $y = \dfrac{1}{1/2} = 2$

Back-substitute $x = -4$: $y = \dfrac{1}{-4} = -\dfrac{1}{4}$

Solutions: $\left(\dfrac{1}{2}, 2\right), \left(-4, -\dfrac{1}{4}\right)$

63. $C = 8650x + 250{,}000$, $R = 9502x$

$$\begin{aligned} R &= C \\ 9502x &= 8650x + 250{,}000 \\ 852x &= 250{,}000 \\ x &\approx 293 \text{ units} \end{aligned}$$

65. $C = 9.45x + 16{,}000$; $R = 55.95x$

(a) $$\begin{aligned} R &= C \\ 55.95x &= 9.45x + 16{,}000 \\ 46.5x &= 16{,}000 \\ x &\approx 344 \end{aligned}$$

About 344 units must be sold to break even.

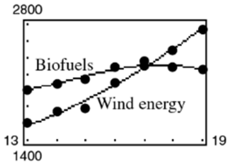
(b) $$\begin{aligned} P &= R - C \\ 100{,}000 &= 55.95x - (9.45x + 16{,}000) \\ 100{,}000 &= 46.5x - 16{,}000 \\ 116{,}000 &= 46.5x \\ x &\approx 2495 \end{aligned}$$

About 2495 units must be sold to earn a $100,000 profit.

67. (a) Biofuels: $C_1 = -3.250t^3 + 141.96t^2 - 1978.3t + 10{,}874$

Wind energy: $C_2 = 11.20t^2 - 162.5t + 1797$

(b)



(c) The graphs intersect at approximately (17.16, 2306.34), which means during 2017, the average rate of consumption of biofuels and wind energy were approximately equal to 2306.34 Btus.

(d) Answers will vary. *Sample answer*: The model for wind energy consumption is increasing and the model for biofuels consumption increases and then decreases.

69. $$\begin{aligned} 2l + 2w &= 56 \;\Rightarrow\; & l + w &= 28 \\ l &= w + 4 \;\Rightarrow & (w + 4) + w &= 28 \\ & & 2w + 4 &= 28 \\ & & 2w &= 24 \\ & & w &= 12 \text{ meters} \end{aligned}$$

$l = w + 4 = 12 + 4 = 16$ meters
Dimensions: 12 meters × 16 meters

71. False. To solve a system of equations by substitution, you can solve for either variable in one of the two equations and then back-substitute.

73. *Sample answer*: If the result is a contradictory equation such as $0 = N$, then you know there are no solutions. When solving a system of equations that is a nonlinear system, there may be an equation with imaginary or extraneous solutions.

75. (a) There are 0, 1, or 2 solutions. The horizontal line $x = a$ passes below the parabola when $a < 0$, through the vertex of the parabola when $a = 0$, or through two points on the parabola when $a > 0$.

(b) There is exactly 1 solution. Any vertical line $x = a$ intersects the parabola at exactly one point.

77. Answers will vary.

79. $$\begin{aligned} 4(5x + y) &= 4(-1) \\ 20x + 4y &= -4 \end{aligned}$$

81. $$\begin{aligned} 3(2x - 4y) &= 3(14) \\ 6x - 12y &= 42 \end{aligned}$$

83. $$\begin{aligned} 100(0.02x - 0.05y) &= 100(-0.38) \\ 2x - 5y &= -38 \end{aligned}$$

85. $$\begin{aligned} 3x + 5x + 2y - 2y &= 4 + 12 \\ 8x &= 16 \\ x &= 2 \end{aligned}$$

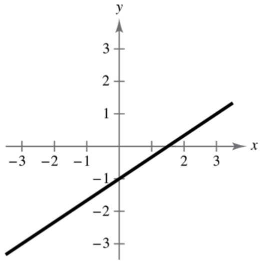
87. $$\begin{aligned} 2x - 4y + 20x + 4y &= (-7) + (-4) \\ 22x &= -11 \\ x &= -\tfrac{1}{2} \end{aligned}$$

89. $$\begin{aligned} 20x + 12y + 6x - 12y &= 4(9) + 3(14) \\ 26x &= 36 + 42 \\ 26x &= 78 \\ x &= 3 \end{aligned}$$

91. $$\begin{aligned} 2(3y + 2x) - 3(2y + 5x) &= 7(2) - 3 \\ 6y + 4x - 6y - 15x &= 14 - 3 \\ 4x - 15x &= 11 \\ -11x &= 11 \\ x &= -1 \end{aligned}$$

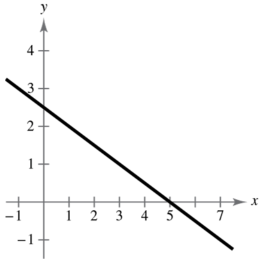
93. $2x - 3y = 3$

Intercepts: $(0, -1), \left(\frac{3}{2}, 0\right)$



95. $x + 2y = 5$

Intercepts: $\left(0, \frac{5}{2}\right), (5, 0)$



Section 7.2 Two-Variable Linear Systems

1. elimination

3. equilibrium point

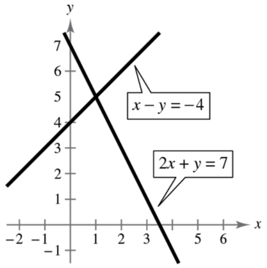
5. A system of linear equations with no solution is inconsistent.

7. $$\begin{cases} 2x + y = 7 & \text{Equation 1} \\ x - y = -4 & \text{Equation 2} \end{cases}$$

Add to eliminate y: $$\begin{array}{rcrcr} 2x & + & y & = & 7 \\ x & - & y & = & -4 \\ \hline 3x & & & = & 3 \Rightarrow x = 1 \end{array}$$

Substitute $x = 1$ in Equation 2: $1 - y = -4 \Rightarrow y = 5$

Solution: $(1, 5)$



9. $\begin{cases} x + y = 0 & \text{Equation 1} \\ 3x + 2y = 1 & \text{Equation 2} \end{cases}$

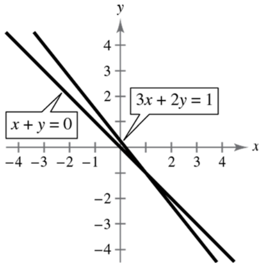
Multiply Equation 1 by −2: $-2x - 2y = 0$

Add this to Equation 2 to eliminate y:

$$\begin{array}{rcrcr} -2x & - & 2y & = & 0 \\ 3x & + & 2y & = & 1 \\ \hline x & & & = & 1 \end{array}$$

Substitute $x = 1$ in Equation 1: $1 + y = 0 \Rightarrow y = -1$

Solution: $(1, -1)$

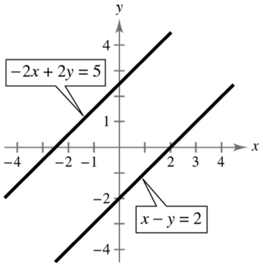


11. $\begin{cases} x - y = 2 & \text{Equation 1} \\ -2x + 2y = 5 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 2: $2x - 2y = 4$

Add this to Equation 2:
$$\begin{array}{rcrcr} 2x & - & 2y & = & 4 \\ -2x & + & 2y & = & 5 \\ \hline & & 0 & = & 9 \end{array}$$

There are no solutions.



13. $\begin{cases} 3x - 2y = 5 & \text{Equation 1} \\ -6x + 4y = -10 & \text{Equation 2} \end{cases}$

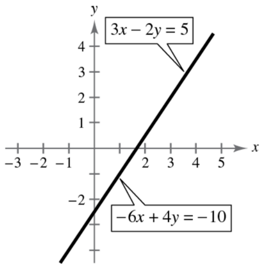
Multiply Equation 1 by 2: $6x - 4y = 10$

Add this to Equation 2:
$$\begin{array}{rcrcr} 6x & - & 4y & = & 10 \\ -6x & + & 4y & = & -10 \\ \hline & & 0 & = & 0 \end{array}$$

The equations are dependent. There are infinitely many solutions.

Let $x = a$, then $y = \dfrac{3a - 5}{2} = \dfrac{3}{2}a - \dfrac{5}{2}$.

Solution: $\left(a, \dfrac{3}{2}a - \dfrac{5}{2}\right)$, where a is any real number.



15. $\begin{cases} x + 2y = 6 & \text{Equation 1} \\ x - 2y = 2 & \text{Equation 2} \end{cases}$

Add the equations to eliminate y:
$$\begin{array}{rcrcl} x & + & 2y & = & 6 \\ x & - & 2y & = & 2 \\ \hline 2x & & & = & 8 \Rightarrow x = 4 \end{array}$$

Substitute $x = 4$ into Equation 1:
$4 + 2y = 6 \Rightarrow y = 1$

Solution: $(4, 1)$

17. $\begin{cases} 5x + 3y = 6 & \text{Equation 1} \\ 3x - y = 5 & \text{Equation 2} \end{cases}$

Multiply Equation 2 by 3: $9x - 3y = 15$

Add this to Equation 1 to eliminate y:
$$\begin{array}{rcrcl} 5x & + & 3y & = & 6 \\ 9x & - & 3y & = & 15 \\ \hline 14x & & & = & 21 \Rightarrow x = \frac{3}{2} \end{array}$$

Substitute $x = \frac{3}{2}$ into Equation 1:

$5\left(\frac{3}{2}\right) + 3y = 6 \Rightarrow y = -\frac{1}{2}$

Solution: $\left(\frac{3}{2}, -\frac{1}{2}\right)$

19. $\begin{cases} 2u + 3v = -1 & \text{Equation 1} \\ 7u + 15v = 4 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by -5 and add to Equation 2.

$$\begin{cases} -10u - 15v = 5 \\ 7u + 15v = 4 \end{cases}$$

Solve for u: $-3u = 9 \Rightarrow u = -3$

Substitute $u = -3$ in Equation 1:

$$2(-3) + 3v = -1 \Rightarrow v = \frac{5}{3}$$

Solution: $\left(-3, \frac{5}{3}\right)$

21. $\begin{cases} 3x + 2y = 10 & \text{Equation 1} \\ 2x + 5y = 3 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 2 and Equation 2 by -3:

$$\begin{cases} 6x + 4y = 20 \\ -6x - 15y = -9 \end{cases}$$

Add to eliminate x: $-11y = 11 \Rightarrow y = -1$

Substitute $y = -1$ in Equation 1:

$3x - 2 = 10 \Rightarrow x = 4$

Solution: $(4, -1)$

23. $\begin{cases} 4b + 3m = 3 & \text{Equation 1} \\ 3b + 11m = 13 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 3 and Equation 2 by -4:

$$\begin{cases} 12b + 9m = 9 \\ -12b - 44m = -52 \end{cases}$$

Add to eliminate b: $-35m = -43 \Rightarrow m = \frac{43}{35}$

Substitute $m = \frac{43}{35}$ in Equation 1:

$$4b + 3\left(\tfrac{43}{35}\right) = 3 \Rightarrow b = -\tfrac{6}{35}$$

Solution: $\left(-\frac{6}{35}, \frac{43}{35}\right)$

25. $\begin{cases} 0.2x - 0.5y = -27.8 & \text{Equation 1} \\ 0.3x + 0.4y = 68.7 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 4 and Equation 2 by 5:

$$\begin{cases} 0.8x - 2y = -111.2 \\ 1.5x + 2y = 343.5 \end{cases}$$

Add these to eliminate y:

$$\begin{array}{rcrcr} 0.8x & - & 2y & = & -111.2 \\ 1.5x & + & 2y & = & 343.5 \\ \hline 2.3x & & & = & 232.3 \\ & & x & = & 101 \end{array}$$

Substitute $x = 101$ in Equation 1:

$0.2(101) - 0.5y = -27.8 \Rightarrow y = 96$

Solution: $(101, 96)$

27. $\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 9x + 6y = 3 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by -3 and add to Equation 2.

$$\begin{cases} -9x - 6y = -12 \\ 9x + 6y = 3 \end{cases}$$

Add:

$$\begin{array}{rcrcr} -9x & - & 6y & = & -12 \\ 9x & + & 6y & = & 3 \\ \hline & & 0 & \neq & -9 \end{array}$$

No solution.

29. $\begin{cases} -5x + 6y = -3 & \text{Equation 1} \\ 20x - 24y = 12 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 4:

$$\begin{cases} -20x + 24y = -12 \\ 20x - 24y = 12 \end{cases}$$

Add these two together: $0 = 0$

The equations are dependent. There are infinitely many solutions.

Let $x = a$, then

$$-5a + 6y = -3 \Rightarrow y = \frac{5a - 3}{6} = \frac{5}{6}a - \frac{1}{2}.$$

Solution: $\left(a, \frac{5}{6}a - \frac{1}{2}\right)$, where a is any real number

31. $\begin{cases} \dfrac{x+3}{4} + \dfrac{y-1}{3} = 1 & \text{Equation 1} \\ 2x - y = 12 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 12 and Equation 2 by 4:

$\begin{cases} 3x + 4y = 7 \\ 8x - 4y = 48 \end{cases}$

Add to eliminate y: $11x = 55 \Rightarrow x = 5$

Substitute $x = 5$ into Equation 2:

$2(5) - y = 12 \Rightarrow y = -2$

Solution: $(5, -2)$

33. $\begin{cases} -7x + 6y = -4 \\ 14x - 12y = 8 \end{cases}$

Multiply Equation 1 by 2:

$\begin{cases} -14x + 12y = -8 \\ 14x - 12y = 8 \end{cases}$

Add this to Equation 2: $0 = 0$

The original equations are dependent.

Matches graph (a).

Number of solutions: Infinite

Consistent

35. $\begin{cases} 7x - 6y = -6 \\ -7x + 6y = -4 \end{cases}$

Add the equations: $0 = -10$

Inconsistent

Matches graph (d).

Number of solutions: None

Inconsistent

37. $\begin{cases} 3x - 5y = 7 & \text{Equation 1} \\ 2x + y = 9 & \text{Equation 2} \end{cases}$

Multiply Equation 2 by 5:

$10x + 5y = 45$

Add this to Equation 1:

$13x = 52 \Rightarrow x = 4$

Back-substitute $x = 4$ into Equation 2:

$2(4) + y = 9 \Rightarrow y = 1$

Solution: $(4, 1)$

39. $\begin{cases} -2x + 8y = 20 & \text{Equation 1} \\ y = x - 5 & \text{Equation 2} \end{cases}$

Substitute Equation 2 into Equation 1:

$$\begin{aligned} -2x + 8(x-5) &= 20 \\ -2x + 8x - 40 &= 20 \\ 6x &= 60 \\ x &= 10 \end{aligned}$$

Back-substitute $x = 10$ into Equation 2:

$y = 10 - 5 = 5$

Solution: $(10, 5)$

41. $\begin{cases} y = -2x - 17 & \text{Equation 1} \\ y = 2 - 3x & \text{Equation 2} \end{cases}$

Use substitution because both equations are solved for y, set them equal to one another and solve for x.

$$\begin{aligned} -2x - 17 &= 2 - 3x \\ x &= 19 \end{aligned}$$

Back-substitute $x = 19$ into Equation 1:

$y = -2(19) - 17 = -55$

Solution: $(19, -15)$

43. Let r_1 = the air speed of the plane and

r_2 = the wind air speed.

$\begin{cases} 3(r_1 - r_2) = 1500 & \text{Equation 1} \\ 2.5(r_1 + r_2) = 1500 & \text{Equation 2} \end{cases}$

Simplify:

$\begin{cases} r_1 - r_2 = 500 & \text{Revised Equation 1} \\ r_1 + r_2 = 600 & \text{Revised Equation 2} \end{cases}$

Add to eliminate r_2: $2r_1 = 1100 \Rightarrow r_1 = 550$

Back-substitute into revised Equation 2:

$r_2 = 600 - r_1 \Rightarrow r_2 = 50$

The air speed of the plane is 550 miles per hour and the speed of wind is 50 miles per hour.

45. Let x = the number of calories in a cheeseburger.

Let y = the number of calories in a small order of french fries.

$\begin{cases} 2x + y = 1460 & \text{Equation 1} \\ 3x + 2y = 2350 & \text{Equation 2} \end{cases}$

Solve for y in Equation 1: $y = 1460 - 2x$

Substituting for y in Equation 2:

$3x + 2(1460 - 2x) = 2350 \Rightarrow -x = -570 \Rightarrow x = 570$

Back-substitute $x = 570$: $y = 1460 - 2(570) = 320$

The cheeseburger contains 570 calories and the fries contain 320 calories.

47. $$\begin{aligned} 500 - 0.4x &= 380 + 0.1x \\ 120 &= 0.5x \\ x &= 240 \text{ units} \\ p &= \$404 \end{aligned}$$

Equilibrium point: $(240, 404)$

49. Let $x =$ the amount of money invested at 3.5%.

Let $y =$ the amount of money invested at 5%.

$$\begin{cases} x + y = 24{,}000 & \text{Equation 1} \\ 0.035x + 0.05y = 930 & \text{Equation 2} \end{cases}$$

Solve Equation 1 for x: $x = 24{,}000 - y$

Substitute this into Equation 2 to eliminate x:

$$\begin{aligned} 0.035(24{,}000 - y) + 0.05y &= 930 \\ 840 + 0.015y &= 930 \\ y &= \$6000 \end{aligned}$$

Back-substitute $y = 6000$ into Equation 1:

$$\begin{aligned} x + 6000 &= 24{,}000 \\ x &= \$18{,}000 \end{aligned}$$

\$18,000 should be invested in the 3.5% bond.

51. (a) $$\begin{cases} 3.00b + 3.70a = 105.00 & \text{Equation 1} \\ 3.70b + 4.69a = 123.90 & \text{Equation 2} \end{cases}$$

Solve Equation 1 for b: $b = 35 - \dfrac{3.70}{3}a$

Substitute this into Equation 2 to eliminate b:

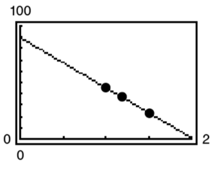
$$\begin{aligned} 3.70\left(35 - \frac{3.70}{3}a\right) + 4.69a &= 123.90 \\ a &\approx -44.21 \end{aligned}$$

Back-substitute $a \approx -44.21$ into Equation 1:

$$3.00b + 3.70(-44.21) = 105.00 \Rightarrow b \approx 89.53$$

Least squares regression line:

$y = -44.21x + 89.53$



(b) $$\begin{aligned} y &= -44.21x + 89.53 \\ y &= -44.21(1.75) + 89.53 \\ y &= 12.16 \end{aligned}$$

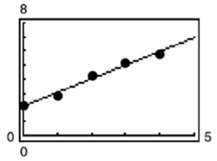
When the price is \$1.75, the demand is about 12 units.

53. $$\begin{cases} 5b + 10a = 20.2 \Rightarrow & b + 2a = 4.04 \\ 10b + 30a = 50.1 \Rightarrow & -b - 3a = -5.01 \end{cases}$$

$$\begin{aligned} -a &= -0.97 \\ a &= 0.97 \end{aligned}$$

$$\begin{aligned} b + 2a &= 4.04 \\ b + 2(0.97) &= 4.04 \\ b &= 2.1 \end{aligned}$$

Least squares regression line: $y = 0.97x + 2.1$



55. (a) $$\begin{cases} 4b + 7.0a = 174 \Rightarrow & 28b + 49.0a = 1218 \\ 7b + 13.5a = 322 \Rightarrow & -28b - 54.0a = -1288 \end{cases}$$

$$\begin{aligned} -5a &= -70 \\ a &= 14 \end{aligned}$$

$$\begin{aligned} 4b + 7.0a &= 174 \\ 4b + 7.0(14) &= 174 \\ 4b &= 76 \\ b &= 19 \end{aligned}$$

Least squares regression line: $y = 14x + 19$

(b) Substitute $x = 1.6$ into $y = 14x + 19$.

$y = 14(1.6) + 19 = 41.4$

The wheat yield is about 41.4 bushels per acre.

57. False. Two lines that coincide have infinitely many points of intersection.

59. $$\begin{cases} 4x - 8y = -3 & \text{Equation 1} \\ 2x + ky = 16 & \text{Equation 2} \end{cases}$$

Multiply Equation 2 by –2: $-4x - 2ky = -32$

Add this to Equation 1: $$\begin{aligned} 4x - 8y &= -3 \\ -4x - 2ky &= -32 \\ \hline -8y - 2ky &= -35 \end{aligned}$$

The system is inconsistent if $-8y - 2ky = 0$.

This occurs when $k = -4$.

61. $\begin{cases} 3x + 2y = 4 \\ 5x - 2y = 12 \end{cases}$

$$2y = -3x + 4$$
$$y = -\tfrac{3}{2}x + 2$$

$$\begin{aligned} 5x - 2\left(-\tfrac{3}{2}x + 2\right) &= 12 \\ 5x + 3x - 4 &= 12 \\ 8x &= 16 \\ x &= 2 \end{aligned}$$

Back substitute $x = 2$: $3(2) + 2y = 4$

$$\begin{aligned} 6 + 2y &= 4 \\ 2y &= -2 \\ y &= -1 \end{aligned}$$

Solution: $(2, -1)$

Answers will vary: *Sample answer*: If the equations can be added or subtracted without having to multiply by any coefficient, elimination of variable may be preferred. If one or both of the equations is already solved for one of the variables, the method of substitution may be more efficient.

63. $\begin{cases} 100y - x = 200 & \text{Equation 1} \\ 99y - x = -198 & \text{Equation 2} \end{cases}$

Subtract Equation 2 from Equation 1 to eliminate x:

$$\begin{array}{rcrcr} 100y & - & x & = & 200 \\ -99y & + & x & = & 198 \\ \hline y & & & = & 398 \end{array}$$

Substitute $y = 398$ into Equation 1:

$100(398) - x = 200 \Rightarrow x = 39{,}600$

Solution: $(39, 600, 398)$

The lines are not parallel. The scale on the axes must be changed to see the point of intersection.

65. No, it is not possible for a consistent system of linear equations to have exactly two solutions. Either the lines will intersect once or they will coincide and then the system would have infinite solutions.

67. $\begin{cases} u \sin x + v \cos x = 0 & \text{Equation 1} \\ u \cos x - v \sin x = \sec x & \text{Equation 2} \end{cases}$

Multiply Equation 1 by $\cos x$ and multiply Equation 2 by $-\sin x$. Then add the equations to eliminate u.

$$\begin{array}{l} u \sin x \cos x + v \cos^2 x = 0 \\ -u \sin x \cos x + v \sin^2 x = -\sin x \sec x \\ \hline v\left(\sin^2 x + \cos^2 x\right) = -\sin x \sec x \end{array}$$

$$v = -\sin x \sec x = -\sin x\left(\frac{1}{\cos x}\right) = -\tan x$$

Back substitute v into Equation 1

$$\begin{aligned} u \sin x + (-\tan x) \cos x &= 0 \\ u \sin x - \left(\frac{\sin x}{\cos x}\right) \cos x &= 0 \\ u \sin x - \sin x &= 0 \\ u \sin x &= \sin x \\ u &= 1 \end{aligned}$$

The solution of this system is: $u = 1, v = -\tan x$.

69. $2x^2 - 3x = \dfrac{8x^2 + 12}{4}$

$$\begin{aligned} 2x^2 - 3x &= 2x^2 + 3 \\ -3x &= 3 \\ x &= -1 \end{aligned}$$

71.
$$\begin{aligned} 6x + 5 &= x(2x + 3) \\ 6x + 5 &= 2x^2 + 3x \\ 2x^2 - 3x - 5 &= 0 \\ (x + 1)(2x - 5) &= 0 \\ x &= -1, \tfrac{5}{2} \end{aligned}$$

73. $$\begin{aligned} 5x^2 - 6 - 3x^4 &= 3x - 5x^2 - 2x^4 \\ 10x^2 - 9 - x^4 &= 0 \\ x^4 - 10x^2 + 9 &= 0 \\ (x^2 - 9)(x^2 - 1) &= 0 \\ x &= \pm 3, \pm 1 \end{aligned}$$

75. $$\begin{aligned} 11x - 4y &= 19 \\ -4y &= -11x + 19 \\ y &= \frac{11}{4}x - \frac{19}{4} \end{aligned}$$

77. $$\begin{aligned} \frac{1}{5}x + 1 &= \frac{1}{2}x + \frac{1}{5}y \\ 2x + 10 &= 5x + 2y \\ -3x + 10 &= 2y \\ y &= \frac{-3}{2}x + 5 \end{aligned}$$

79. $$\begin{aligned} f(x) &= x^2 - 12x + 24 = 0 \\ x &= \frac{12 \pm \sqrt{(-12)^2 - 4(24)}}{2} \\ &= 6 \pm \frac{\sqrt{48}}{2} \\ &= 6 \pm 2\sqrt{3} \end{aligned}$$

81. $$\begin{aligned} f(x) &= 6x^3 - 7x^2 + 6x + 7 = 0 \\ x^2(6x - 7) - (6x - 7) &= 0 \\ (x^2 - 1)(6x - 7) &= 0 \\ x &= \pm 1, \tfrac{7}{6} \end{aligned}$$

Section 7.3 Multivariable Linear Systems

1. row-echelon

3. Gaussian

5. To produce an equivalent system of linear equations, you can interchange two equations, multiply one of the equations by a nonzero constant, or add a multiple of one equation to another equation to replace the linear equation.

7. $$\begin{cases} 6x - y + z = -1 \\ 4x - 3z = -19 \\ 2y + 5z = 25 \end{cases}$$

(a) $(0, 3, 1)$

$6(0) - (3) + (1) \neq 1$

$(0, 3, 1)$ *is not* a solution.

(b) $(-3, 0, 5)$

$6(-3) - 0 + 5 \neq -1$

$(-3, 0, 5)$ *is not* a solution

(c) $(0, -1, 4)$

$4(0) - 3(4) \neq -19$

$(0, -1, 4)$ *is not* a solution.

(d) $(-1, 0, 5)$

$$\begin{aligned} 6(-1) - 0 + 5 &= -1 \\ 4(-1) - 3(5) &= -19 \\ 2(0) + 5(5) &= 25 \end{aligned}$$

$(-1, 0, 5)$ *is* a solution.

9. $$\begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$$

(a) $4\left(\frac{1}{2}\right) + \left(-\frac{3}{4}\right) - \left(-\frac{7}{4}\right) \neq 0$

$\left(\frac{1}{2}, -\frac{3}{4}, -\frac{7}{4}\right)$ *is not* a solution.

(b) $4\left(\frac{3}{2}\right) + \left(-\frac{2}{5}\right) - \left(\frac{3}{5}\right) \neq 0$

$\left(\frac{3}{2}, -\frac{2}{5}, \frac{3}{5}\right)$ *is not* a solution.

(c) $$\begin{aligned} 4\left(-\tfrac{1}{2}\right) + \left(\tfrac{3}{4}\right) - \left(-\tfrac{5}{4}\right) &= 0 \\ -8\left(-\tfrac{1}{2}\right) - 6\left(\tfrac{3}{4}\right) + \left(-\tfrac{5}{4}\right) &= -\tfrac{7}{4} \\ 3\left(-\tfrac{1}{2}\right) - \left(\tfrac{3}{4}\right) &= -\tfrac{9}{4} \end{aligned}$$

$\left(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4}\right)$ *is* a solution.

(d) $4\left(-\frac{1}{2}\right) + \left(\frac{1}{6}\right) - \left(-\frac{3}{4}\right) \neq 0$

$\left(-\frac{1}{2}, \frac{1}{6}, -\frac{3}{4}\right)$ *is not* a solution.

11. $$\left\{\begin{array}{rcrcrcrl} x & - & y & + & 5z & = & 37 & \text{Equation 1} \\ & & y & + & 2z & = & 6 & \text{Equation 2} \\ & & & & z & = & 8 & \text{Equation 3} \end{array}\right.$$

Back-substitute $z = 8$ into Equation 2:

$$\begin{aligned} y + 2(8) &= 6 \\ y &= -10 \end{aligned}$$

Back-substitute $y = -10$ and $z = 8$ into Equation 1:

$$\begin{aligned} x - (-10) + 5(8) &= 37 \\ x + 10 + 40 &= 37 \\ x &= -13 \end{aligned}$$

Solution: $(-13, -10, 8)$

13. $$\left\{\begin{array}{rcrcrcrl} x & + & y & - & 3z & = & 7 & \text{Equation 1} \\ & & y & + & z & = & 12 & \text{Equation 2} \\ & & & & z & = & 2 & \text{Equation 3} \end{array}\right.$$

Back-substitute $z = 2$ into Equation 2: $y + 2 = 12 \Rightarrow y = 10$

Back-substitute $y = 10$ and $z = 2$ into Equation 1:

$$\begin{aligned} x + (10) - 3(2) &= 7 \\ x + 4 &= 7 \\ x &= 3 \end{aligned}$$

Solution: $(3, 10, 2)$

15. $$\left\{\begin{array}{rcrcrcrl} x & - & 2y & + & z & = & -\frac{1}{4} & \text{Equation 1} \\ & & y & - & z & = & -4 & \text{Equation 2} \\ & & & & z & = & 11 & \text{Equation 3} \end{array}\right.$$

Back-substitute $z = 11$ into Equation 2:

$$\begin{aligned} y - 11 &= -4 \\ y &= 7 \end{aligned}$$

Back-substitute $y = 7$ and $z = 11$ into Equation 1:

$$\begin{aligned} x - 2(7) + (11) &= -\tfrac{1}{4} \\ x - 3 &= -\tfrac{1}{4} \\ x &= \tfrac{11}{4} \end{aligned}$$

Solution: $\left(\frac{11}{4}, 7, 11\right)$

17. $$\left\{\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 5 & \text{Equation 1} \\ -x & + & 3y & - & 5z & = & 4 & \text{Equation 2} \\ 2x & & & - & 3z & = & 0 & \text{Equation 3} \end{array}\right.$$

Add Equation 1 to Equation 2:

$$\left\{\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 5 \\ & & y & - & 2z & = & 9 \\ 2x & & & - & 3z & = & 0 \end{array}\right.$$

This is the first step in putting the system in row-echelon form.

19. $$\left\{\begin{array}{rcrcr} x & + & y & = & 0 \\ -2x & + & 3y & = & 10 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrl} x & + & y & = & 0 & \\ & & 5y & = & 10 & \quad 2\,\text{Eq. }1 + \text{Eq. }2 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrl} x & + & y & = & 0 & \\ & & y & = & 2 & \quad \tfrac{1}{5}\,\text{Eq. }2 \end{array}\right.$$

$$\begin{aligned} x + (2) &= 0 \\ x &= -2 \end{aligned}$$

Solution: $(-2, 2)$

21. $$\left\{\begin{array}{rcrcr} x & - & 2y & = & -2 \\ 3x & - & y & = & 9 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrl} x & - & 2y & = & -2 & \\ & & 5y & = & 15 & \quad (-3)\text{Eq. }1 + \text{Eq. }2 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrl} x & - & 2y & = & -2 & \\ & & y & = & 3 & \quad \tfrac{1}{5}\,\text{Eq. }2 \end{array}\right.$$

$$\begin{aligned} x - 2(3) &= -2 \\ x &= 4 \end{aligned}$$

Solution: $(4, 3)$

23.
$$\left\{\begin{array}{rcl} x + y + z & = & 7 \\ 2x - y + z & = & 9 \\ 3x - z & = & 10 \end{array}\right. \quad \begin{array}{l}\text{Equation 1}\\ \text{Equation 2}\\ \text{Equation 3}\end{array}$$

$$\left\{\begin{array}{rcl} x + y + z & = & 7 \\ 3x + 2z & = & 16 \\ 3x - z & = & 10 \end{array}\right. \quad \begin{array}{l}\\ \text{Eq. 2} + \text{Eq. 1}\\ \\ \end{array}$$

$$\left\{\begin{array}{rcl} x + y + z & = & 7 \\ 3x + 2z & = & 16 \\ 9x & = & 36 \end{array}\right. \quad \begin{array}{l}\\ \\ \text{Eq. 2} + 2\text{Eq. 3}\end{array}$$

$$\left\{\begin{array}{rcl} x + y + z & = & 7 \\ 3x + 2z & = & 16 \\ x & = & 4 \end{array}\right. \quad \begin{array}{l}\\ \\ \tfrac{1}{4}\text{ Eq. 3}\end{array}$$

$$\begin{aligned} 3(4) + 2z &= 16 \\ 2z &= 4 \\ z &= 2 \\ 4 + y + 2 &= 7 \\ y &= 1 \end{aligned}$$

Solution: $(4, 1, 2)$

25.
$$\left\{\begin{array}{rcl} 2x + 4y - z & = & 7 \\ 2x - 4y + 2z & = & -6 \\ x + 4y + z & = & 0 \end{array}\right.$$

$$\left\{\begin{array}{rcl} x + 4y + z & = & 0 \\ 2x - 4y + 2z & = & -6 \\ 2x + 4y - z & = & 7 \end{array}\right. \quad \begin{array}{l}\text{Interchange equations.}\\ \\ \\ \end{array}$$

$$\left\{\begin{array}{rcl} x + 4y + z & = & 0 \\ -12y & = & -6 \\ -4y - 3z & = & 7 \end{array}\right. \quad \begin{array}{l}\\ (-2)\text{Eq. 1} + \text{Eq. 2}\\ (-2)\text{Eq. 1} + \text{Eq. 3}\end{array}$$

$$\left\{\begin{array}{rcl} x + 4y + z & = & 0 \\ y & = & \tfrac{1}{2} \\ -4y - 3z & = & 7 \end{array}\right. \quad \begin{array}{l}\\ -\tfrac{1}{12}\text{Eq. 2}\\ \\ \end{array}$$

$$\begin{aligned} y &= \tfrac{1}{2} \\ -4\left(\tfrac{1}{2}\right) - 3z &= 7 \Rightarrow z = -3 \\ x + 4\left(\tfrac{1}{2}\right) + (-3) &= 0 \\ x &= 1 \end{aligned}$$

Solution: $\left(1, \tfrac{1}{2}, -3\right)$

27.
$$\left\{\begin{array}{rcl} x - 2y + 2z & = & -9 \\ 2x + y - z & = & 7 \\ 3x - y + z & = & 5 \end{array}\right. \quad \begin{array}{l}\text{Interchange equations.}\\ \\ \\ \end{array}$$

$$\left\{\begin{array}{rcl} x - 2y + 2z & = & -9 \\ 5y - 5z & = & 25 \\ 5y - 5z & = & 32 \end{array}\right. \quad \begin{array}{l}\\ -2\text{Eq. 1} + \text{Eq. 2}\\ -3\text{Eq. 1} + \text{Eq. 3}\end{array}$$

$$\left\{\begin{array}{rcl} x - 2y + 2z & = & -9 \\ 5y - 5z & = & 25 \\ 0 & = & 7 \end{array}\right. \quad \begin{array}{l}\\ \\ -\text{Eq. 2} + \text{Eq. 3}\end{array}$$

Inconsistent, no solution

29.
$$\begin{cases} 3x - 5y + 5z = 1 & \text{Equation 1}\\ 2x - 2y + 3z = 0 & \text{Equation 2}\\ 7x - y + 3z = 0 & \text{Equation 3}\end{cases}$$

$$\begin{cases} x - 3y + 2z = 1 & \text{Eq. 1} - \text{Eq. 2}\\ 2x - 2y + 3z = 0 & \\ 7x - y + 3z = 0 & \end{cases}$$

$$\begin{cases} x - 3y + 2z = 1 & \\ -4y + z = 2 & 2\text{Eq. 1} - \text{Eq. 2}\\ 7x - y + 3z = 0 & \end{cases}$$

$$\begin{cases} x - 3y + 2z = 1 & \\ -4y + z = 2 & \\ -20y + 11z = 7 & 7\text{Eq. 1} - \text{Eq. 3}\end{cases}$$

$$\begin{cases} x - 3y + 2z = 1 & \\ -4y + z = 2 & \\ 6z = -3 & -5\text{Eq. 2} + \text{Eq. 3}\end{cases}$$

$$\begin{aligned} 6z &= -3 \Rightarrow z = -\tfrac{1}{2}\\ -4y + \left(-\tfrac{1}{2}\right) &= 2 \Rightarrow -4y = \tfrac{5}{2} \Rightarrow y = -\tfrac{5}{8}\\ x - 3\left(-\tfrac{5}{8}\right) + 2\left(-\tfrac{1}{2}\right) &= 1 \Rightarrow x + \tfrac{7}{8} = 1 \Rightarrow x = \tfrac{1}{8}\end{aligned}$$

Solution: $\left(\frac{1}{8}, -\frac{5}{8}, -\frac{1}{2}\right)$

31.
$$\begin{cases} 2x + 3y = 0 & \text{Equation 1}\\ 4x + 3y - z = 0 & \text{Equation 2}\\ 8x + 3y + 3z = 0 & \text{Equation 3}\end{cases}$$

$$\begin{cases} 2x + 3y = 0 & \\ -3y - z = 0 & -2\text{Eq. 1} + \text{Eq. 2}\\ -9y + 3z = 0 & -4\text{Eq. 1} + \text{Eq. 3}\end{cases}$$

$$\begin{cases} 2x + 3y = 0 & \\ -3y - z = 0 & \\ 6z = 0 & -3\text{Eq. 2} + \text{Eq. 3}\end{cases}$$

$$\begin{aligned} 6z = 0 &\Rightarrow z = 0\\ -3y - 0 = 0 &\Rightarrow y = 0\\ 2x + 3(0) = 0 &\Rightarrow x = 0\end{aligned}$$

Solution: $(0, 0, 0)$

33.
$$\begin{cases} x + 4z = 1 & \text{Equation 1}\\ x + y + 10z = 10 & \text{Equation 2}\\ 2x - y + 2z = -5 & \text{Equation 3}\end{cases}$$

$$\begin{cases} x + 4z = 1 & \\ y + 6z = 9 & -\text{Eq. 1} + \text{Eq. 2}\\ -y - 6z = -7 & -2\text{Eq. 1} + \text{Eq. 3}\end{cases}$$

$$\begin{cases} x + 4z = 1 & \\ y + 6z = 9 & \\ 0 = 2 & \text{Eq. 2} + \text{Eq. 3}\end{cases}$$

No solution, inconsistent

35.
$$\left\{\begin{array}{rcrcrcrl} 3x & - & 3y & + & 6z & = & 6 & \quad \text{Equation 1} \\ x & + & 2y & - & z & = & 5 & \quad \text{Equation 2} \\ 5x & - & 8y & + & 13z & = & 7 & \quad \text{Equation 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & - & y & + & 2z & = & 2 & \quad \tfrac{1}{3}\text{Eq. 1} \\ x & + & 2y & - & z & = & 5 & \\ 5x & - & 8y & + & 13z & = & 7 & \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & - & y & + & 2z & = & 2 & \\ & & 3y & - & 3z & = & 3 & \quad -\text{Eq. 1} + \text{Eq. 2} \\ & & -3y & + & 3z & = & -3 & \quad -5\text{Eq. 1} + \text{Eq. 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & - & y & + & 2z & = & 2 & \\ & & y & - & z & = & 1 & \quad \tfrac{1}{3}\text{Eq. 2} \\ & & & & 0 & = & 0 & \quad \text{Eq. 2} + \text{Eq. 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & & & + & z & = & 3 & \quad \text{Eq. 2} + \text{Eq. 1} \\ & & y & - & z & = & 1 & \end{array}\right.$$

Let $z = a$, then:

$$\begin{aligned} y &= a + 1 \\ x &= -a + 3 \end{aligned}$$

Solution: $(-a + 3, a + 1, a)$

37.
$$\left\{\begin{array}{rcrcrcrl} x & + & 2y & - & 7z & = & -4 & \quad \text{Equation 1} \\ 2x & + & y & + & z & = & 13 & \quad \text{Equation 2} \\ 3x & + & 9y & - & 36z & = & -33 & \quad \text{Equation 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & + & 2y & - & 7z & = & -4 & \\ & & -3y & + & 15z & = & 21 & \quad -2\text{Eq. 1} + \text{Eq. 2} \\ & & 3y & - & 15z & = & -21 & \quad -3\text{Eq. 1} + \text{Eq. 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & + & 2y & - & 7z & = & -4 & \\ & & -3y & + & 15z & = & 21 & \\ & & & & 0 & = & 0 & \quad \text{Eq. 2} + \text{Eq. 3} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & + & 2y & - & 7z & = & -4 & \\ & & y & - & 5z & = & -7 & \quad -\tfrac{1}{3}\text{Eq. 2} \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} x & & & + & 3z & = & 10 & \quad -2\text{Eq. 2} + \text{Eq. 1} \\ & & y & - & 5z & = & -7 & \end{array}\right.$$

Let $z = a$, then:

$$\begin{aligned} y &= 5a - 7 \\ x &= -3a + 10 \end{aligned}$$

Solution: $(-3a + 10, 5a - 7, a)$

39.
$$\begin{cases} x + 3w = 4 & \text{Equation 1} \\ 2y - z - w = 0 & \text{Equation 2} \\ 3y - 2w = 1 & \text{Equation 3} \\ 2x - y + 4z = 5 & \text{Equation 4} \end{cases}$$

$$\begin{cases} x + 3w = 4 & \\ 2y - z - w = 0 & \\ 3y - 2w = 1 & \\ -y + 4z - 6w = -3 & -2\text{Eq. }1 + \text{Eq. }4 \end{cases}$$

$$\begin{cases} x + 3w = 4 & \\ y - 4z + 6w = 3 & -\text{Eq. 4 and interchange} \\ 2y - z - w = 0 & \text{the equations.} \\ 3y - 2w = 1 & \end{cases}$$

$$\begin{cases} x + 3w = 4 & \\ y - 4z + 6w = 3 & \\ 7z - 13w = -6 & -\text{Eq. }2 + \text{Eq. }3 \\ 12z - 20w = -8 & -3\text{Eq. }2 + \text{Eq. }4 \end{cases}$$

$$\begin{cases} x + 3w = 4 & \\ y - 4z + 6w = 3 & \\ z - 3w = -2 & -\tfrac{1}{2}\text{Eq. }4 + \text{Eq. }3 \\ 12z - 20w = -8 & \end{cases}$$

$$\begin{cases} x + 3w = 4 & \\ y - 4z + 6w = 3 & \\ z - 3w = -2 & \\ 16w = 16 & -12\text{Eq. }3 + \text{Eq. }4 \end{cases}$$

$$\begin{aligned} 16w &= 16 \Rightarrow w = 1 \\ z - 3(1) &= -2 \Rightarrow z = 1 \\ y - 4(1) + 6(1) &= 3 \Rightarrow y = 1 \\ x + 3(1) &= 4 \Rightarrow x = 1 \end{aligned}$$

Solution: $(1, 1, 1, 1)$

41.
$$\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$$

Let $z = a$, then: $x = \frac{1}{4}a$.

$$\begin{aligned} \tfrac{1}{4}a - 2y + 5a &= 2 \\ a - 8y + 20a &= 8 \\ -8y &= -21a + 8 \\ y &= \tfrac{21}{8}a - 1 \end{aligned}$$

Answer: $\left(\frac{1}{4}a, \frac{21}{8}a - 1, a\right)$

To avoid fractions, we could go back and let $z = 8a$, then $4x - 8a = 0 \Rightarrow x = 2a$.

$$\begin{aligned} 2a - 2y + 5(8a) &= 2 \\ -2y + 42a &= 2 \\ y &= 21a - 1 \end{aligned}$$

Solution: $(2a, 21a - 1, 8a)$

43.
$$\begin{cases} 2x - 3y + z = -2 & \text{Equation 1} \\ -4x + 9y = 7 & \text{Equation 2} \end{cases}$$

$$\begin{cases} 2x - 3y + z = -2 & \\ 3y + 2z = 3 & 2\text{Eq. }1 + \text{Eq. }2 \end{cases}$$

$$\begin{cases} 2x + 3z = 1 & \text{Eq. }2 + \text{Eq. }1 \\ 3y + 2z = 3 & \end{cases}$$

Let $z = a$, then:

$y = -\frac{2}{3}a + 1$

$x = -\frac{3}{2}a + \frac{1}{2}$

Solution: $\left(-\frac{3}{2}a + \frac{1}{2}, -\frac{2}{3}a + 1, a\right)$

45. $s = \frac{1}{2}at^2 + v_0t + s_0$

$(1, 128), (2, 80), (3, 0)$

$$\begin{aligned} 128 &= \tfrac{1}{2}a + v_0 + s_0 \Rightarrow a + 2v_0 + 2s_0 = 256 \\ 80 &= 2a + 2v_0 + s_0 \Rightarrow 2a + 2v_0 + s_0 = 80 \\ 0 &= \tfrac{9}{2}a + 3v_0 + s_0 \Rightarrow 9a + 6v_0 + 2s_0 = 0 \end{aligned}$$

Solving this system yields $a = -32, v_0 = 0, s_0 = 144$.

So, $s = \frac{1}{2}(-32)t^2 + (0)t + 144 = -16t^2 + 144$.

47. $s = \frac{1}{2}at^2 + v_0t + s_0$

$(1, 352), (2, 272), (3, 160)$

$$\begin{cases} 352 = \frac{1}{2}a + v_0 + s_0 \Rightarrow a + 2v_0 + 2s_0 = 704 \\ 272 = 2a + 2v_0 + s_0 \Rightarrow 2a + 2v_0 + s_0 = 272 \\ 160 = \frac{9}{2}a + 3v_0 + s_0 \Rightarrow 9a + 6v_0 + 2s_0 = 320 \end{cases}$$

Solving the system, $a = -32, v_0 = -32, s_0 = 400$.

$$\begin{aligned} \text{Thus, } s &= \tfrac{1}{2}(-32)t^2 - 32t + 400 \\ &= -16t^2 - 32t + 400. \end{aligned}$$

49. $y = ax^2 + bx + c$ passing through

$(0, 0), (2, -2), (4, 0)$

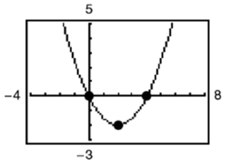
$(0, 0)$: $0 = c$

$(2, -2)$: $-2 = 4a + 2b + c \Rightarrow -1 = 2a + b$

$(4, 0)$: $0 = 16a + 4b + c \Rightarrow 0 = 4a + b$

Solution: $a = \frac{1}{2}, b = -2, c = 0$

The equation of the parabola is $y = \frac{1}{2}x^2 - 2x$.



51. $y = ax^2 + bx + c$ passing through

$(2, 0), (3, -1), (4, 0)$

$(2, 0)$: $0 = 4a + 2b + c$

$(3, -1)$: $-1 = 9a + 3b + c$

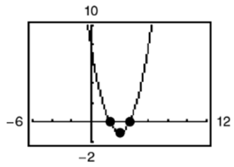
$(4, 0)$: $0 = 16a + 4b + c$

$$\begin{cases} 0 = 4a + 2b + c & \\ -1 = 5a + b & -\text{Eq. }1 + \text{Eq. }2 \\ 0 = 12a + 2b & -\text{Eq. }1 + \text{Eq. }3 \end{cases}$$

$$\begin{cases} 0 = 4a + 2b + c & \\ -1 = 5a + b & \\ 2 = 2a & -2\text{Eq. }2 + \text{Eq. }3 \end{cases}$$

Solution: $a = 1, b = -6, c = 8$

The equation of the parabola is $y = x^2 - 6x + 8$.



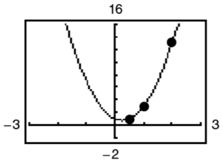
53. $y = ax^2 + bx + c$ passing through $\left(\frac{1}{2}, 1\right), (1, 3), (2, 13)$

$\left(\frac{1}{2}, 1\right)$: $1 = a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + c$

$(1, 3)$: $3 = a(1)^2 + b(1) + c$

$(2, 13)$: $13 = a(2)^2 + b(2) + c$

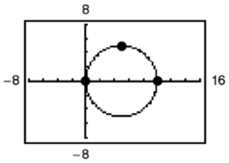
$$\begin{cases} a + 2b + 4c = 4 \\ a + b + c = 3 \\ 4a + 2b + c = 13 \end{cases}$$



Solution: $a = 4, b = -2, c = 1$

The equation of the parabola is $y = 4x^2 - 2x + 1$.

55. $x^2 + y^2 + Dx + Ey + F = 0$ passing through $(0, 0), (5, 5), (10, 0)$



$(0, 0)$: $0^2 + 0^2 + D(0) + E(0) + F = 0 \Rightarrow F = 0$

$(5, 5)$: $5^2 + 5^2 + D(5) + E(5) + F = 0 \Rightarrow 5D + 5E + F = -50$

$(10, 0)$: $10^2 + 0^2 + D(10) + E(0) + F = 0 \Rightarrow 10D + F = -100$

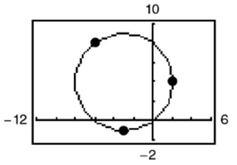
Solution: $D = -10, E = 0, F = 0$

The equation of the circle is $x^2 + y^2 - 10x = 0$. To graph, complete the square first, then solve for y.

$$\begin{aligned} (x^2 - 10x + 25) + y^2 &= 25 \\ (x - 5)^2 + y^2 &= 25 \\ y^2 &= 25 - (x - 5)^2 \\ y &= \pm\sqrt{25 - (x - 5)^2} \end{aligned}$$

Let $y_1 = \sqrt{25 - (x - 5)^2}$ and $y_2 = -\sqrt{25 - (x - 5)^2}$.

57. $x^2 + y^2 + Dx + Ey + F = 0$ passing through $(-3, -1), (2, 4), (-6, 8)$



$(-3, -1)$: $10 - 3D - E + F = 0 \Rightarrow 10 = 3D + E - F$

$(2, 4)$: $20 + 2D + 4E + F = 0 \Rightarrow 20 = -2D - 4E - F$

$(-6, 8)$: $100 - 6D + 8E + F = 0 \Rightarrow 100 = 6D - 8E - F$

Solution: $D = 6, E = -8, F = 0$

The equation of the circle is $x^2 + y^2 + 6x - 8y = 0$. To graph, complete the squares first, then solve for y.

$$\begin{aligned} (x^2 + 6x + 9) + (y^2 - 8y + 16) &= 0 + 9 + 16 \\ (x + 3)^2 + (y - 4)^2 &= 25 \\ (y - 4)^2 &= 25 - (x + 3)^2 \\ y - 4 &= \pm\sqrt{25 - (x + 3)^2} \\ y &= 4 \pm \sqrt{25 - (x + 3)^2} \end{aligned}$$

Let $y_1 = 4 + \sqrt{25 - (x + 3)^2}$ and $y_2 = 4 - \sqrt{25 - (x + 3)^2}$.

59. Let x = pounds of brand X.

Let y = pounds of brand Y.

Let z = pounds of brand Z.

$$\begin{array}{ll} \text{Fertilizer A:} & \frac{1}{3}y + \frac{2}{9}z = 5 \\ \text{Fertilizer B:} & \frac{1}{2}x + \frac{2}{3}y + \frac{5}{9}z = 13 \\ \text{Fertilizer C:} & \frac{1}{2}x + \frac{2}{9}z = 4 \end{array}$$

$$\begin{cases} \frac{1}{2}x + \frac{2}{3}y + \frac{5}{9}z = 13 & \text{Interchange Eq. 1 and Eq. 2.} \\ \frac{1}{3}y + \frac{2}{9}z = 5 \\ \frac{1}{2}x + \frac{2}{9}z = 4 \end{cases}$$

$$\begin{cases} \frac{1}{2}x + \frac{2}{3}y + \frac{5}{9}z = 13 \\ \frac{1}{3}y + \frac{2}{9}z = 5 \\ -\frac{2}{3}y - \frac{1}{3}z = -9 & -\text{Eq. 1} + \text{Eq. 3} \end{cases}$$

$$\begin{cases} \frac{1}{2}x + \frac{2}{3}y + \frac{5}{9}z = 13 \\ \frac{1}{3}y + \frac{2}{9}z = 5 \\ \frac{1}{9}z = 1 & 2\text{Eq. 2} + \text{Eq. 3} \end{cases}$$

$z = 9$

$\frac{1}{3}y + \frac{2}{9}(9) = 5 \Rightarrow y = 9$

$\frac{1}{2}x + \frac{2}{3}(9) + \frac{5}{9}(9) = 13 \Rightarrow x = 4$

4 pounds of brand X, 9 pounds of brand Y, and 9 pounds of brand Z are needed to obtain the desired mixture.

61.

$$\begin{cases} x + y + z = 180 \\ 2x + 7 + z = 180 \\ y + 2x - 7 = 180 \end{cases}$$

$$\begin{cases} x + y + z = 180 \\ 2x + z = 173 \\ 2x + y = 187 \end{cases}$$

$$\begin{cases} -x + y = 7 & -\text{Eq. 2} + \text{Eq. 1} \\ 2x + z = 173 \\ 2x + y = 187 \end{cases}$$

$$\begin{cases} -x + y = 7 \\ 2x + z = 173 \\ 3x = 180 & -\text{Eq. 1} + \text{Eq. 3} \end{cases}$$

$x = 60°$

$2(60) + z = 173 \Rightarrow z = 53°$

$-60 + y = 7 \Rightarrow y = 67°$

63.

$$\begin{cases} I_1 - I_2 + I_3 = 0 & \text{Equation 1} \\ 3I_1 + 2I_2 = 7 & \text{Equation 2} \\ 2I_2 + 4I_3 = 8 & \text{Equation 3} \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 5I_2 - 3I_3 = 7 & (-3)\text{Eq. 1} + \text{Eq. 2} \\ 2I_2 + 4I_3 = 8 \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 10I_2 - 6I_3 = 14 & 2\text{Eq. 2} \\ 10I_2 + 20I_3 = 40 & 5\text{Eq. 3} \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 10I_2 - 6I_3 = 14 \\ 26I_3 = 26 & (-1)\text{Eq. 2} + \text{Eq. 3} \end{cases}$$

$26I_3 = 26 \Rightarrow I_3 = 1$

$10I_2 - 6(1) = 14 \Rightarrow I_2 = 2$

$I_1 - 2 + 1 = 0 \Rightarrow I_1 = 1$

Solution: $I_1 = 1, I_2 = 2, I_3 = 1$

65.
$$\left\{\begin{array}{rcrcrcr} 4c & + & 9b & + & 29a & = & 20 \\ 9c & + & 29b & + & 99a & = & 70 \\ 29c & + & 99b & + & 353a & = & 254 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcr} 9c & + & 29b & + & 99a & = & 70 \\ 4c & + & 9b & + & 29a & = & 20 \\ 29c & + & 99b & + & 353a & = & 254 \end{array}\right. \quad \text{Interchange equations.}$$

$$\left\{\begin{array}{rcrcrcrl} c & + & 11b & + & 41a & = & 30 & \quad -2\text{Eq. }2 + \text{Eq. }1 \\ & & -35b & - & 135a & = & -100 & \quad -4\text{Eq. }1 + \text{Eq. }2 \\ & & -220b & - & 836a & = & -616 & \quad -29\text{Eq. }1 + \text{Eq. }3 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrr} c & + & 11b & + & 41a & = & 30 & \\ & & 1540b & + & 5940a & = & 4400 & \quad -44\text{Eq. }2 \\ & & -1540b & - & 5852a & = & -4312 & \quad 7\text{Eq. }3 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} c & + & 11b & + & 41a & = & 30 & \\ & & 1540b & + & 5940a & = & 4400 & \\ & & & & 88a & = & 88 & \quad \text{Eq. }2 + \text{Eq. }3 \end{array}\right.$$

$$\begin{array}{rclcrcr} 88a & = & 88 & \Rightarrow & a & = & 1 \\ 1540b + 5940(1) & = & 4400 & \Rightarrow & b & = & -1 \\ c + 11(-1) + 41(1) & = & 30 & \Rightarrow & c & = & 0 \end{array}$$

Least squares regression parabola: $y = x^2 - x$

67. (a)
$$\left\{\begin{array}{rcrcrcr} 5c & + & 250b & + & 13{,}500a & = & 923 \\ 250c & + & 13{,}500b & + & 775{,}000a & = & 52{,}170 \\ 13{,}500c & + & 775{,}000b & + & 46{,}590{,}000a & = & 3{,}101{,}300 \end{array}\right.$$

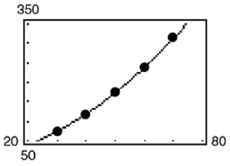
$$\left\{\begin{array}{rcrcrcrl} 5c & + & 250b & + & 13{,}500a & = & 923 & \\ & & 1000b & + & 100{,}000a & = & 6020 & \quad (-50)\text{Eq. }1 + \text{Eq. }2 \\ & & 100{,}000b & + & 10{,}140{,}000a & = & 609{,}200 & \quad (-2700)\text{Eq. }1 + (3)\text{Eq. }3 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcrl} 5c & + & 250b & + & 13{,}500a & = & 923 & \\ & & 1000b & + & 100{,}000a & = & 6020 & \\ & & & & 140{,}000a & = & 7200 & \quad (-100)\text{Eq. }2 + \text{Eq. }3 \end{array}\right.$$

$$\begin{array}{rclcrcr} 140{,}000a & = & 7200 & \Rightarrow & a & \approx & 0.0514 \\ 1000b + 100{,}000(0.0514) & = & 6020 & \Rightarrow & b & \approx & 0.8771 \\ 5c + 250(0.8771) + 13{,}500(0.0514) & = & 923 & \Rightarrow & c & \approx & 1.8857 \end{array}$$

Least-squares regression parabola: $y = 0.0514x^2 + 0.8771x + 1.8857$

(b)



The model fits the data well.

(c) When $x = 75$, $y = 0.0514(75)^2 + 0.8771(75) + 1.8857 \approx 356$ feet.

69. $$\begin{cases} 2x - 2x\lambda = 0 \Rightarrow 2x(1-\lambda) = 0 \Rightarrow \lambda = 1 \text{ or } x = 0 \\ -2y + \lambda = 0 \\ y - x^2 = 0 \end{cases}$$

If $\lambda = 1$: $2y = \lambda \Rightarrow y = \frac{1}{2}$

$x^2 = y \Rightarrow x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

If $x = 0$: $x^2 = y \Rightarrow y = 0$

$2y = \lambda \Rightarrow \lambda = 0$

Solution: $x = \pm\frac{\sqrt{2}}{2}$ or $x = 0$

$y = \frac{1}{2}$ $\quad y = 0$

$\lambda = 1$ $\quad \lambda = 0$

71. False. For example, refer to Example 6 on page 495,

$$\begin{cases} x - 2y + z = 2 \\ 2x - y - z = 1 \end{cases}$$

has the solution set of all ordered triples of the form $(a, a-1, a)$ where a is a real number. Therefore, it is not an unique solution.

73. Answers will vary. *Sample answer.*

$$\begin{cases} x + 3z = 1 \\ y - 4z = -2 \end{cases}$$

75. Answers will vary. *Sample answer:*

$(2, 0, -1)$ is a solution to $\begin{cases} x + y + z = 1 \\ 3x + y + z = 5 \\ -x + 2y + 3z = -5 \end{cases}$

77. Answers will vary. *Sample answer:*

$\left(\frac{1}{2}, -3, 0\right)$ is a solution to

$$\begin{cases} 2x + y + z = -2 \\ 4x + y + z = -1 \\ -2x + 2y + 3z = -7 \end{cases}$$

79. $\dfrac{x^2 + 2x - 8}{-20 - 5x} = \dfrac{(x+4)(x-2)}{-5(x+4)} = \dfrac{2-x}{5}, x \neq -4$

81. $$\frac{3x^2 + 9x + 6}{x^3 - 2x^2 - x - 2} = \frac{3(x^2 + 3x + 2)}{x^3 - 2x^2 - x - 2} = \frac{3(x+1)(x+2)}{x^3 - 2x^2 - x - 2}$$

83. $$\begin{array}{r} -3x + 1 \\ x - 6\overline{)-3x^2 + 19x - 6} \\ \underline{-3x^2 + 18x} \\ x - 6 \\ \underline{x - 6} \\ 0 \end{array} \quad \text{or} \quad \begin{array}{r|rrr} 6 & -3 & 19 & -6 \\ & & -18 & 6 \\ \hline & -3 & 1 & 0 \end{array}$$

$(-3x^2 + 19x - 6) \div (x - 6) = -3x + 1, x \neq 6$

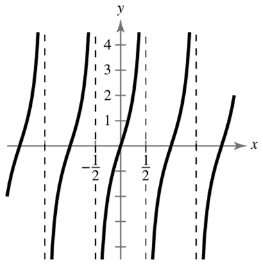
85. $$\begin{array}{r} x^2 + x + 2 \\ x - 3\overline{)x^3 - 2x^2 - x + 7} \\ \underline{x^3 - 3x^2} \\ x^2 - x \\ \underline{x^2 - 3x} \\ 2x + 7 \\ \underline{2x - 6} \\ 13 \end{array} \quad \text{or} \quad \begin{array}{r|rrrr} 3 & 1 & -2 & -1 & 7 \\ & & 3 & 3 & 6 \\ \hline & 1 & 1 & 2 & 13 \end{array}$$

$(x^3 - 2x^2 - x + 7) \div (x - 3) = x^2 + x + 2 + \dfrac{13}{x-3}$

87. $y = 2\tan \pi x$

Asymptotes: $\pi x = \frac{-\pi}{2} \Rightarrow x = -\frac{1}{2}$

$\pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{2}$



Section 7.4 Partial Fractions

1. improper

3. two

5. $\dfrac{3}{x^2 - 2x} = \dfrac{3}{x(x - 2)} = \dfrac{A}{x} + \dfrac{B}{x - 2}$

7. $\dfrac{6x + 5}{(x + 2)^4} = \dfrac{6x + 5}{(x + 2)(x + 2)(x + 2)(x + 2)} = \dfrac{A}{x + 2} + \dfrac{B}{(x + 2)^2} + \dfrac{C}{(x + 2)^3} + \dfrac{D}{(x + 2)^4}$

9. $\dfrac{8x}{x^2(x^2 + 3)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{x^2 + 3} + \dfrac{Ex + F}{(x^2 + 3)^2}$

11. $\dfrac{1}{x^2 + x} = \dfrac{A}{x} + \dfrac{B}{x + 1}$

$1 = A(x + 1) + Bx$

Let $x = 0$: $1 = A$

Let $x = -1$: $1 = -B \Rightarrow B = -1$

$\dfrac{1}{x^2 + x} = \dfrac{1}{x} - \dfrac{1}{x + 1}$

13. $\dfrac{3}{x^2 + x - 2} = \dfrac{A}{x - 1} + \dfrac{B}{x + 2}$

$3 = A(x + 2) + B(x - 1)$

Let $x = 1$: $3 = 3A \Rightarrow A = 1$

Let $x = -2$: $3 = -3B \Rightarrow B = -1$

$\dfrac{3}{x^2 + x - 2} = \dfrac{1}{x - 1} - \dfrac{1}{x + 2}$

15. $\dfrac{1}{x^2 - 1} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1}$

$1 = A(x - 1) + B(x + 1)$

Let $x = -1$: $1 = -2A \Rightarrow A = -\dfrac{1}{2}$

Let $x = 1$: $1 = 2B \Rightarrow B = \dfrac{1}{2}$

$\dfrac{1}{x^2 - 1} = \dfrac{1/2}{x - 1} - \dfrac{1/2}{x + 1} = \dfrac{1}{2}\left(\dfrac{1}{x - 1} - \dfrac{1}{x + 1}\right)$

17. $\dfrac{x^2 + 12x + 12}{x^3 - 4x} = \dfrac{A}{x} + \dfrac{B}{x + 2} + \dfrac{C}{x - 2}$

$x^2 + 12x + 12 = A(x + 2)(x - 2) + Bx(x - 2) + Cx(x + 2)$

Let $x = 0$: $12 = -4A \Rightarrow A = -3$

Let $x = -2$: $-8 = 8B \Rightarrow B = -1$

Let $x = 2$: $40 = 8C \Rightarrow C = 5$

$\dfrac{x^2 + 12x + 12}{x^3 - 4x} = -\dfrac{3}{x} - \dfrac{1}{x + 2} + \dfrac{5}{x - 2}$

19. $\dfrac{3x}{(x - 3)^2} = \dfrac{A}{x - 3} + \dfrac{B}{(x - 3)^2}$

$3x = A(x - 3) + B$

Let $x = 3$: $9 = B$

Let $x = 0$: $0 = -3A + B$

$0 = -3A + 9$

$3 = A$

$\dfrac{3x}{(x - 3)^2} = \dfrac{3}{x - 3} + \dfrac{9}{(x - 3)^2}$

21. $\dfrac{4x^2 + 2x - 1}{x^2(x + 1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x + 1}$

$4x^2 + 2x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$

Let $x = 0$: $-1 = B$

Let $x = -1$: $1 = C$

Let $x = 1$: $5 = 2A + 2B + C$

$5 = 2A - 2 + 1$

$6 = 2A$

$3 = A$

$\dfrac{4x^2 + 2x - 1}{x^2(x + 1)} = \dfrac{3}{x} - \dfrac{1}{x^2} + \dfrac{1}{x + 1}$

23. $\dfrac{2x}{x^3 - 1} = \dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 + x + 1}$

$$2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$$

$$2x = (A + B)x^2 + (A - B + C)x + (A - C)$$

Equating coefficients of like terms gives $A + B = 0$,

$A - B + C = 2$ and $A - C = 0$.

Let $x = 1$:

$$3A = 2 \Rightarrow A = \frac{2}{3}$$

$$A - C = 0 \Rightarrow \frac{2}{3} - C = 0 \Rightarrow C = \frac{2}{3}$$

$$A + B = 0 \Rightarrow \frac{2}{3} + B = 0 \Rightarrow B = -\frac{2}{3}$$

$$\frac{2x}{x^3 - 1} = \frac{2}{3}\left(\frac{1}{x - 1} - \frac{x - 1}{x^2 + x + 1}\right)$$

25. $\dfrac{x}{x^3 - x^2 - 2x + 2} = \dfrac{x}{(x - 1)(x^2 - 2)} = \dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 - 2}$

$$\begin{aligned} x &= A(x^2 - 2) + (Bx + C)(x - 1) \\ &= Ax^2 - 2A + Bx^2 - Bx + Cx - C \\ &= (A + B)x^2 + (C - B)x - (2A + C) \end{aligned}$$

Equating coefficients of like terms gives $0 = A + B, 1 = C - B$, and $0 = 2A + C$. So, $A = -1, B = 1$, and $C = 2$.

$$\frac{x}{x^3 - x^2 - 2x + 2} = -\frac{1}{x - 1} + \frac{x + 2}{x^2 - 2}$$

27. $\dfrac{x}{16x^4 - 1} = \dfrac{x}{(4x^2 - 1)(4x^2 + 1)} = \dfrac{x}{(2x + 1)(2x - 1)(4x^2 + 1)} = \dfrac{A}{2x + 1} + \dfrac{B}{2x - 1} + \dfrac{Cx + D}{4x^2 + 1}$

$$\begin{aligned} x &= A(2x - 1)(4x^2 + 1) + B(2x + 1)(4x^2 + 1) + (Cx + D)(2x + 1)(2x - 1) \\ &= A(8x^3 - 4x^2 + 2x - 1) + B(8x^3 + 4x^2 + 2x + 1) + (Cx + D)(4x^2 - 1) \\ &= 8Ax^3 - 4Ax^2 + 2Ax - A + 8Bx^3 + 4Bx^2 + 2Bx + B + 4Cx^3 + 4Dx^2 - Cx - D \\ &= (8A + 8B + 4C)x^3 + (-4A + 4B + 4D)x^2 + (2A + 2B - C)x + (-A + B - D) \end{aligned}$$

Equating coefficients of like terms gives $0 = 8A + 8B + 4C, 0 = -4A + 4B + 4D, 1 = 2A + 2B - C$, and $0 = -A + B - D$.

Using the first and third equations, $2A + 2B + C = 0$ and $2A + 2B - C = 1$; by subtraction, $2C = -1$, so $C = -\frac{1}{2}$.

Using the second and fourth equations, $-A + B + D = 0$ and $-A + B - D = 0$; by subtraction $2D = 0$, so $D = 0$.

Substituting $-\frac{1}{2}$ for C and 0 for D in the first and second equations, $8A + 8B = 2$ and $-4A + 4B = 0$, so $A = \frac{1}{8}$ and $B = \frac{1}{8}$.

$$\frac{x}{16x^4 - 1} = \frac{\frac{1}{8}}{2x + 1} + \frac{\frac{1}{8}}{2x - 1} + \frac{\left(-\frac{1}{2}\right)x}{4x^2 + 1} = \frac{1}{8(2x + 1)} + \frac{1}{8(2x - 1)} - \frac{x}{2(4x^2 + 1)} = \frac{1}{8}\left(\frac{1}{2x + 1} + \frac{1}{2x - 1} - \frac{4x}{4x^2 + 1}\right)$$

29. $$\frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$\begin{aligned} x^2 + 5 &= A(x^2 - 2x + 3) + (Bx + C)(x + 1) = Ax^2 - 2Ax + 3A + Bx^2 + Bx + Cx + C \\ &= (A + B)x^2 + (-2A + B + C)x + (3A + C) \end{aligned}$$

Equating coefficients of like terms gives $1 = A + B, 0 = -2A + B + C,$ and $5 = 3A + C.$

Subtracting both sides of the second equation from the first gives $1 = 3A - C;$ combining this with the third equation gives $A = 1$ and $C = 2.$ Because $A + B = 1, B = 0.$

$$\frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{1}{x + 1} + \frac{2}{x^2 - 2x + 3}$$

31. $$\frac{2x^2 + x + 8}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}$$

$$2x^2 + x + 8 = (Ax + B)(x^2 + 4) + Cx + D$$

$$2x^2 + x + 8 = Ax^3 + Bx^2 + (4A + C)x + (4B + D)$$

Equating coefficients of like terms gives

$0 = A$

$2 = B$

$1 = 4A + C \Rightarrow C = 1$

$8 = 4B + D \Rightarrow D = 0$

$$\frac{2x^2 + x + 8}{(x^2 + 4)^2} = \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2}$$

33. $$\frac{8x - 12}{x^2(x^2 + 2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2} + \frac{Ey + F}{(x^2 + 2)^2}$$

$$\begin{aligned} 8x - 12 &= Ax(x^2 + 2)^2 + B(x^2 + 2)^2 + (Cx + D)x^2(x^2 + 2) + (Ex + F)x^2 \\ &= Ax^5 + 4Ax^3 + 4Ax + Bx^4 + 4Bx^2 + 4B + Cx^5 + 2Cx^3 + Dx^4 + 2Dx^2 + Ex^3 + Fx^2 \\ &= (A + C)x^5 + (B + D)x^4 + (4A + 2C + E)x^3 + (4B + 2D + F)x^2 + 4Ax + 4B \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} A + C &= 0, \\ B + D &= 0, \\ 4A + 2C + E &= 0, \\ 4B + 2D + F &= 0, \\ 4A &= 8, \text{ and} \\ 4B &= -12. \end{aligned}$$

So, $A = 2, B = -3, C = -2, D = 3, E = -4,$ and $F = 6.$

$$\frac{8x - 12}{x^2(x^2 + 2)^2} = \frac{2}{x} + \frac{-3}{x^2} + \frac{-2x + 3}{x^2 + 2} + \frac{-4x + 6}{(x^2 + 2)^2}$$

35. $$\frac{x^2 - x}{x^2 + x + 1} = 1 + \frac{-2x - 1}{x^2 + x + 1} = 1 - \frac{2x + 1}{x^2 + x + 1}$$

37. $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2} = 2x - 7 + \dfrac{18x + 19}{(x + 1)(x + 2)}$

$$\frac{18x + 19}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$18x + 19 = A(x + 2) + B(x + 1)$$

Let $x = -1$: $1 = A$

Let $x = -2$: $-17 = -B \Rightarrow B = 17$

$$\frac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2} = 2x - 7 + \frac{1}{x + 1} + \frac{17}{x + 2}$$

39. $\dfrac{x^4}{(x - 1)^3} = \dfrac{x^4}{x^3 - 3x^2 + 3x - 1} = x + 3 + \dfrac{6x^2 - 8x + 3}{(x - 1)^3}$

$$\frac{6x^2 - 8x + 3}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$$

$$6x^2 - 8x + 3 = A(x - 1)^2 + B(x - 1) + C$$

Let $x = 1$: $1 = C$

$$\left.\begin{array}{l} \text{Let } x = 0\text{: } \; 3 = A - B + 1 \\ \text{Let } x = 2\text{: } \; 11 = A + B + 1 \end{array}\right\} \begin{array}{l} A - B = 2 \\ A + B = 10 \end{array}$$

So, $A = 6$ and $B = 4$.

$$\frac{x^4}{(x - 1)^3} = x + 3 + \frac{6}{x - 1} + \frac{4}{(x - 1)^2} + \frac{1}{(x - 1)^3}$$

41. $\dfrac{x^4 + 2x^3 + 4x^2 + 8x + 2}{x^3 + 2x^2 + x} = x + \dfrac{3x^2 + 8x + 2}{x^3 + 2x^2 + x} = x + \dfrac{3x^2 + 8x + 2}{x(x + 1)^2}$

$$\frac{3x^2 + 8x + 2}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

$$3x^2 + 8x + 2 = A(x + 1)^2 + B(x)(x + 1) + C(x)$$

$$3x^2 + 8x + 2 = Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$3x^2 + 8x + 2 = (A + B)x^2 + (2A + B + C)x + A$$

Equating coefficients of like terms gives $A + B = 3, 2A + B + C = 8,$ and $A = 2$.

So, $A = 2, B = 1,$ and $C = 3$.

$$\frac{x^4 + 2x^3 + 4x^2 + 8x + 2}{x^3 + 2x^2 + x} = x + \frac{2}{x} + \frac{1}{x + 1} + \frac{3}{(x + 1)^2}$$

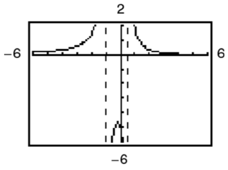
43. $\dfrac{5 - x}{2x^2 + x - 1} = \dfrac{A}{2x - 1} + \dfrac{B}{x + 1}$

$$-x + 5 = A(x + 1) + B(2x - 1)$$

Let $x = \frac{1}{2}$: $\frac{9}{2} = \frac{3}{2}A \Rightarrow A = 3$

Let $x = -1$: $6 = -3B \Rightarrow B = -2$

$$\frac{5 - x}{2x^2 + x - 1} = \frac{3}{2x - 1} - \frac{2}{x + 1}$$



45. $\dfrac{3x^2 - 7x - 2}{x^3 - x} = \dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{x - 1}$

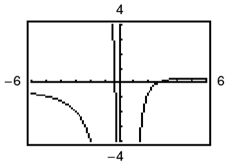
$3x^2 - 7x - 2 = A(x^2 - 1) + Bx(x - 1) + Cx(x + 1)$

Let $x = 0$: $-2 = -A \Rightarrow A = 2$

Let $x = -1$: $8 = 2B \Rightarrow B = 4$

Let $x = 1$: $-6 = 2C \Rightarrow C = -3$

$\dfrac{3x^2 - 7x - 2}{x^3 - x} = \dfrac{2}{x} + \dfrac{4}{x + 1} - \dfrac{3}{x - 1}$



47. $\dfrac{x^2 + x + 2}{(x^2 + 2)^2} = \dfrac{Ax + B}{x^2 + 2} + \dfrac{Cx + D}{(x^2 + 2)^2}$

$x^2 + x + 2 = (Ax + B)(x^2 + 2) + Cx + D$

$x^2 + x + 2 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$

Equating coefficients of like terms gives

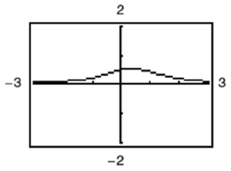
$0 = A$

$1 = B$

$1 = 2A + C \Rightarrow C = 1$

$2 = 2B + D \Rightarrow D = 0$

$\dfrac{x^2 + x + 2}{(x^2 + 2)^2} = \dfrac{1}{x^2 + 2} + \dfrac{x}{(x^2 + 2)^2}$



49. $\dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \dfrac{x + 5}{(x + 2)(x - 4)}$

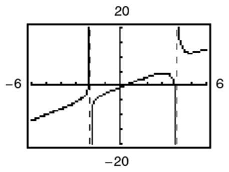
$\dfrac{x + 5}{(x + 2)(x - 4)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 4}$

$x + 5 = A(x - 4) + B(x + 2)$

Let $x = -2$: $3 = -6A \Rightarrow A = -\dfrac{1}{2}$

Let $x = 4$: $9 = 6B \Rightarrow B = \dfrac{3}{2}$

$\dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \dfrac{1}{2}\left(\dfrac{3}{x - 4} - \dfrac{1}{x + 2}\right)$



51. $C = \dfrac{120p}{10{,}000 - p^2} = \dfrac{120p}{(100 + p)(100 - p)} = \dfrac{A}{100 + p} + \dfrac{B}{100 - p}$

$120p = A(100 - p) + B(100 + p)$

Let $p = 100$: $200B = 12{,}000$

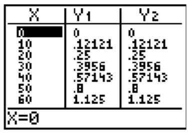
$B = 60$

Let $p = -100$: $200A = -12{,}000$

$A = -60$

$C = \dfrac{120p}{10{,}000 - p^2} = -\dfrac{60}{100 + p} + \dfrac{60}{100 - p}$

Let $y_1 = \dfrac{120p}{10{,}000 - p^2}$ and $y_2 = -\dfrac{60}{100 + p} + \dfrac{60}{100 - p}$.



X	Y1	Y2
0	0	0
10	.12121	.12121
20	.25	.25
30	.3956	.3956
40	.57143	.57143
50	.8	.8
60	1.125	1.125

X=0

53. True. The expression is an improper rational expression.

55. The expression is improper, $\frac{x^2+1}{x(x-1)} = \frac{x^2+1}{x^2-x}$ so first divide the denominator into the numerator to obtain

$$\frac{x^2+1}{x^2-x} = 1 + \frac{x+1}{x^2-x} = 1 + \frac{x+1}{x(x-1)}.$$

Then find the partial fraction decomposition of

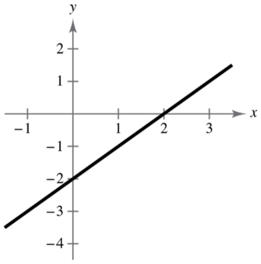
$$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

57. Answers will vary. *Sample answer:* One way to find the constants is to choose values of the variable that eliminate one or more of the constants in the basic equation so that you can solve for another constant. If necessary, you can then use these constants with other chosen values of the variable to solve for any remaining constants. Another way is to expand the basic equation and collect like terms. Then you can equate coefficients of the like terms on each side of the equation to obtain simple equations involving the constants. If necessary, you can solve these equations using substitution.

59. $x - y = 2$

$y = x - 2$

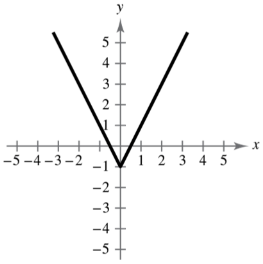
Intercepts: $(0, -2), (2, 0)$



61. $|2x| - y = 1$

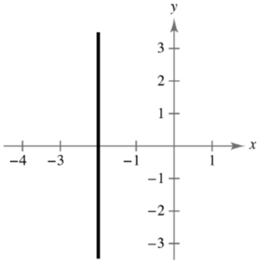
$y = |2x| - 1$

Points on graph: $(0, -1), \left(\frac{1}{2}, 0\right), \left(-\frac{1}{2}, 0\right)$



63. $x = -2$

Vertical line

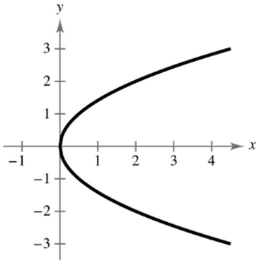


65. $2x - y^2 = 0$

Parabola that opens to the right

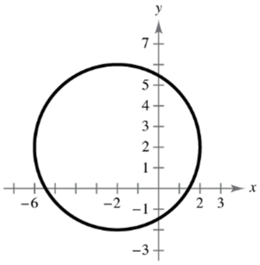
Intercept: $(0, 0)$

Points on graph: $\left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)$



67. $(x+2)^2 + (y-2)^2 = 16$

Circle with center $(-2, 2)$ and radius 4



69. $1 \geq x^2 - 1$

$2 \geq x^2$

(a) $x = -1$: $2 \overset{?}{\geq} (-1)^2$

$x = -1$ is a solution.

(b) $x = \sqrt{2}$: $2 \overset{?}{\geq} \left(\sqrt{2}\right)^2$

$x = \sqrt{2}$ is a solution.

71. $|x - 1| < 2$

(a) $x = 0$: $|0 - 1| \overset{?}{<} 2$

$|-1| \overset{?}{<} 2$

$1 < 2$

$x = 0$ is a solution.

(b) $x = 4$: $|4 - 1| \overset{?}{<} 2$

$|3| \overset{?}{<} 2$

$3 \not< 2$

$x = 4$ is not a solution.

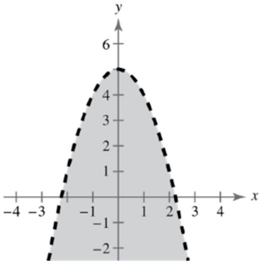
Section 7.5 Systems of Inequalities

1. graph

3. The graph of an inequality with a $<$ sign does not include the points on the graph of the corresponding equation. The graph of an inequality with a $\le$ sign does include those points.

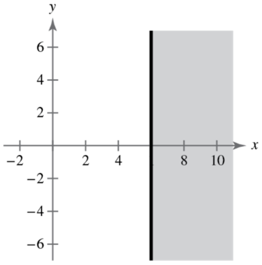
5. $y < 5 - x^2$

Using a dashed line, graph $y = 5 - x^2$, and shade the region inside the parabola.



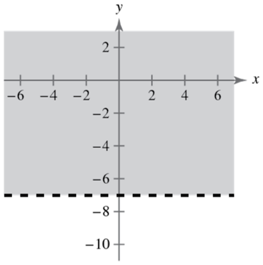
7. $x \ge 6$

Using a solid line, graph the vertical line $x = 6$, and shade to the right of this line.



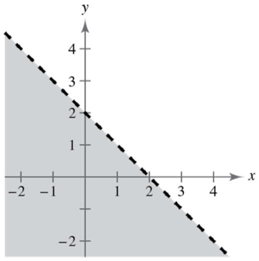
9. $y > -7$

Using a dashed line, graph the horizontal line $y = -7$, and shade above the line.



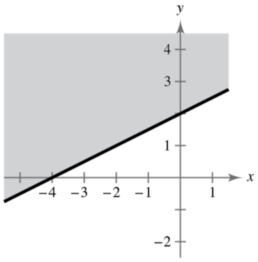
11. $y < 2 - x$

Using a dashed line, graph $y = 2 - x$, and then shade below the line. (Use $(0, 0)$ as a test point.)



13. $2y - x \ge 4$

Using a solid line, graph $2y - x = 4$, and then shade above the line. (Use $(0, 0)$ as a test point.)



15. $x^2 + (y - 3)^2 < 4$

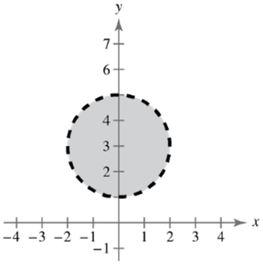
Using a dashed line, sketch the circle $x^2 + (y - 3)^2 = 4$.

Center: $(0, 3)$

Radius: 2

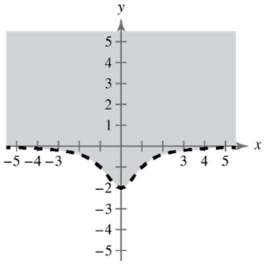
Test point: $(0, 0)$

Shade the inside of the circle.

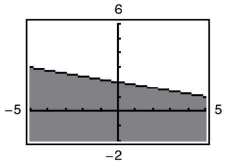


17. $y > -\dfrac{2}{x^2 + 1}$

Using a solid line, graph $y = -\dfrac{2}{x^2 + 1}$. Use $(0, 0)$ as a test point. Then shade above the curve.

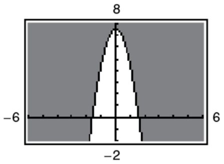


19. $y \le 2 - \frac{1}{5}x$

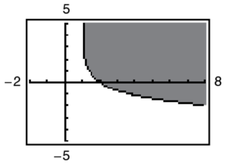


21. $\frac{2}{3}y + 2x^2 - 5 \ge 0$

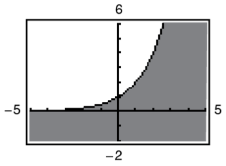
$$\frac{2}{3}y \ge 5 - 2x^2$$
$$y \ge \frac{3}{2}\left(5 - 2x^2\right)$$
$$y \ge \frac{15}{2} - 3x^2$$



23. $y \ge -\ln(x - 1)$



25. $y < 2^x$

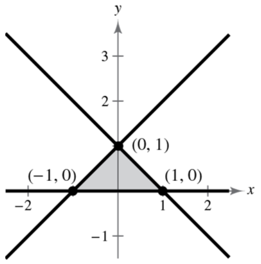


27. The line through $(-5, 0)$ and $(-1, 0)$ is $y = 5x + 5$. The shaded region below the line gives $y < 5x + 5$.

29. The line through $(0, 2)$ and $(3, 0)$ is $y = -\frac{2}{3}x + 2$. The shaded region above the line gives $y \ge -\frac{2}{3}x + 2$.

31. $\begin{cases} x + y \le 1 \\ -x + y \le 1 \\ y \ge 0 \end{cases}$

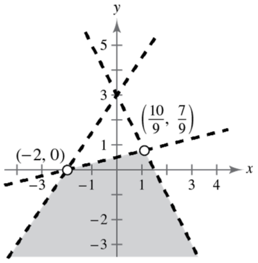
First, find the points of intersection of each pair of equations.



Vertex A	Vertex B	Vertex C
$x + y = 1$	$x + y = 1$	$-x + y = 1$
$-x + y = 1$	$y = 0$	$y = 0$
$(0, 1)$	$(1, 0)$	$(-1, 0)$

33. $\begin{cases} -3x + 2y < 6 \\ x - 4y > -2 \\ 2x + y < 3 \end{cases}$

First, find the points of intersection of each pair of equations.



Vertex A	Vertex B
$-3x + 2y = 6$	$-3x + 2y = 6$
$x - 4y = -2$	$2x + y = 3$
$(-2, 0)$	$(0, 3)$

Vertex C

$x - 4y = -2$

$2x + y = 3$

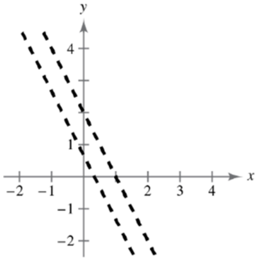
$\left(\frac{10}{9}, \frac{7}{9}\right)$

Note that B is not a vertex of the solution region.

35. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$

The graphs of $2x + y = 2$ and $6x + 3y = 2$ are parallel lines. The first inequality has the region above the line shaded. The second inequality has the region below the line shaded. There are no points that satisfy both inequalities.

No solution

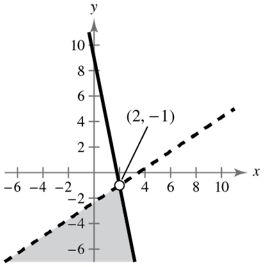


37. $\begin{cases} 2x - 3y > 7 \\ 5x + y \le 9 \end{cases}$

$$2x - 3y = 7$$
$$5x + y = 9 \Rightarrow y = -5x + 9$$

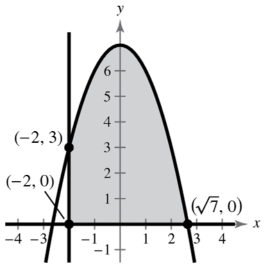
$$\begin{aligned} 2x - 3(-5x + 9) &= 7 \\ 2x + 15x - 27 &= 7 \\ 17x &= 34 \\ x &= 2 \\ y &= -5(2) + 9 = -1 \end{aligned}$$

Point of intersection: $(2, -1)$



39. $\begin{cases} x^2 + y \le 7 \\ x \ge -2 \\ y \ge 0 \end{cases}$

First, find the points of intersection of each pair of equations.



Vertex A

$x^2 + y = 7, x = -2$

$4 + y = 7$

$y = 3$

$(-2, 3)$

Vertex B

$x^2 + y = 7, y = 0$

$x^2 = 7$

$x = \sqrt{7}$

$(\sqrt{7}, 0)$

Vertex C

$x = -2, y = 0$

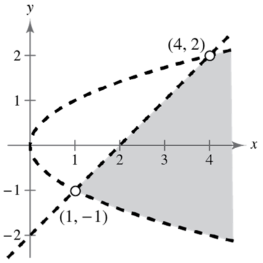
$(-2, 0)$

41. $\begin{cases} x - y^2 > 0 \\ x - y > 2 \end{cases}$

Points of intersection:

$$\begin{aligned} y^2 &= y + 2 \\ y^2 - y - 2 &= 0 \\ (y + 1)(y - 2) &= 0 \\ y &= -1, 2 \end{aligned}$$

$(1, -1), (4, 2)$



43. $$\begin{cases} x^2 + y^2 \le 25 \\ 4x - 3y < 0 \end{cases}$$

$$x = \frac{3}{4}y \Rightarrow \left(\frac{3}{4}y\right)^2 + y^2 = 25$$

$$\frac{9}{16}y^2 + y^2 = 25$$

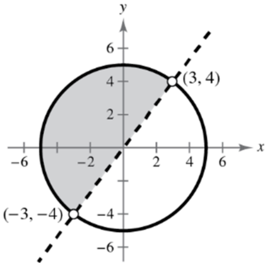
$$9y^2 + 16y^2 = 25(16)$$

$$y^2 = 16$$

$$y = \pm 4$$

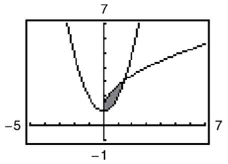
$$x = \frac{3}{4}(4) \Rightarrow x = 3$$

$$x = \frac{3}{4}(-4) \Rightarrow x = -3$$

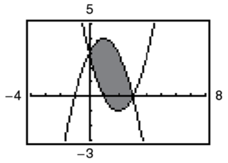


Points of intersection: $(3, 4), (-3, -4)$

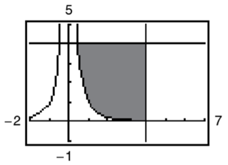
45. $$\begin{cases} y \le \sqrt{3x} + 1 \\ y \ge x^2 + 1 \end{cases}$$



47. $$\begin{cases} y < -x^2 + 2x + 3 \\ y > x^2 - 4x + 3 \end{cases}$$



49. $$\begin{cases} x^2y \ge 1 \Rightarrow y \ge \dfrac{1}{x^2} \\ 0 < x \le 4 \\ y \le 4 \end{cases}$$



51. Line through points $(6, 0)$ and $(0, 6)$: $y = 6 - x$

$$\begin{cases} x \ge 0 \\ y \ge 0 \\ y \le 6 - x \end{cases}$$

53. $(8, 0), (0, 8)$

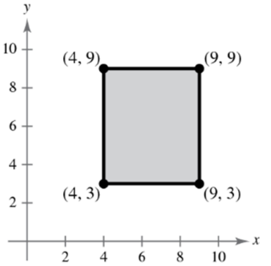
$$\begin{cases} x \ge 0 \\ y \ge 0 \\ x^2 + y^2 < 64 \end{cases}$$

55. Rectangular region with vertices at $(4, 3), (9, 3), (9, 9), (4, 9)$

$$\begin{cases} x \ge 4 \\ x \le 9 \\ y \ge 3 \\ y \le 9 \end{cases}$$

This system may be written as:

$$\begin{cases} 4 \le x \le 9 \\ 3 \le y \le 9 \end{cases}$$



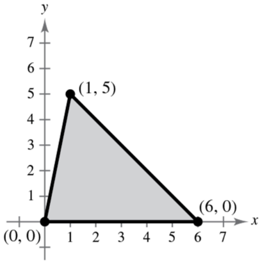
57. Triangle with vertices at $(0, 0), (6, 0), (1, 5)$

$(0, 0), (6, 0)$: $y = 0$

$(0, 0), (1, 5)$: $y = 5x$

$(6, 0), (1, 5)$: $y = -x + 6$

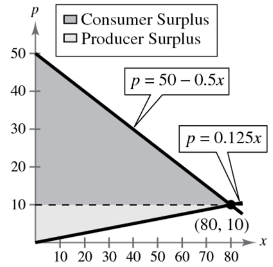
$$\begin{cases} y \ge 0 \\ y \le 5x \\ y \le -x + 6 \end{cases}$$



59. (a)

$$\begin{aligned} \text{Demand} &= \text{Supply} \\ 50 - 0.5x &= 0.125x \\ 50 &= 0.625x \\ 80 &= x \\ 10 &= p \end{aligned}$$

Point of equilibrium: $(80, 10)$



(b) The consumer surplus is the area of the triangular region defined by

$$\begin{cases} p \le 50 - 0.5x \\ p \ge 10 \\ x \ge 0. \end{cases}$$

$$\text{Consumer surplus} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(80)(40) = \$1600$$

The producer surplus is the area of the triangular region defined by

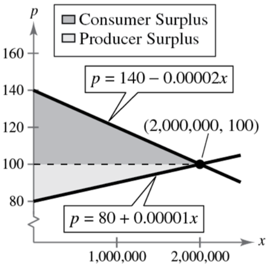
$$\begin{cases} p \ge 0.125x \\ p \le 10 \\ x \ge 0. \end{cases}$$

$$\text{Producer surplus} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(80)(10) = \$400$$

61. (a)

$$\begin{aligned} \text{Demand} &= \text{Supply} \\ 140 - 0.00002x &= 80 + 0.00001x \\ 60 &= 0.00003x \\ 2{,}000{,}000 &= x \\ 100 &= p \end{aligned}$$

Point of equilibrium: $(2{,}000{,}000, 100)$



(b) The consumer surplus is the area of the triangular region defined by

$$\begin{cases} p \le 140 - 0.00002x \\ p \ge 100 \\ x \ge 0. \end{cases}$$

$$\begin{aligned} \text{Consumer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(2{,}000{,}000)(40) \\ &= \$40{,}000{,}000 \end{aligned}$$

The producer surplus is the area of the triangular region defined by

$$\begin{cases} p \ge 80 + 0.00001x \\ p \le 100 \\ x \ge 0. \end{cases}$$

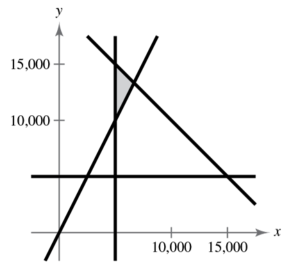
$$\begin{aligned} \text{Producer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(2{,}000{,}000)(20) \\ &= \$20{,}000{,}000 \end{aligned}$$

63. x = amount in smaller account

y = amount in larger account

Account constraints:

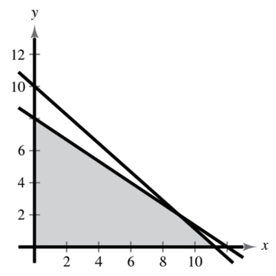
$$\begin{cases} x + y \le 20{,}000 \\ y \ge 2x \\ x \ge 5{,}000 \\ y \ge 5{,}000 \end{cases}$$



65. x = number of tables

y = number of chairs

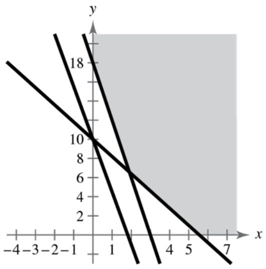
$$\begin{cases} x + \tfrac{3}{2}y \le 12 & \text{Assembly center} \\ \tfrac{4}{3}x + \tfrac{3}{2}y \le 15 & \text{Finishing center} \\ x \ge 0 \\ y \ge 0 \end{cases}$$



67. (a) x = number of ounces of food X

y = number of ounces of food Y

$$\begin{cases} 180x + 100y \geq 1000 \text{ (calcium)} \\ 6x + y \geq 18 \text{ (iron)} \\ 220x + 40y \geq 400 \text{ (magnesium)} \\ x \geq 0 \\ y \geq 0 \end{cases}$$



(b) Answers will vary. Some possible solutions which would satisfy the minimum daily requirements for calcium, iron, and magnesium:

$(5, 10) \Rightarrow$ 5 ounces of food X and 10 ounces of food Y

$(4, 12) \Rightarrow$ 4 ounces of food X and 12 ounces of food Y

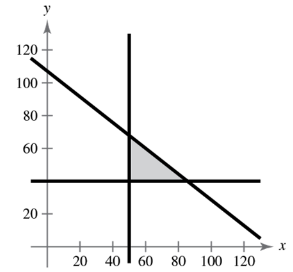
Either of these will satisfy the minimum daily requirements of the dietician's special dietary diet plan.

69. (a) Let x = number of bags of gravel

Let y = number of bags of stone.

The delivery requirements are:

$$\begin{cases} x \geq 50 \\ y \geq 40 \\ 55x + 70y \leq 7500 \end{cases}$$



(b) The points $(60, 60)$ and $(70, 52)$ lie in the solution region. These values would represent the number of bags of each type of fill while maintaining the maximum weight capacity of the truck. The first $(60, 60)$ is to ship 60 bags of gravel and 60 bags of stone. The second $(70, 52)$ is to is to ship 70 bags of gravel and 52 bags of stone.

71. True. The figure is a rectangle with a length of 9 units and a width of 11 units.

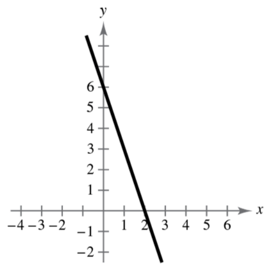
73. False. For example, $(0, 0)$ satisfies both inequalities.

75. Test a point on each side of the line $y = -x + 3$. Because the origin $(0, 0)$ satisfies the inequality, the solution set of the inequality lies below the dashed line.

77. $y = -3x + 6$

Slope: -3

y-intercept: $(0, 6)$



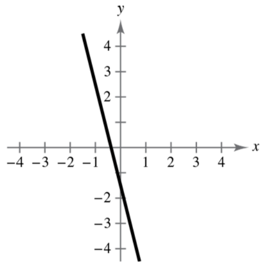
79. $8x + 2y = -3$

$2y = -8x - 3$

$y = -4x - \frac{3}{2}$

Slope: -4

y-intercept: $(0, -\frac{3}{2})$



81. $f(x) = 2x^2 + 5x$

Relative minimum at $(-1.25, -3.13)$

83. $v(x) = x\sqrt{x + 1}$

Relative minimum at $(-0.67, -0.38)$

85. $15° = \dfrac{30°}{2} \Rightarrow$ angle is in Quadrant I.

$$\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\cos 15° = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\tan 15° = \frac{1-\cos 30°}{\sin 30°} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$$

87. $\dfrac{-5\pi}{12} = \dfrac{-5\pi/6}{2} \Rightarrow$ angle is in Quadrant IV.

$$\sin\left(\frac{-5\pi}{12}\right) = -\sqrt{\frac{1-\cos(-5\pi/6)}{2}} = -\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

Note: $-\dfrac{\sqrt{2+\sqrt{3}}}{2} = -\left(\dfrac{\sqrt{2}+\sqrt{6}}{4}\right)$

$$\cos\left(\frac{-5\pi}{12}\right) = \sqrt{\frac{1+\cos(-5\pi/6)}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\tan\left(\frac{-5\pi}{12}\right) = \frac{1-\cos(-5\pi/6)}{\sin(-5\pi/6)} = \frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{-\frac{1}{2}} = -\left(2+\sqrt{3}\right)$$

89. $-22°30' = -22.5° = \dfrac{-45°}{2}$ angle is in Quadrant IV.

$$\sin(-22.5°) = -\sqrt{\frac{1-\cos(-45°)}{2}} = -\sqrt{\frac{1-\left(\frac{\sqrt{2}}{2}\right)}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos(-22.5°) = \sqrt{\frac{1+\cos(-45°)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\tan(-22.5°) = \frac{1-\cos(-45°)}{\sin(-45°)} = \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1-\sqrt{2}$$

Section 7.6 Linear Programming

1. objective

3. To find the vertices of the region corresponding to the system of constraints in a linear programming problem, find the points of intersection of the graphs of the equations.

5. $z = 4x + 3y$

At $(0, 5)$: $z = 4(0) + 3(5) = 15$

At $(0, 0)$: $z = 4(0) + 3(0) = 0$

At $(5, 0)$: $z = 4(5) + 3(0) = 20$

The minimum value is 0 at $(0, 0)$.

The maximum value is 20 at $(5, 0)$.

7. $z = 2x + 5y$

At $(1, 0)$: $z = 2(1) + 5(0) = 2$

At $(4, 0)$: $z = 2(4) + 5(0) = 8$

At $(3, 4)$: $z = 2(3) + 5(4) = 26$

At $(0, 5)$: $z = 2(0) + 5(5) = 25$

The minimum value is 2 at $(1, 0)$.

The maximum value is 26 at $(3, 4)$.

9. $z = 10x + 7y$

At $(0, 20)$: $z = 10(0) + 7(20) = 140$

At $(30, 45)$: $z = 10(30) + 7(45) = 615$

At $(60, 20)$: $z = 10(60) + 7(20) = 740$

At $(60, 0)$: $z = 10(60) + 7(0) = 600$

At $(0, 45)$: $z = 10(0) + 7(45) = 315$

The minimum value is 140 at $(0, 20)$.

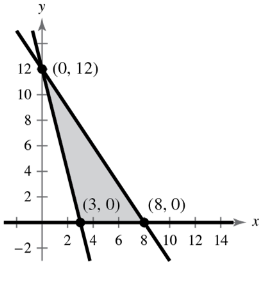
The maximum value is 740 at $(60, 20)$.

11. $z = 3x + 2y$

At $(3, 0)$: $z = 3(3) + 2(0) = 9$

The minimum value is 9 at $(3, 0)$.

The maximum value is 24 at any point on the line $3x + 2y = 24$, that is any point on the line segment between $(0, 12)$ and $(8, 0)$.



13. $z = 4x + 5y$

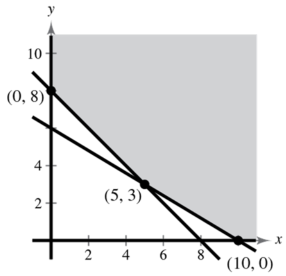
At $(10, 0)$: $z = 4(10) + 5(0) = 40$

At $(5, 3)$: $z = 4(5) + 5(3) = 35$

At $(0, 8)$: $z = 4(0) + 5(8) = 40$

The minimum value is 35 at $(5, 3)$.

The region is unbounded. There is no maximum.



15. $z = 3x + y$

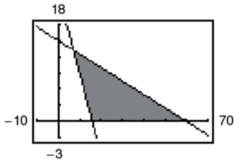
At $(16, 0)$: $z = 3(16) + 0 = 48$

At $(60, 0)$: $z = 3(60) + 0 = 180$

At $(7.2, 13.2)$: $z = 3(7.2) + 13.2 = 34.8$

The minimum value is 34.8 at $(7.2, 13.2)$.

The maximum value is 180 at $(60, 0)$.



17. $z = x$

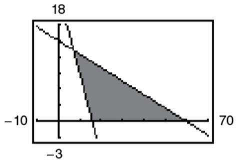
At $(60, 0)$: $z = 60$

At $(7.2, 13.2)$: $z = 7.2$

At $(16, 0)$: $z = 16$

The minimum value is 7.2 at $(7.2, 13.2)$.

The maximum value is 60 at $(60, 0)$.



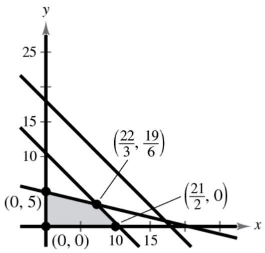


Figure for Exercises 19–21

19. $z = x + 5y$

At $(0, 5)$: $z = 0 + 5(5) = 25$

At $\left(\frac{22}{3}, \frac{19}{6}\right)$: $z = \frac{22}{3} + 5\left(\frac{19}{6}\right) = \frac{139}{6}$

At $\left(\frac{21}{2}, 0\right)$: $z = \frac{21}{2} + 5(0) = \frac{21}{2}$

At $(0, 0)$: $z = 0 + 5(0) = 0$

The minimum value is 0 at $(0, 0)$.

The maximum value is 25 at $(0,5)$.

21. $z = 4x + 5y$

At $(0, 5)$: $z = 4(0) + 5(5) = 25$

At $\left(\frac{22}{3}, \frac{19}{6}\right)$: $z = 4\left(\frac{22}{3}\right) + 5\left(\frac{19}{6}\right) = \frac{271}{6}$

At $\left(\frac{21}{2}, 0\right)$: $z = 4\left(\frac{21}{2}\right) + 5(0) = 42$

At $(0, 0)$: $z = 4(0) + 5(0) = 0$

The minimum value is 0 at $(0, 0)$.

The maximum value is $\frac{271}{6}$ at $\left(\frac{22}{3}, \frac{19}{6}\right)$.

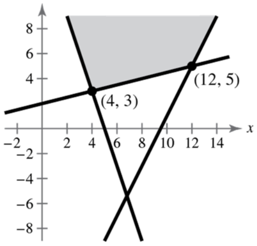


Figure for Exercises 23–25

23. $z = x + 2y$

At $(4, 3)$: $z = 4 + 2(3) = 10$

At $(12, 5)$: $z = 12 + 2(5) = 22$

The minimum value is 10 at $(4, 3)$.

There is no maximum value, the region is unbounded.

25. $z = x - y$

At $(4, 3)$: $z = 4 - 3 = 1$

At $(12, 5)$: $z = 12 - 5 = 7$

There is no minimum value.

The maximum value is 7 at $(12, 5)$.

27. Objective function: $z = 2.5x + y$

Constraints:

$x \geq 0, y \geq 0, 3x + 5y \leq 15, 5x + 2y \leq 10$

At $(0, 0)$: $z = 0$

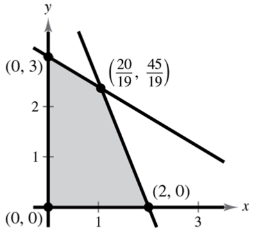
At $(2, 0)$: $z = 5$

At $\left(\frac{20}{19}, \frac{45}{19}\right)$: $z = \frac{95}{19} = 5$

At $(0, 3)$: $z = 3$

The minimum value is 0 at $(0, 0)$.

The maximum value of 5 occurs at any point on the line segment connecting $(2, 0)$ and $\left(\frac{20}{19}, \frac{45}{19}\right)$.



29. Objective function: $z = -x + 2y$

Constraints: $x \geq 0, y \geq 0, x \leq 10, x + y \leq 7$

At $(0, 0)$: $z = -0 + 2(0) = 0$

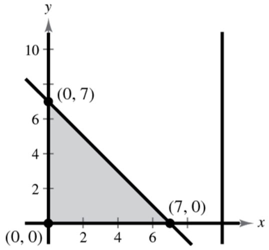
At $(0, 7)$: $z = -0 + 2(7) = 14$

At $(7, 0)$: $z = -7 + 2(0) = -7$

The constraint $x \leq 10$ is extraneous.

The minimum value is -7 at $(7, 0)$.

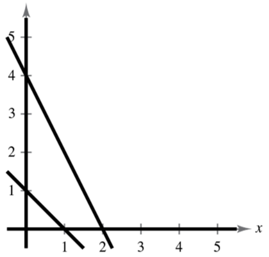
The maximum value is 14 at $(0, 7)$.



31. Objective function: $z = 3x + 4y$

Constraints: $x \geq 0, y \geq 0, x + y \leq 1, 2x + y \geq 4$

The feasible set is empty.



33. Objective function: $z = x + y$

Constraints: $x \ge 9, 0 \le y \le 7, -x + 3y \le -6$

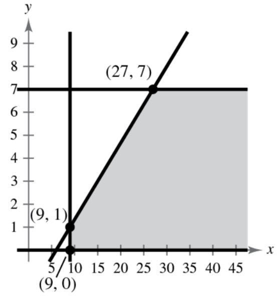
At $(9, 0)$: $z = 9 + 0 = 9$

At $(9, 1)$: $z = 9 + 1 = 10$

At $(27, 7)$: $z = 27 + 7 = 34$

The solution region is unbounded.
The minimum value is 9 at $(9, 0)$.

There is no maximum value.



35. $x =$ number of audits
$y =$ number of tax returns

Constraints:

$60x + 10y \le 780$
$16x + 4y \le 272$
$x \ge 0$
$y \ge 0$

Objective function:
$R = 1600x + 250y$

Vertices: $(0, 0), (13, 0), (5, 48), (0, 68)$

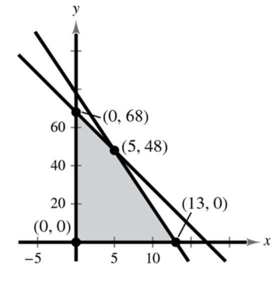
At $(0, 0)$: $R = 1600(0) + 250(0) = 0$

At $(13, 0)$: $R = 1600(13) + 250(0) = 20{,}800$

At $(5, 48)$: $R = 1600(5) + 250(48) = 20{,}000$

At $(0, 68)$: $R = 1600(0) + 250(68) = 17{,}000$

A maximum revenue of \$20,800 occurs when the firm conducts 13 audits and 0 tax returns.



37. $x =$ number of bags of Brand X
$y =$ number of bags of Brand Y

Constraints: $3x + 9y \ge 30$
$3x + 2y \ge 16$
$7x + 2y \ge 24$
$x \ge 0$
$y \ge 0$

Objective function: $C = 25x + 15y$

Vertices: $(0, 12), (4, 2), (2, 5), (10, 0)$

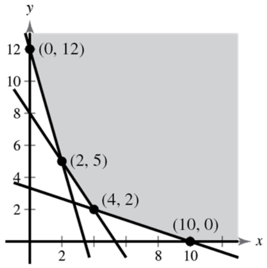
At $(0, 12)$: $C = 25(0) + 15(12) = 180$

At $(4, 2)$: $C = 25(4) + 15(2) = 130$

At $(2, 5)$: $C = 25(2) + 15(5) = 125$

At $(10, 0)$: $C = 25(10) + 15(0) = 250$

To minimize cost, use two bags of Brand X and five bags of Brand Y for a minimal cost of \$125.

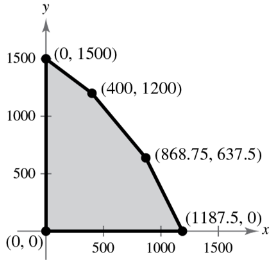


39. x = number of model X

y = number of model Y

Constraints:

$$\begin{aligned} 3x + 4y &\le 6000 && \text{Assembling} \\ 3x + 2.5y &\le 4200 && \text{Finishing} \\ 0.8x + 0.4y &\le 950 && \text{Packaging} \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$



Objective function: $P = 300x + 375y$

Vertices: $(0, 0), (1187.5, 0), (868.75, 637.5), (400, 1200), (0, 1500)$

At $(0, 0)$: $P = 300(0) + 375(0) = 0$

At $(1187.5, 0)$: $P = 300(1187.5) + 375(0) = 356{,}250$

At $(868.75, 637.5)$: $P = 300(868.75) + 375(637.5) = 499{,}687.5$

At $(400, 1200)$: $P = 300(400) + 375(1200) = 570{,}000$

At $(0, 1500)$: $P = 300(0) + 375(1500) = 562{,}500$

For a maximum profit of \$570,000, the company should produce 400 units of model X and 1200 units of model Y.

41. True. The objective function has a maximum value at any point on the line segment connecting the two vertices. Both of these points are on the line $y = -x + 11$ and lie between $(4, 7)$ and $(8, 3)$.

43. The solution region is unbounded $(x + y \ge 1)$, so no maximum value exists.

45. If a linear programming problem has an objective function $z = 3x + 5y$ and an infinite number of optimal solutions then the slope of the line connecting two points is $m = -\frac{3}{5}$, that is $z = 3x + 5y \Rightarrow y = -\frac{3}{5}x - \frac{1}{5}z$.

47.
$$\begin{cases} -x + y + z = -3 & \text{Equation 1} \\ 2x - 3y + 2z = -4 & \text{Equation 2} \\ x - y - 2z = 7 & \text{Equation 3} \end{cases}$$

$$\begin{cases} -x + y + z = -3 & \\ -y + 4z = -10 & 2(\text{Eq. } 1) + \text{Eq. } 2 \\ x - y - 2z = 7 & \end{cases}$$

$$\begin{cases} -x + y + z = -3 & \\ -y + 4z = -10 & \\ -z = 4 & \text{Eq. } 1 + \text{Eq. } 3 \end{cases}$$

$$\begin{aligned} z &= -4 \\ -y + 4(-4) &= -10 \\ -y &= 6 \\ y &= -6 \\ -x - 6 - 4 &= -3 \\ -x &= 7 \\ x &= -7 \end{aligned}$$

Solution: $(-7, -6, -4)$

49.
$$\begin{cases} x - y - z = 0 & \text{Equation 1} \\ x + 2y - z = 8 & \text{Equation 2} \\ 2x - z = 5 & \text{Equation 3} \end{cases}$$

$$\begin{cases} x - y - z = 0 & \\ 3y = 8 & -\text{Eq. } 1 + \text{Eq. } 2 \\ 2y + z = 5 & -2\text{Eq. } 1 + \text{Eq. } 3 \end{cases}$$

$y = \frac{8}{3}$

$z = 5 - 2y = 5 - 2\left(\frac{8}{3}\right) = -\frac{1}{3}$

$x = y + z = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Solution: $\left(\frac{7}{3}, \frac{8}{3}, -\frac{1}{3}\right)$

51. $$\begin{cases} -x + 2w = 1 & \text{Equation 1} \\ 4y - z - w = 2 & \text{Equation 2} \\ y - w = 0 & \text{Equation 3} \\ 3x - 2y + 3z = 4 & \text{Equation 4} \end{cases}$$

$$\begin{cases} -x + 2w = 1 & \\ y - w = 0 & \text{Interchange Eq. 2 and Eq. 3} \\ 4y - z - w = 2 & \\ -2y + 3z + 6w = 7 & \text{3Eq. 1 + Eq. 4} \end{cases}$$

$$\begin{cases} x - 2w = -1 & \\ y - w = 0 & \\ -z + 3w = 2 & -4\text{Eq. 2 + Eq. 3} \\ 3z + 4w = 7 & \text{2Eq. 2 + Eq. 4} \end{cases}$$

$$\begin{cases} x - 2w = -1 & \\ y - w = 0 & \\ z - 3w = -2 & -\text{Eq. 3} \\ 13w = 13 & \text{3Eq. 3 + Eq. 4} \end{cases}$$

$w = 1$

$z = -2 + 3w = 1$

$y = w = 1$

$x = 2w - 1 = 1$

Solution: $(1, 1, 1, 1)$

Review Exercises for Chapter 7

1. $$\begin{cases} x + y = 2 \\ x - y = 0 \Rightarrow x = y \end{cases}$$

$x + x = 2$

$2x = 2$

$x = 1$

$y = 1$

Solution: $(1, 1)$

3. $$\begin{cases} 4x - y - 1 = 0 \Rightarrow y = 4x - 1 \\ 8x + y - 17 = 0 \end{cases}$$

$8x + (4x - 1) - 17 = 0$

$12x = 18$

$x = \frac{3}{2}$

$4\left(\frac{3}{2}\right) - y - 1 = 0$

$-y + 5 = 0$

$y = 5$

Solution: $\left(\frac{3}{2}, 5\right)$

5. $$\begin{cases} 0.5x + y = 0.75 \Rightarrow y = 0.75 - 0.5x \\ 1.25x - 4.5y = -2.5 \end{cases}$$

$1.25x - 4.5(0.75 - 0.5x) = -2.5$

$1.25x - 3.375 + 2.25x = -2.5$

$3.50x = 0.875$

$x = 0.25$

$y = 0.625$

Solution: $(0.25, 0.625)$

7. $$\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \Rightarrow x = y + 1 \end{cases}$$

$(y + 1)^2 - y^2 = 9$

$2y + 1 = 9$

$y = 4$

$x = 5$

Solution: $(5, 4)$

9. $\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases} \Rightarrow 2x^2 = x^4 - 2x^2$

$0 = x^4 - 4x^2$

$0 = x^2(x^2 - 4)$

$0 = x^2(x + 2)(x - 2) \Rightarrow x = 0, -2, 2$

$x = 0: y = 2(0)^2 = 0$

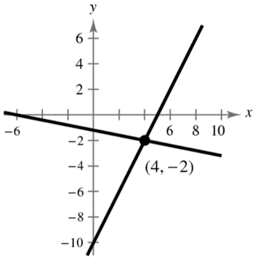
$x = -2: y = 2(-2)^2 = 8$

$x = 2: y = 2(2)^2 = 8$

Solutions: $(0, 0), (-2, 8), (2, 8)$

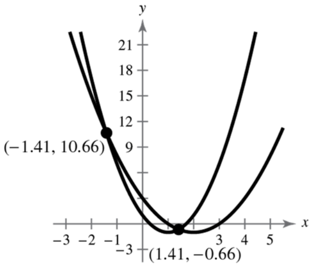
11. $\begin{cases} 2x - y = 10 \\ x + 5y = -6 \end{cases}$

Point of intersection: $(4, -2)$



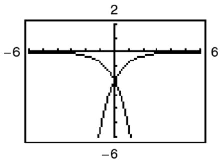
13. $\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$

Points of intersection: $(1.41, -0.66), (-1.41, 10.66)$



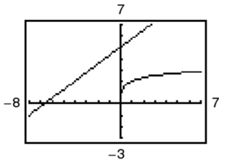
15. $\begin{cases} y = -2e^{-x} \\ 2e^x + y = 0 \Rightarrow y = -2e^x \end{cases}$

Point of intersection: $(0, -2)$

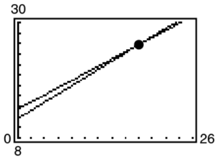


17. $\begin{cases} y = 2 + \log x \\ y = \frac{3}{4}x + 5 \end{cases}$

No Solution



19.



$0.68a + 13.5 > 0.78a + 11.7$

$1.8 > 0.1a$

$18 > a$

The BMI for males exceeds the BMI for females after age 18.

21. $\begin{cases} 2l + 2w = 68 \\ w = \frac{8}{9}l \end{cases}$

$2l + 2\left(\frac{8}{9}\right)l = 68$

$\frac{34}{9}l = 68$

$l = 18$

$w = \frac{8}{9}l = 16$

The width of the rectangle is 16 feet, and the length is 18 feet.

23. $\begin{cases} 2x - y = 2 \Rightarrow 16x - 8y = 16 \\ 6x + 8y = 39 \Rightarrow 6x + 8y = 39 \end{cases}$

$22x = 55$

$x = \frac{55}{22} = \frac{5}{2}$

Back-substitute $x = \frac{5}{2}$ into Equation 1.

$2\left(\frac{5}{2}\right) - y = 2$

$y = 3$

Solution: $\left(\frac{5}{2}, 3\right)$

25. $\begin{cases} 3x - 2y = 0 \\ 3x + 2y = 0 \end{cases}$

Add the equations $6x = 0 \Rightarrow x = 0$.

Back-substitute into Equation 1.

$3(0) - 2y = 0$

$2y = 0$

$y = 0$

Solution: $(0, 0)$

27. $$\left\{\begin{array}{l} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{array}\right. \begin{array}{l} \Rightarrow \\ \Rightarrow \\ \end{array} \begin{array}{r} 5x - 8y = 14 \\ \underline{-5x + 8y = -14} \\ 0 = 0 \end{array}$$

There are infinitely many solutions.

Let $y = a$, then $5x - 8a = 14 \Rightarrow x = \frac{8}{5}a + \frac{14}{5}$.

Solution: $\left(\frac{8}{5}a + \frac{14}{5}, a\right)$ where a is any real number.

29. $$\left\{\begin{array}{l} x + 5y = 4 \\ x - 3y = 6 \end{array}\right. \begin{array}{l} \Rightarrow \\ \Rightarrow \\ \end{array} \begin{array}{r} x + 5y = 4 \\ \underline{-x + 3y = -6} \\ 8y = -2 \end{array} \begin{array}{l} \\ \\ \Rightarrow y = -\frac{1}{4} \end{array}$$

Matches graph (d). The system has one solution and is consistent.

31. $$\left\{\begin{array}{l} 3x - y = 7 \\ -6x + 2y = 8 \end{array}\right. \begin{array}{l} \Rightarrow \\ \Rightarrow \\ \end{array} \begin{array}{r} 6x - 2y = 14 \\ \underline{-6x + 2y = 8} \\ 0 \neq 22 \end{array}$$

Matches graph (b). The system has no solution and is inconsistent.

33. $$\begin{aligned} 22 + 0.00001x &= 43 - 0.0002x \\ 0.00021x &= 21 \\ x &= 100{,}000,\ p = 2^3 \end{aligned}$$

Point of Equilibrium: $(100{,}000, 23)$

35. $$\left\{\begin{array}{rrrcr} x & -\ 4y & +\ 3z & = & 3 \\ & -y & +\ z & = & -1 \\ & & z & = & -5 \end{array}\right.$$

$$\begin{aligned} -y + (-5) &= -1 \Rightarrow y = -4 \\ x - 4(-4) + 3(-5) &= 3 \Rightarrow x = 2 \end{aligned}$$

Solution: $(2, -4, -5)$

37. $$\left\{\begin{array}{rrrcr} 4x & -\ 3y & -\ 2z & = & -65 \\ & 8y & -\ 7z & = & -14 \\ & & z & = & 10 \end{array}\right.$$

$$\begin{aligned} 8y - 7(10) &= -14 \Rightarrow y = 7 \\ 4x - 3(7) - 2(10) &= -65 \Rightarrow x = -6 \end{aligned}$$

Solution: $(-6, 7, 10)$

39. $$\left\{\begin{array}{rrrcr} x & +\ 2y & +\ 6z & = & 4 \\ -3x & +\ 2y & -\ z & = & -4 \\ 4x & & +\ 2z & = & 16 \end{array}\right. \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} x & +\ 2y & +\ 6z & = & 4 \\ & 8y & +\ 17z & = & 8 \\ & -8y & -\ 22z & = & 0 \end{array}\right. \begin{array}{l} \\ \text{3Eq. 1 + Eq. 2} \\ \text{–4 Eq. 1 + Eq. 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} x & +\ 2y & +\ 6z & = & 4 \\ & 8y & +\ 17z & = & 8 \\ & & -5z & = & 8 \end{array}\right. \begin{array}{l} \\ \\ \text{Eq. 2 + Eq. 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} x & +\ 2y & +\ 6z & = & 4 \\ & 8y & +\ 17z & = & 8 \\ & & z & = & -\frac{8}{5} \end{array}\right. \begin{array}{l} \\ \\ -\frac{1}{5}\text{ Eq. 3} \end{array}$$

$$\begin{aligned} 8y + 17\left(-\tfrac{8}{5}\right) &= 8 \Rightarrow y = \tfrac{22}{5} \\ x + 2\left(\tfrac{22}{5}\right) + 6\left(-\tfrac{8}{5}\right) &= 4 \Rightarrow x = \tfrac{24}{5} \end{aligned}$$

Solution: $\left(\frac{24}{5}, \frac{22}{5}, -\frac{8}{5}\right)$

41. $$\left\{\begin{array}{rrrcr} 2x & & +\ 6z & = & -9 \\ 3x & -\ 2y & +\ 11z & = & -16 \\ 3x & -\ y & +\ 7z & = & -11 \end{array}\right. \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} -x & +\ 2y & -\ 5z & = & 7 \\ 3x & -\ 2y & +\ 11z & = & -16 \\ 3x & -\ y & +\ 7z & = & -11 \end{array}\right. \begin{array}{l} (-1)\text{Eq. 2 + Eq. 1} \\ \\ \\ \end{array}$$

$$\left\{\begin{array}{rrrcr} -x & +\ 2y & -\ 5z & = & 7 \\ & 4y & -\ 4z & = & 5 \\ & 5y & -\ 8z & = & 10 \end{array}\right. \begin{array}{l} \\ \text{3Eq. 1 + Eq. 2} \\ \text{3Eq. 1 + Eq. 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} -x & +\ 2y & -\ 5z & = & 7 \\ & 4y & -\ 4z & = & 5 \\ & -3y & & = & 0 \end{array}\right. \begin{array}{l} \\ \\ (-2)\text{Eq. 2 + Eq. 3} \end{array}$$

$$\left\{\begin{array}{rrrcr} -x & +\ 2y & -\ 5z & = & 7 \\ & y & -\ z & = & \frac{5}{4} \\ & y & & = & 0 \end{array}\right. \begin{array}{l} \\ \left(\frac{1}{4}\right)\text{Eq. 2} \\ \left(-\frac{1}{3}\right)\text{Eq. 3} \end{array}$$

$$\begin{aligned} 0 - z &= \tfrac{5}{4} \Rightarrow z = -\tfrac{5}{4} \\ -x + 2(0) - 5\left(-\tfrac{5}{4}\right) &= 7 \Rightarrow x = -\tfrac{3}{4} \end{aligned}$$

Solution: $\left(-\frac{3}{4}, 0, -\frac{5}{4}\right)$

43. $$\left\{\begin{array}{rrrcr} 5x & -\ 12y & +\ 7z & = & 16 \\ 3x & -\ 7y & +\ 4z & = & 9 \end{array}\right. \begin{array}{l} \Rightarrow \\ \Rightarrow \end{array} \left\{\begin{array}{rrrcr} 15x & -\ 36y & +\ 21z & = & 48 \\ -15x & +\ 35y & -\ 20z & = & -45 \\ \hline & -y & +\ z & = & 3 \end{array}\right.$$

Let $z = a$. Then $y = a - 3$ and $5x - 12(a - 3) + 7a = 16 \Rightarrow x = a - 4$.

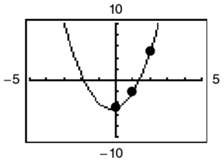
Solution: $(a - 4, a - 3, a)$ where a is any real number.

45. $y = ax^2 + bx + c$ through $(0, -5), (1, -2)$, and $(2, 5)$.

$$\begin{array}{llrrrcl} (0,-5): & -5 = & & & c \Rightarrow & & c = -5 \\ (1,-2): & -2 = & a + & b + & c \Rightarrow & \left\{\begin{array}{l} a + b = 3 \\ 2a + b = 5 \end{array}\right. & \\ (2,\ 5): & 5 = & 4a + & 2b + & c \Rightarrow & & \end{array}$$

$$\begin{cases} 2a + b = 5 \\ \underline{-a - b = -3} \end{cases}$$

$$\begin{aligned} a &= 2 \\ b &= 1 \end{aligned}$$



The equation of the parabola is $y = 2x^2 + x - 5$.

47. $x^2 + y^2 + Dx + Ey + F = 0$ through $(-1, -2), (5, -2)$, and $(2, 1)$.

$$\begin{array}{lrcl} (-1,-2): & 5 - D - 2E + F = 0 & \Rightarrow & \left\{\begin{array}{rcr} D + 2E - F &=& 5 \\ 5D - 2E + F &=& -29 \\ 2D + E + F &=& -5 \end{array}\right. \\ (5,-2): & 29 + 5D - 2E + F = 0 & \Rightarrow & \\ (2,1): & 5 + 2D + E + F = 0 & \Rightarrow & \end{array}$$

From the first two equations

$$\begin{aligned} 6D &= -24 \\ D &= -4. \end{aligned}$$

Substituting $D = -4$ into the second and third equations yields:

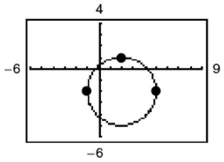
$$\begin{array}{rcl} -20 - 2E + F = -29 & \Rightarrow & \left\{\begin{array}{rcr} -2E + F &=& -9 \\ \underline{-E - F} &=& \underline{-3} \end{array}\right. \\ -8 + E + F = -5 & \Rightarrow & \end{array}$$

$$\begin{aligned} -3E &= -12 \\ E &= 4 \\ F &= -1 \end{aligned}$$

The equation of the circle is $x^2 + y^2 - 4x + 4y - 1 = 0$.

To verify the result using a graphing utility, solve the equation for y.

$$\begin{aligned} \left(x^2 - 4x + 4\right) + \left(y^2 + 4y + 4\right) &= 1 + 4 + 4 \\ (x-2)^2 + (y+2)^2 &= 9 \\ (y+2)^2 &= 9 - (x-2)^2 \\ y &= -2 \pm \sqrt{9 - (x-2)^2} \end{aligned}$$



Let $y_1 = -2 + \sqrt{9 - (x-2)^2}$ and $y_2 = -2 - \sqrt{9 - (x-2)^2}$.

49. From the following chart we obtain our system of equations.

	A	B	C
Mixture X	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
Mixture Y	0	0	1
Mixture Z	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Desired Mixture	$\frac{6}{27}$	$\frac{8}{27}$	$\frac{13}{27}$

$$\left.\begin{aligned} \tfrac{1}{5}x + \tfrac{1}{3}z &= \tfrac{6}{27} \\ \tfrac{2}{5}x + \tfrac{1}{3}z &= \tfrac{8}{27} \end{aligned}\right\} x = \tfrac{10}{27},\ z = \tfrac{12}{27}$$

$$\tfrac{2}{5}x + y + \tfrac{1}{3}z = \tfrac{13}{27} \Rightarrow y = \tfrac{5}{27}$$

To obtain the desired mixture, use 10 gallons of spray X, 5 gallons of spray Y, and 12 gallons of spray Z.

51. Let x = amount invested at 7%

y = amount invested at 9%

z = amount invested at 11%.

$y = x - 3000$ and $z = x - 5000 \Rightarrow y + z = 2x - 8000$

$$\begin{cases} x + y + z = 40{,}000 \\ 0.07x + 0.09y + 0.11z = 3500 \\ y + z = 2x - 8000 \end{cases}$$

$$\begin{aligned} x + (2x - 8000) = 40{,}000 &\Rightarrow x = 16{,}000 \\ y = 16{,}000 - 3000 &\Rightarrow y = 13{,}000 \\ z = 16{,}000 - 5000 &\Rightarrow z = 11{,}000 \end{aligned}$$

So, \$16,000 was invested at 7%, \$13,000 at 9%, and \$11,000 at 11%.

53. $s = \frac{1}{2}at^2 + v_0t + s_0$

When $t = 1$: $s = 134$: $\frac{1}{2}a(1)^2 + v_0(1) + s_0 = 134 \Rightarrow a + 2v_0 + 2s_0 = 268$

When $t = 2$: $s = 86$: $\frac{1}{2}a(2)^2 + v_0(2) + s_0 = 86 \Rightarrow 2a + 2v_0 + s_0 = 86$

When $t = 3$: $s = 6$: $\frac{1}{2}a(3)^2 + v_0(3) + s_0 = 6 \Rightarrow 9a + 6v_0 + 2s_0 = 12$

$$\begin{cases} a + 2v_0 + 2s_0 = 268 \\ 2a + 2v_0 + s_0 = 86 \\ 9a + 6v_0 + 2s_0 = 12 \end{cases}$$

$$\begin{cases} a + 2v_0 + 2s_0 = 268 & \\ -2v_0 - 3s_0 = -450 & (-2)\text{Eq. } 1 + \text{Eq. } 2 \\ -12v_0 - 16s_0 = -2400 & (-9)\text{Eq. } 1 + \text{Eq. } 3 \end{cases}$$

$$\begin{cases} a + 2v_0 + 2s_0 = 268 & \\ -2v_0 - 3s_0 = -450 & \\ 3v_0 + 4s_0 = 600 & \left(-\frac{1}{4}\right)\text{Eq. } 3 \end{cases}$$

$$\begin{cases} a + 2v_0 + 2s_0 = 268 & \\ -2v_0 - 3s_0 = -450 & \\ -s_0 = -150 & 3\text{Eq. } 2 + 2\text{Eq. } 3 \end{cases}$$

$$\begin{aligned} -s_0 = -150 &\Rightarrow s_0 = 150 \\ -2v_0 - 3(150) = -450 &\Rightarrow v_0 = 0 \\ a + 2(0) + 2(150) = 268 &\Rightarrow a = -32 \end{aligned}$$

The position equation is $s = \frac{1}{2}(-32)t^2 + (0)t + 150$, or $s = -16t^2 + 150$.

55. $\dfrac{3}{x^2 + 20x} = \dfrac{3}{x(x + 20)} = \dfrac{A}{x} + \dfrac{B}{x + 20}$

57. $\dfrac{3x - 4}{x^3 - 5x^2} = \dfrac{3x - 4}{x^2(x - 5)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x - 5}$

59. $\dfrac{4 - x}{x^2 + 6x + 8} = \dfrac{A}{x + 2} + \dfrac{B}{x + 4}$

$4 - x = A(x + 4) + B(x + 2)$

Let $x = -2$: $6 = 2A \Rightarrow A = 3$

Let $x = -4$: $8 = -2B \Rightarrow B = -4$

$$\frac{4 - x}{x^2 + 6x + 8} = \frac{3}{x + 2} - \frac{4}{x + 4}$$

61. $$\frac{x^2}{x^2+2x-15} = 1 - \frac{2x-15}{x^2+2x-15}$$

$$\frac{-2x+15}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3}$$

$$-2x+15 = A(x-3) + B(x+5)$$

Let $x = -5$: $25 = -8A \Rightarrow A = -\frac{25}{8}$

Let $x = 3$: $9 = 8B \Rightarrow B = \frac{9}{8}$

$$\frac{x^2}{x^2+2x-15} = 1 - \frac{25}{8(x+5)} + \frac{9}{8(x-3)}$$

63. $$\begin{aligned}\frac{x^2+2x}{x^3-x^2+x-1} &= \frac{x^2+2x}{(x-1)(x^2+1)}\\ &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1}\\ x^2+2x &= A(x^2+1) + (Bx+C)(x-1)\\ &= Ax^2 + A + Bx^2 - Bx + Cx - C\\ &= (A+B)x^2 + (-B+C)x + (A-C)\end{aligned}$$

Equating coefficients of like terms gives $1 = A + B$, $2 = -B + C$, and $0 = A - C$.

Adding both sides of all three equations gives $3 = 2A$. So, $A = \frac{3}{2}$, $B = -\frac{1}{2}$, and $C = \frac{3}{2}$.

$$\begin{aligned}\frac{x^2+2x}{x^3-x^2+x-1} &= \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}x+\frac{3}{2}}{x^2+1}\\ &= \frac{1}{2}\left(\frac{3}{x-1} - \frac{x-3}{x^2+1}\right)\end{aligned}$$

65. $$\frac{3x^2+4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$\begin{aligned}3x^2+4x &= (Ax+B)(x^2+1) + Cx + D\\ &= Ax^3 + Bx^2 + (A+C)x + (B+D)\end{aligned}$$

Equating coefficients of like terms gives

$0 = A$

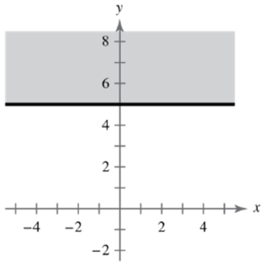
$3 = B$

$4 = 0 + C \Rightarrow C = 4$

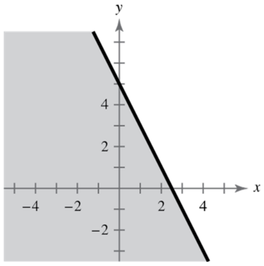
$0 = B + D \Rightarrow D = -3$

$$\frac{3x^2+4x}{(x^2+1)^2} = \frac{3}{x^2+1} + \frac{4x-3}{(x^2+1)^2}$$

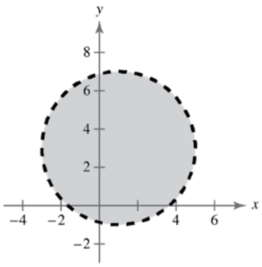
67. $y \geq 5$



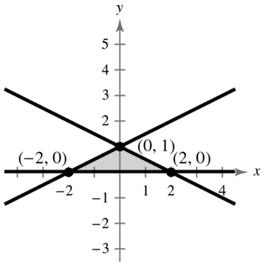
69. $y \leq 5 - 2x$



71. $(x-1)^2 + (y-3)^2 < 16$



73. $\begin{cases} x + 2y \le 2 \\ -x + 2y \le 2 \\ y \ge 0 \end{cases}$



Vertex A

$\begin{cases} x + 2y = 2 \\ -x + 2y = 2 \end{cases}$

$4y = 4 \Rightarrow y = 1$

$x + 2(1) = 2 \Rightarrow x = 0$

$(0, 1)$

Vertex B

$\begin{cases} x + 2y = 2 \\ y = 0 \end{cases}$

$x + 2(0) = 2$

$x = 2$

$(2, 0)$

Vertex C

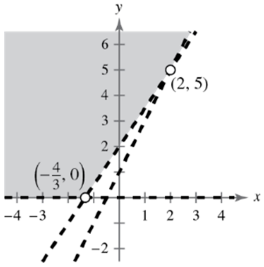
$\begin{cases} -x + 2y = 2 \\ y = 0 \end{cases}$

$-x + 2(0) = 2$

$x = -2$

$(-2, 0)$

75. $\begin{cases} 2x - y < -1 \\ -3x + 2y > 4 \\ y > 0 \end{cases}$



Vertex A

$\begin{cases} 2x - y = -1 \Rightarrow 4x - 2y = -2 \\ -3x + 2y = 4 \Rightarrow -3x + 2y = 4 \end{cases}$

$x = 2$

$2(2) - y = -1 \Rightarrow -y = -5 \Rightarrow y = 5$

$(2, 5)$

Vertex B

$\begin{cases} -3x + 2y = 4 \\ y = 0 \end{cases}$

$-3x + 2(0) = 4 \Rightarrow x = -\frac{4}{3}$

$\left(-\frac{4}{3}, 0\right)$

77. $\begin{cases} y < x + 1 \\ y > x^2 - 1 \end{cases}$

Vertices:

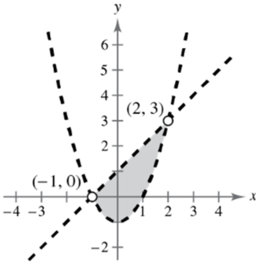
$x + 1 = x^2 - 1$

$0 = x^2 - x - 2 = (x+1)(x-2)$

$x = -1$ or $x = 2$

$y = 0 \quad y = 3$

$(-1, 0) \quad (2, 3)$

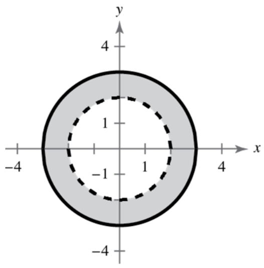


79. $\begin{cases} x^2 + y^2 > 4 \Rightarrow y^2 > 4 - x^2: \\ x^2 + y^2 \le 9 \Rightarrow y^2 \le 9 - x^2: \end{cases}$ The region outside the circle centered at $(0, 0)$ with radius of 2. The region inside and on the circle centered at $(0, 0)$ with radius of 3.

Vertices: $4 - x^2 = 9 - x^2$

$0 \ne 5$

The circles do not intersect, so there are no vertices.



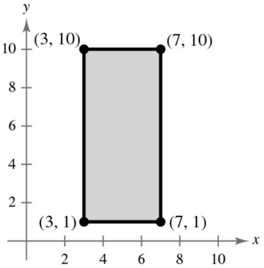
81. Rectangular region with vertices at:

$(3, 1), (7, 1), (7, 10)$, and $(3, 10)$

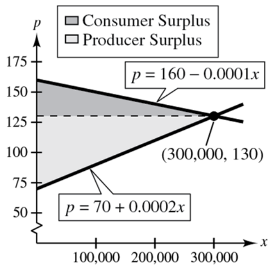
$$\begin{cases} x \geq 3 \\ x \leq 7 \\ y \geq 1 \\ y \leq 10 \end{cases}$$

This system may be written as:

$$\begin{cases} 3 \leq x \leq 7 \\ 1 \leq y \leq 10 \end{cases}$$



83. (a)



$$\begin{aligned} 160 - 0.0001x &= 70 + 0.0002x \\ 90 &= 0.0003x \\ x &= 300{,}000 \text{ units} \\ p &= \$130 \end{aligned}$$

Point of equilibrium: $(300{,}000, 130)$

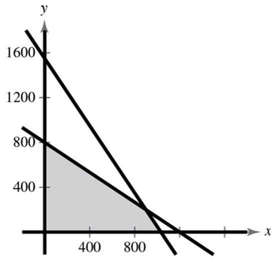
(b) Consumer surplus: $\frac{1}{2}(300{,}000)(30) = \$4{,}500{,}000$

Producer surplus: $\frac{1}{2}(300{,}000)(60) = \$9{,}000{,}000$

85. x = number of units of Product I

y = number of units of Product II

$$\begin{cases} 20x + 30y \leq 24{,}000 \\ 12x + 8y \leq 12{,}400 \\ x \geq 0 \\ y \geq 0 \end{cases}$$



87. Objective function: $z = 3x + 4y$

Constraints: $$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 5y \leq 50 \\ 4x + y \leq 28 \end{cases}$$

At $(0, 0)$: $z = 0$

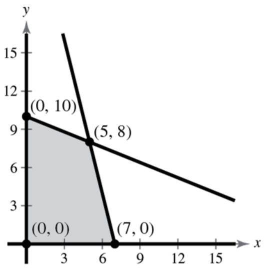
At $(0, 10)$: $z = 40$

At $(5, 8)$: $z = 47$

At $(7, 0)$: $z = 21$

The minimum value is 0 at $(0, 0)$.

The maximum value is 47 at $(5, 8)$.



89. Objective function: $z = 1.75x + 2.25y$

Constraints: $$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \geq 25 \\ 3x + 2y \geq 45 \end{cases}$$

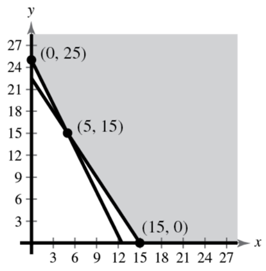
At $(0, 25)$: $z = 56.25$

At $(5, 15)$: $z = 42.5$

At $(15, 0)$: $z = 26.25$

The minimum value is 26.25 at $(15, 0)$.

Because the region is unbounded, there is no maximum value.



91. x = number of haircuts

y = number of permanents

Objective function: Optimize $R = 25x + 70y$ subject to the following constraints:

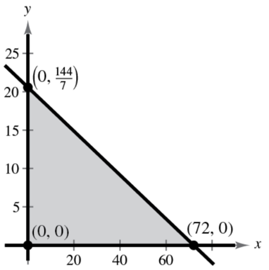
$$\begin{cases} x \geq 0 \\ y \geq 0 \\ \left(\frac{20}{60}\right)x + \left(\frac{70}{60}\right)y \leq 24 \Rightarrow 2x + 7y \leq 144 \end{cases}$$

At $(0, 0)$: $R = 0$

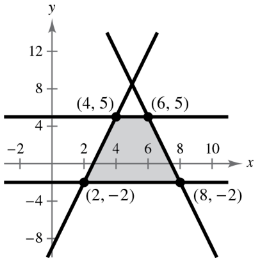
At $(72, 0)$: $R = 1800$

At $\left(0, \frac{144}{7}\right)$: $R = 1440$

The revenue is optimal if the student does 72 haircuts and no permanents. The maximum revenue is \$1800.



93. True. Because $y = 5$ and $y = -2$ are horizontal lines, exactly one pair of opposite sides are parallel. The non-parallel sides of the trapezoid are equal in length. Therefore, the trapezoid is isosceles as shown below.



The distance from $(-4, 5)$ to $(2, -2)$ is equal to the distance from $(6, 5)$ to $(8, -2)$.

$$d_1 = \sqrt{(4-2)^2 + \left[5-(-2)\right]^2} = \sqrt{53}$$

$$d_2 = \sqrt{(8-6)^2 + (-2-5)^2} = \sqrt{53}$$

95. There are an infinite number of linear systems with the solution $(-8, 10)$. One possible system is:

$$\begin{cases} 4x + y = -22 \\ \frac{1}{2}x + y = 6 \end{cases}$$

97. There are infinite linear systems with the solution $\left(\frac{4}{3}, 3\right)$.

One possible system is:

$$\begin{cases} 3x + y = 7 \\ -6x + 3y = 1 \end{cases}$$

99. There are an infinite number of linear systems with the solution $(4, -1, 3)$. One possible system is as follows:

$$\begin{cases} x + y + z = 6 \\ x + y - z = 0 \\ x - y - z = 2 \end{cases}$$

101. There are an infinite number of linear systems with the solution $\left(5, \frac{3}{2}, 2\right)$. One possible system is:

$$\begin{cases} 2x + 2y - 3z = 7 \\ x - 2y + z = 4 \\ -x + 4y - z = -1 \end{cases}$$

103. A system of linear equations is inconsistent if it has no solution.

Problem Solving for Chapter 7

1. The longest side of the triangle is a diameter of the circle and has a length of 20.

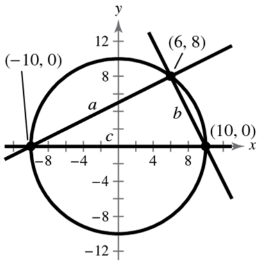
The lines $y = \frac{1}{2}x + 5$ and $y = -2x + 20$ intersect at the point $(6, 8)$.

The distance between $(-10, 0)$ and $(6, 8)$ is:

$$d_1 = \sqrt{\left(6-(-10)\right)^2 + (8-0)^2} = \sqrt{320} = 8\sqrt{5}$$

The distance between $(6, 8)$ and $(10, 0)$ is:

$$d_2 = \sqrt{(10-6)^2 + (0-8)^2} = \sqrt{80} = 4\sqrt{5}$$



Because $\left(\sqrt{320}\right)^2 + \left(\sqrt{80}\right)^2 = (20)^2$

$$400 = 400,$$

the sides of the triangle satisfy the Pythagorean Theorem. So, the triangle is a right triangle.

3. The system will have exactly one solution when the slopes of the line are *not* equal.

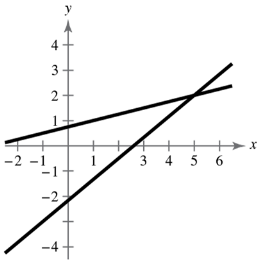
$$\begin{cases} ax + by = e \Rightarrow y = -\dfrac{a}{b}x + \dfrac{e}{b} \\ cx + dy = f \Rightarrow y = -\dfrac{c}{d}x + \dfrac{f}{d} \end{cases}$$

$$-\frac{a}{b} \neq -\frac{c}{d}$$

$$\frac{a}{b} \neq \frac{c}{d}$$

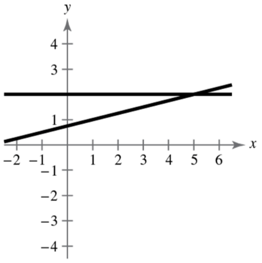
$$ad \neq bc$$

5. (a) $\begin{cases} x - 4y = -3 & \text{Eq. 1} \\ 5x - 6y = 13 & \text{Eq. 2} \end{cases}$

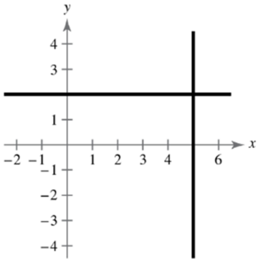


$\begin{cases} x - 4y = -3 & \\ 14y = 28 & -5\text{Eq. 1} + \text{Eq. 2} \end{cases}$

$\begin{cases} x - 4y = -3 & \\ y = 2 & \frac{1}{14}\text{Eq. 2} \end{cases}$

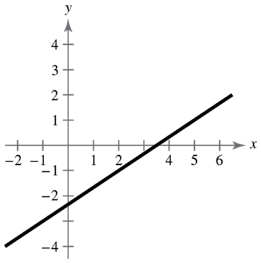


$\begin{cases} x = 5 & 4\text{Eq. 2} + \text{Eq. 1} \\ y = 2 & \end{cases}$



Solution: $(5, 2)$

(b) $\begin{cases} 2x - 3y = 7 & \text{Eq. 1} \\ -4x + 6y = -14 & \text{Eq. 2} \end{cases}$



$\begin{cases} 2x - 3y = 7 & \\ 0 = 0 & 2\text{Eq. 1} + \text{Eq. 2} \end{cases}$

The lines coincide. Infinite solutions.

Let $y = a$, then $2x - 3a = 7 \Rightarrow x = \frac{3}{2}a + \frac{7}{2}$

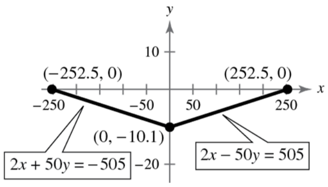
Solution: $\left(\frac{3}{2}a + \frac{7}{2}, a\right)$

The solution(s) remain the same at each step of the process.

7. The point where the two sections meet is at a depth of 10.1 feet. The distance between $(0, -10.1)$ and $(252.5, 0)$ is:

$$d = \sqrt{(252.5 - 0)^2 + (0 - (-10.1))^2} = \sqrt{63{,}858.26}$$
$$d \approx 252.7$$

Each section is approximately 252.7 feet long.



9. Let x = cost of the cable, per foot.

Let y = cost of a connector.

$$\begin{cases} 6x + 2y = 15.50 \Rightarrow & 6x + 2y = 15.50 \\ 3x + 2y = 10.25 \Rightarrow & \underline{-3x - 2y = -10.25} \end{cases}$$

$$\begin{aligned} 3x &= 5.25 \\ x &= 1.75 \\ y &= 2.50 \end{aligned}$$

For a four-foot cable with a connector on each end, the cost should be $4(1.75) + 2(2.50) = \$12.00$.

11. Let $X = \frac{1}{x}, Y = \frac{1}{y}$, and $Z = \frac{1}{z}$.

(a) $$\begin{cases} \frac{12}{x} - \frac{12}{y} = 7 \Rightarrow 12X - 12Y = 7 \Rightarrow & 12X - 12Y = 7 \\ \frac{3}{x} - \frac{4}{y} = 0 \Rightarrow 3X + 4Y = 0 \Rightarrow & \underline{9X + 12Y = 0} \\ & 21X = 7 \\ & X = \frac{1}{3} \\ & Y = -\frac{1}{4} \end{cases}$$

So, $\frac{1}{x} = \frac{1}{3} \Rightarrow x = 3$ and $\frac{1}{y} = -\frac{1}{4} \Rightarrow y = -4$.

Solution: $(3, -4)$

(b) $$\begin{cases} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 4 \Rightarrow 2X + Y - 3Z = 4 & \text{Eq.1} \\ \frac{4}{x} + \frac{2}{z} = 10 \Rightarrow 4X + 2Z = 10 & \text{Eq.2} \\ -\frac{2}{x} + \frac{3}{y} - \frac{13}{z} = -8 \Rightarrow -2X + 3Y - 13Z = -8 & \text{Eq.3} \end{cases}$$

$$\begin{cases} 2X + Y - 3Z = 4 & \\ -2Y + 8Z = 2 & -2\text{Eq. }1 + \text{Eq. }2 \\ 4Y - 16Z = -4 & \text{Eq. }1 + \text{Eq. }3 \end{cases}$$

$$\begin{cases} 2X + Y - 3Z = 4 & \\ -2Y + 8Z = 2 & \\ 0 = 0 & 2\text{Eq. }2 + \text{Eq. }3 \end{cases}$$

The system has infinite solutions.

Let $Z = a$, then $Y = 4a - 1$ and $X = \frac{-a + 5}{2}$.

Then $\frac{1}{z} = a \Rightarrow z = \frac{1}{a}, \frac{1}{y} = 4a - 1 \Rightarrow y = \frac{1}{4a - 1}$, and $\frac{1}{x} = \frac{-a + 5}{2} \Rightarrow x = \frac{2}{-a + 5}$.

Solution: $\left(\frac{2}{-a + 5}, \frac{1}{4a - 1}, \frac{1}{a}\right), a \neq 5, \frac{1}{4}, 0$

13. Solution: $(1, -1, 2)$

$$\begin{cases} 4x - 2y + 5z = 16 \\ x + y = 0 \\ -x - 3y + 2z = 6 \end{cases} \qquad \begin{matrix} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{matrix}$$

(a) $\begin{cases} 4x - 2y + 5z = 16 \\ x + y = 0 \end{cases}$

$$\begin{cases} x + y = 0 \\ 4x - 2y + 5z = 16 \end{cases} \qquad \text{Interchange the equations.}$$

$$\begin{cases} x + y = 0 \\ -6y + 5z = 16 \end{cases} \qquad -4\text{Eq. } 1 + \text{Eq. } 2$$

Let $z = a$, then $y = \dfrac{5a - 16}{6}$ and $x = \dfrac{-5a + 16}{6}$.

Solution: $\left(\dfrac{-5a + 16}{6}, \dfrac{5a - 16}{6}, a\right)$

When $a = 2$, we have the original solution.

(b) $\begin{cases} 4x - 2y + 5z = 16 \\ -x - 3y + 2z = 6 \end{cases}$

$$\begin{cases} -x - 2y + 2z = 6 \\ 4x - 3y + 5z = 16 \end{cases} \qquad \text{Interchange the equations.}$$

$$\begin{cases} -x - 3y + 2z = 6 \\ -14y + 13z = 40 \end{cases} \qquad 4\text{Eq. } 1 + \text{Eq. } 2$$

Let $z = a$, then $y = \dfrac{13a - 40}{14}$ and $x = \dfrac{-11a + 36}{14}$.

Solution: $\left(\dfrac{-11a + 36}{14}, \dfrac{13a - 40}{14}, a\right)$

When $a = 2$, we have the original solution.

(c) $\begin{cases} x + y = 0 \\ -x - 3y + 2z = 6 \end{cases}$

$$\begin{cases} x + y = 0 \\ -2y + 2z = 6 \end{cases} \qquad \text{Eq. } 1 + \text{Eq. } 2$$

Let $z = a$, then $y = a - 3$ and $x = -a + 3$.

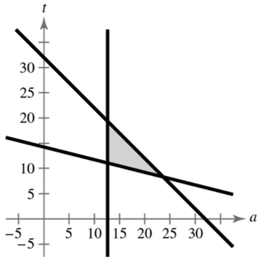
Solution: $(-a + 3, a - 3, a)$

When $a = 2$, we have the original solution.

15. t = amount of terrestrial vegetation in kilograms

a = amount of aquatic vegetation in kilograms

$$\begin{cases} a + t \le 32 \\ 0.15a \ge 1.9 \\ 193a + 772t \ge 11{,}000 \end{cases}$$

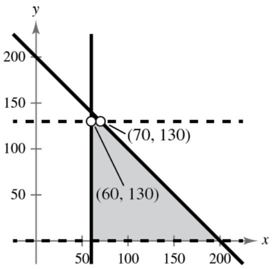


17. x = milligrams of HDL cholestrol

y = milligrams of LDL/VLDL cholestrol

(a) $\begin{cases} 0 < y < 130 \\ x \ge 60 \\ x + y \le 200 \end{cases}$

(b)



(c) $y = 120$ is in the region because $0 < y < 130$.

$x = 90$ is in the region because $60 \le x \le 200$.

$x + y = 210$ is not the region because $x + y \le 200$.

(d) *Sample answer*: If the LDL/VLDL reading is 135 and the HDL reading is 65, then $x \ge 60$ and $x + y \le 200$, but $y \not< 130$.

(e) $\dfrac{x + y}{x} < 4$

$x + y < 4x$

$y < 3x$

Sample answer: The point $(75, 90)$ is in the region, and $\dfrac{165}{75} = 2.2 < 4$.

Practice Test for Chapter 7

For Exercises 1–3, solve the given system by the method of substitution.

1. $\begin{cases} x + y = 1 \\ 3x - y = 15 \end{cases}$

2. $\begin{cases} x - 3y = -3 \\ x^2 + 6y = 5 \end{cases}$

3. $\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 0 \\ 5x + 2y - z = -3 \end{cases}$

4. Find the two numbers whose sum is 110 and product is 2800.

5. Find the dimensions of a rectangle if its perimeter is 170 feet and its area is 1500 square feet.

For Exercises 6–8, solve the linear system by elimination.

6. $\begin{cases} 2x + 15y = 4 \\ x - 3y = 23 \end{cases}$

7. $\begin{cases} x + y = 2 \\ 38x - 19y = 7 \end{cases}$

8. $\begin{cases} 0.4x + 0.5y = 0.112 \\ 0.3x - 0.7y = -0.131 \end{cases}$

9. Herbert invests \$17,000 in two funds that pay 11% and 13% simple interest, respectively. If he receives \$2080 in yearly interest, how much is invested in each fund?

10. Find the least squares regression line for the points $(4, 3), (1, 1), (-1, -2),$ and $(-2, -1)$.

For Exercises 11–12, solve the system of equations.

11. $\begin{cases} x + y = -2 \\ 2x - y + z = 11 \\ 4y - 3z = -20 \end{cases}$

12. $\begin{cases} 3x + 2y - z = 5 \\ 6x - y + 5z = 2 \end{cases}$

13. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points $(0, -1), (1, 4)$ and $(2, 13)$.

For Exercises 14–15, write the partial fraction decomposition of the rational functions.

14. $\dfrac{10x - 17}{x^2 - 7x - 8}$

15. $\dfrac{x^2 + 4}{x^4 + x^2}$

16. Graph $x^2 + y^2 \geq 9$.

17. Graph the solution of the system.

$$\begin{cases} x + y \leq 6 \\ x \geq 2 \\ y \geq 0 \end{cases}$$

18. Derive a set of inequalities to describe the triangle with vertices $(0, 0), (0, 7),$ and $(2, 3)$.

19. Find the maximum value of the objective function, $z = 30x + 26y,$ subject to the following constraints.

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \leq 21 \\ 5x + 3y \leq 30 \end{cases}$$

20. Graph the system of inequalities.

$$\begin{cases} x^2 + y^2 \leq 4 \\ (x - 2)^2 + y^2 \geq 4 \end{cases}$$

For Exercises 21–22, write the partial fraction decomposition for the rational expression.

21. $\dfrac{1 - 2x}{x^2 + x}$

22. $\dfrac{6x - 17}{(x - 3)^2}$

CHAPTER 8
Matrices and Determinants

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CHAPTER 8
Matrices and Determinants

Section 8.1 Matrices and Systems of Equations

1. row-equivalent

3. The coefficient matrix is the 2×2 matrix

$$\begin{bmatrix} -2 & 3 \\ 6 & 7 \end{bmatrix}.$$

5. Because the matrix has one row and two columns, its order is 1×2.

7. Because the matrix has three rows and one column, its order is 3×1.

9. Because the matrix has two rows and two columns, its order is 2×2.

11. Because the matrix has three rows and three columns, its order is 3×3.

13. $$\left\{\begin{array}{rcrcr} 2x & - & y & = & 7 \\ x & + & y & = & 2 \end{array}\right.$$

$$\left[\begin{array}{rrcr} 2 & -1 & \vdots & 7 \\ 1 & 1 & \vdots & 2 \end{array}\right]$$

15. $$\left\{\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 2 \\ 4x & - & 3y & + & z & = & -1 \\ 2x & + & y & & & = & 0 \end{array}\right.$$

$$\left[\begin{array}{rrrcr} 1 & -1 & 2 & \vdots & 2 \\ 4 & -3 & 1 & \vdots & -1 \\ 2 & 1 & 0 & \vdots & 0 \end{array}\right]$$

17. $$\left\{\begin{array}{rcrcrcr} 3x & - & 5y & + & 2z & = & 12 \\ 12x & & & - & 7z & = & 10 \end{array}\right.$$

$$\left[\begin{array}{rrrcr} 3 & -5 & 2 & \vdots & 12 \\ 12 & 0 & -7 & \vdots & 10 \end{array}\right]$$

19. $$\left[\begin{array}{rrcr} 1 & 1 & \vdots & 3 \\ 5 & -3 & \vdots & -1 \end{array}\right]$$

$$\left\{\begin{array}{rcrcr} x & + & y & = & 3 \\ 5x & - & 3y & = & -1 \end{array}\right.$$

21. $$\left[\begin{array}{rrrcr} 2 & 0 & 5 & \vdots & -12 \\ 0 & 1 & -2 & \vdots & 7 \\ 6 & 3 & 0 & \vdots & 2 \end{array}\right]$$

$$\left\{\begin{array}{rcrcrcr} 2x & & & + & 5z & = & -12 \\ & & y & - & 2z & = & 7 \\ 6x & + & 3y & & & = & 2 \end{array}\right.$$

23. $$\left[\begin{array}{rrrrcr} 9 & 12 & 3 & 0 & \vdots & 0 \\ -2 & 18 & 5 & 2 & \vdots & 10 \\ 1 & 7 & -8 & 0 & \vdots & -4 \\ 3 & 0 & 2 & 0 & \vdots & -10 \end{array}\right]$$

$$\left\{\begin{array}{rcrcrcrcr} 9x & + & 12y & + & 3z & & & = & 0 \\ -2x & + & 18y & + & 5z & + & 2w & = & 10 \\ x & + & 7y & - & 8z & & & = & -4 \\ 3x & & & + & 2z & & & = & -10 \end{array}\right.$$

25. $$\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$$

Add 5 times Row 2 to Row 1.

27. $$\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$$

Interchange Row 1 and Row 2. Then add 4 times the new Row 1 to Row 3.

29. $$\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$$

$$\tfrac{1}{3}R_1 \rightarrow \begin{bmatrix} 1 & \boxed{2} & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$$

31. $$\begin{bmatrix} 1 & 1 & 1 \\ 5 & -2 & 4 \end{bmatrix}$$

$$-5R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \boxed{-7} & -1 \end{bmatrix}$$

33. $$\begin{bmatrix}1&5&4&-1\\0&1&-2&2\\0&0&1&-7\end{bmatrix}$$

$$-5R_2+R_1\to\begin{bmatrix}1&0&\boxed{14}&\boxed{-11}\\0&1&-2&2\\0&0&1&-7\end{bmatrix}$$

35. $$\begin{bmatrix}1&1&4&-1\\3&8&10&3\\-2&1&12&6\end{bmatrix}$$

$$\begin{array}{r}\\-3R_1+R_2\to\\2R_1+R_3\to\end{array}\begin{bmatrix}1&1&4&-1\\0&5&\boxed{-2}&\boxed{6}\\0&3&\boxed{20}&\boxed{4}\end{bmatrix}$$

$$\begin{array}{r}\\\frac{1}{5}R_2\to\\\end{array}\begin{bmatrix}1&1&4&-1\\0&1&-\frac{2}{5}&\frac{6}{5}\\0&3&\boxed{20}&\boxed{4}\end{bmatrix}$$

37. (a) $\begin{bmatrix}-3&4&\vdots&22\\6&-4&\vdots&-28\end{bmatrix}$

(i) $R_1+R_2\to\begin{bmatrix}3&0&\vdots&-6\\6&-4&\vdots&-28\end{bmatrix}$

(ii) $-2R_1+R_2\to\begin{bmatrix}3&0&\vdots&-6\\0&-4&\vdots&-16\end{bmatrix}$

(iii) $-\frac{1}{4}R_2\to\begin{bmatrix}3&0&\vdots&-6\\0&1&\vdots&4\end{bmatrix}$

(iv) $\frac{1}{3}R_1\to\begin{bmatrix}1&0&\vdots&-2\\0&1&\vdots&4\end{bmatrix}$

The solution is $x=-2$ and $y=4$.

(b) $$\begin{cases}-3x+4y=22\\6x-4y=-28\end{cases}$$

$$\begin{aligned}3x&=-6\\x&=-2\end{aligned}$$

Back-substitute $x=-2$ into $-3x+4y=22$.

$$\begin{aligned}-3(-2)+4y&=22\\4y&=16\\y&=4\end{aligned}$$

The solution is $x=-2$ and $y=4$.

(c) Answers vary. *Sample answer:* In this case, solving the system of linear equations using the elimination method was more efficient.

39. $$\begin{bmatrix}1&0&0&0\\0&1&1&5\\0&0&0&0\end{bmatrix}$$

This matrix is in reduced row-echelon form.

41. $$\begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&0&2\end{bmatrix}$$

This matrix is not in row-echelon form.

43. $$\begin{bmatrix}1&1&0&5\\-2&-1&2&-10\\3&6&7&14\end{bmatrix}$$

$$\begin{array}{r}\\2R_1+R_2\to\\-3R_1+R_3\to\end{array}\begin{bmatrix}1&1&0&5\\0&1&2&0\\0&3&7&-1\end{bmatrix}$$

$$\begin{array}{r}\\\\-3R_2+R_3\to\end{array}\begin{bmatrix}1&1&0&5\\0&1&2&0\\0&0&1&-1\end{bmatrix}$$

45. $$\begin{bmatrix}1&-1&-1&1\\5&-4&1&8\\-6&8&18&0\end{bmatrix}$$

$$\begin{array}{r}\\-5R_1+R_2\to\\6R_1+R_3\to\end{array}\begin{bmatrix}1&-1&-1&1\\0&1&6&3\\0&2&12&6\end{bmatrix}$$

$$\begin{array}{r}\\\\-2R_2+R_3\to\end{array}\begin{bmatrix}1&-1&-1&1\\0&1&6&3\\0&0&0&0\end{bmatrix}$$

47. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{bmatrix}1&2&3&-5\\1&2&4&-9\\-2&-4&-4&3\\4&8&11&-14\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

49. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{bmatrix}-1&2&1\\3&4&9\\2&1&-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

51. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{bmatrix}-3&5&1&12\\1&-1&1&4\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&3&16\\0&1&2&12\end{bmatrix}$$

53. $\begin{cases} x - 2y = 4 \\ y = -1 \end{cases}$

$$x - 2(-1) = 4$$
$$x = 2$$

Solution: $(2, -1)$

55. $\begin{cases} x - y + 2z = 4 \\ \phantom{x - {}} y - z = 2 \\ z = -2 \end{cases}$

$$y - (-2) = 2$$
$$y = 0$$
$$x - 0 + 2(-2) = 4$$
$$x = 8$$

Solution: $(8, 0, -2)$

57. $\begin{cases} x + 2y = 7 \\ -x + y = 8 \end{cases}$

$$\left[\begin{array}{rrcr} 1 & 2 & \vdots & 7 \\ -1 & 1 & \vdots & 8 \end{array}\right]$$

$$R_1 + R_2 \rightarrow \left[\begin{array}{rrcr} 1 & 2 & \vdots & 7 \\ 0 & 3 & \vdots & 15 \end{array}\right]$$

$$\tfrac{1}{3}R_2 \rightarrow \left[\begin{array}{rrcr} 1 & 2 & \vdots & 7 \\ 0 & 1 & \vdots & 5 \end{array}\right]$$

$\begin{cases} x + 2y = 7 \\ y = 5 \end{cases}$

$$x + 2(5) = 7 \Rightarrow x = -3$$

Solution: $(-3, 5)$

59. $\begin{cases} 3x - 2y = -27 \\ x + 3y = 13 \end{cases}$

$$\left[\begin{array}{rrcr} 3 & -2 & \vdots & -27 \\ 1 & 3 & \vdots & 13 \end{array}\right]$$

$$\begin{matrix} \curvearrowleft R_1 \\ R_2 \end{matrix} \left[\begin{array}{rrcr} 1 & 3 & \vdots & 13 \\ 3 & -2 & \vdots & -27 \end{array}\right]$$

$$-3R_1 + R_2 \rightarrow \left[\begin{array}{rrcr} 1 & 3 & \vdots & 13 \\ 0 & -11 & \vdots & -66 \end{array}\right]$$

$$-\tfrac{1}{11}R_2 \rightarrow \left[\begin{array}{rrcr} 1 & 3 & \vdots & 13 \\ 0 & 1 & \vdots & 6 \end{array}\right]$$

$\begin{cases} x + 3y = 13 \\ y = 6 \end{cases}$

$$y = 6$$
$$x + 3(6) = 13 \Rightarrow x = -5$$

Solution: $(-5, 6)$

61. $\begin{cases} x + 2y - 3z = -28 \\ \phantom{x + {}} 4y + 2z = 0 \\ -x + y - z = -5 \end{cases}$

$$\left[\begin{array}{rrrcr} 1 & 2 & -3 & \vdots & -28 \\ 0 & 4 & 2 & \vdots & 0 \\ -1 & 1 & -1 & \vdots & -5 \end{array}\right]$$

$$\begin{array}{r} \\ \frac{1}{4}R_2 \rightarrow \\ R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{rrrcr} 1 & 2 & -3 & \vdots & -28 \\ 0 & 1 & \frac{1}{2} & \vdots & 0 \\ 0 & 3 & -4 & \vdots & -33 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -3R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{rrrcr} 1 & 2 & -3 & \vdots & -28 \\ 0 & 1 & \frac{1}{2} & \vdots & 0 \\ 0 & 0 & -\frac{11}{2} & \vdots & -33 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{2}{11}R_3 \rightarrow \end{array} \left[\begin{array}{rrrcr} 1 & 2 & -3 & \vdots & -28 \\ 0 & 1 & \frac{1}{2} & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 6 \end{array}\right]$$

$\begin{cases} x + 2y - 3z = -28 \\ y + \frac{1}{2}z = 0 \\ z = 6 \end{cases}$

$$z = 6$$
$$y + \frac{1}{2}(6) = 0 \Rightarrow y = -3$$
$$x + 2(-3) - 3(6) = -28 \Rightarrow x = -4$$

Solution: $(-4, -3, 6)$

63. $\begin{cases} -3x + 2y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

$$\begin{bmatrix} -3 & 2 & \vdots & -22 \\ 3 & 4 & \vdots & 4 \\ 4 & -8 & \vdots & 32 \end{bmatrix}$$

$$\begin{array}{r} R_1 + R_3 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & -6 & \vdots & 10 \\ 3 & 4 & \vdots & 4 \\ 4 & -8 & \vdots & 32 \end{bmatrix}$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ -4R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -6 & \vdots & 10 \\ 0 & 22 & \vdots & -26 \\ 0 & 16 & \vdots & -8 \end{bmatrix}$$

$$\begin{array}{r} \\ \frac{1}{22}R_2 \rightarrow \\ \frac{1}{16}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & \vdots & 22 \\ 0 & 1 & \vdots & -\frac{13}{10} \\ 0 & 1 & \vdots & -\frac{1}{2} \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & \vdots & 22 \\ 0 & 1 & \vdots & -\frac{13}{10} \\ 0 & 0 & \vdots & \frac{9}{5} \end{bmatrix}$$

The system is inconsistent and there is no solution.

65. Use the reduced row-echelon form feature of a graphing utility.

$\begin{cases} 3x + 2y - z + w = 0 \\ x - y + 4z + 2w = 25 \\ -2x + y + 2z - w = 2 \\ x + y + z + w = 6 \end{cases}$

$$\begin{bmatrix} 3 & 2 & -1 & 1 & \vdots & 0 \\ 1 & -1 & 4 & 2 & \vdots & 25 \\ -2 & 1 & 2 & -1 & \vdots & 2 \\ 1 & 1 & 1 & 1 & \vdots & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 3 \\ 0 & 1 & 0 & 0 & \vdots & -2 \\ 0 & 0 & 1 & 0 & \vdots & 5 \\ 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

$x = 3$

$y = -2$

$z = 5$

$w = 0$

Solution: $(3, -2, 5, 0)$

67. $\begin{bmatrix} 1 & 0 & \vdots & 3 \\ 0 & 1 & \vdots & -4 \end{bmatrix}$

$x = 3$

$y = -4$

Solution: $(3, -4)$

69. $\begin{cases} -2x + 6y = -22 \\ x + 2y = -9 \end{cases}$

$$\begin{bmatrix} -2 & 6 & \vdots & -22 \\ 1 & 2 & \vdots & -9 \end{bmatrix}$$

$$\circlearrowleft \begin{array}{r} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 2 & \vdots & -9 \\ -2 & 6 & \vdots & -22 \end{bmatrix}$$

$$\begin{array}{r} \\ 2R_1 + R_2 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & \vdots & -9 \\ 0 & 10 & \vdots & -40 \end{bmatrix}$$

$$\begin{array}{r} \\ \frac{1}{10}R_2 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & \vdots & -9 \\ 0 & 1 & \vdots & -4 \end{bmatrix}$$

$\begin{cases} x + 2y = -9 \\ y = -4 \end{cases}$

$y = -4$

$x + 2(-4) = -9 \Rightarrow x = -1$

Solution: $(-1, -4)$

71. $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

$$\begin{bmatrix} 1 & 0 & -3 & \vdots & -2 \\ 3 & 1 & -2 & \vdots & 5 \\ 2 & 2 & 1 & \vdots & 4 \end{bmatrix}$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -3 & \vdots & -2 \\ 0 & 1 & 7 & \vdots & 11 \\ 0 & 2 & 7 & \vdots & 8 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -2R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -3 & \vdots & -2 \\ 0 & 1 & 7 & \vdots & 11 \\ 0 & 0 & -7 & \vdots & -14 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -\frac{1}{7}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -3 & \vdots & -2 \\ 0 & 1 & 7 & \vdots & 11 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

$\begin{cases} x - 3z = -2 \\ y + 7z = 11 \\ z = 2 \end{cases}$

$z = 2$

$y + 7(2) = 11 \Rightarrow y = -3$

$x - 3(2) = -2 \Rightarrow x = 4$

Solution: $(4, -3, 2)$

73. $\begin{cases} x + 2y + z = 8 \\ 3x + 7y + 6z = 26 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 1 & \vdots & 8 \\ 3 & 7 & 6 & \vdots & 26 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & 1 & \vdots & 8 \\ 0 & 1 & 3 & \vdots & 2 \end{bmatrix}$$

$$\begin{cases} x + 2y + z = 8 \\ y + 3z = 2 \end{cases}$$

Let $z = a$.

$$y + 3a = 2 \Rightarrow y = -3a + 2$$

$$x + 2(-3a + 2) + a = 8 \Rightarrow x = 5a + 4$$

Solution: $(5a + 4, -3a + 2, a)$ where a is a real number

75. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

$$\begin{bmatrix} -1 & 1 & -1 & \vdots & -14 \\ 2 & -1 & 1 & \vdots & 21 \\ 3 & 2 & 1 & \vdots & 19 \end{bmatrix}$$

$$-R_1 \rightarrow \begin{bmatrix} 1 & -1 & 1 & \vdots & 14 \\ 2 & -1 & 1 & \vdots & 21 \\ 3 & 2 & 1 & \vdots & 19 \end{bmatrix}$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -3R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 1 & \vdots & 14 \\ 0 & 1 & -1 & \vdots & -7 \\ 0 & 5 & -2 & \vdots & -23 \end{bmatrix}$$

$$-5R_2 + R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & \vdots & 14 \\ 0 & 1 & -1 & \vdots & -7 \\ 0 & 0 & 3 & \vdots & 12 \end{bmatrix}$$

$$\tfrac{1}{3}R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & \vdots & 14 \\ 0 & 1 & -1 & \vdots & -7 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix}$$

$$\begin{cases} x - y + z = 14 \\ y - z = -7 \\ z = 4 \end{cases}$$

$$z = 4$$

$$y - 4 = -7 \Rightarrow y = -3$$

$$x - (-3) + 4 = 14 \Rightarrow x = 7$$

Solution: $(7, -3, 4)$

77. Use the reduced row-echelon form feature of a graphic utility.

$$\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases} \begin{bmatrix} 3 & 3 & 12 & \vdots & 6 \\ 1 & 1 & 4 & \vdots & 2 \\ 2 & 5 & 20 & \vdots & 10 \\ -1 & 2 & 8 & \vdots & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 4 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 0 \\ y + 4z = 2 \end{cases}$$

Let $z = a$.

$y = 2 - 4a$

$x = 0$

Solution: $(0, 2 - 4a, a)$ where a is any real number

79. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{cases} 2x + y - z + 2w = -6 \\ 3x + 4y + w = 1 \\ x + 5y + 2z + 6w = -3 \\ 5x + 2y - z - w = 3 \end{cases}$$

$$\left[\begin{array}{cccccc} 2 & 1 & -1 & 2 & \vdots & -6 \\ 3 & 4 & 0 & 1 & \vdots & 1 \\ 1 & 5 & 2 & 6 & \vdots & -3 \\ 5 & 2 & -1 & -1 & \vdots & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 4 \\ 0 & 0 & 0 & 1 & \vdots & -2 \end{array}\right]$$

$x = 1$

$y = 0$

$z = 4$

$w = -2$

Solution: $(1, 0, 4, -2)$

81. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{cases} x + y + z + w = 0 \\ 2x + 3y + z - 2w = 0 \\ 3x + 5y + z = 0 \end{cases}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & \vdots & 0 \\ 2 & 3 & 1 & -2 & \vdots & 0 \\ 3 & 5 & 1 & 0 & \vdots & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 2 & 0 & \vdots & 0 \\ 0 & 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \end{array}\right]$$

$$\begin{cases} x + 2z = 0 \\ y - z = 0 \\ w = 0 \end{cases}$$

Let $z = a$. Then $x = -2a$ and $y = a$.

Solution: $(-2a, a, a, 0)$ where a is a real number

83. $$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = -1 \\ 9a + 3b + c = -5 \end{cases}$$

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 4 & 2 & 1 & \vdots & -1 \\ 9 & 3 & 1 & \vdots & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{array}\right]$$

$a = 1$

$b = 1$

$c = 1$

So, $f(x) = -x^2 + x + 1$.

85. $$\begin{cases} 4a - 2b + c = -15 \\ a - b + c = 7 \\ a + b + c = -3 \end{cases}$$

$$\left[\begin{array}{ccccc} 4 & -2 & 1 & \vdots & -15 \\ 4 & -1 & 1 & \vdots & 7 \\ 1 & 1 & 1 & \vdots & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & -9 \\ 0 & 1 & 0 & \vdots & -5 \\ 0 & 0 & 1 & \vdots & 11 \end{array}\right]$$

$a = -9$

$b = -5$

$c = 11$

So, $f(x) = -9x^2 - 5x + 11$.

87. x = amount at 8%

y = amount at 9%

z = amount at 12%

$$\begin{cases} x + y + z = 2{,}000{,}000 \\ 0.08x + 0.09y + 0.12z = 186{,}000 \\ x - 2z = 0 \end{cases}$$

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 2{,}000{,}000 \\ 0.08 & 0.09 & 0.12 & \vdots & 186{,}000 \\ 1 & 0 & -2 & \vdots & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -0.08R_1 + R_2 \\ -R_1 + R_3 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 2{,}000{,}000 \\ 0 & 0.01 & 0.04 & \vdots & 26{,}000 \\ 0 & -1 & -3 & \vdots & -2{,}000{,}000 \end{array}\right]$$

$$\begin{array}{r} \\ 100R_2 \rightarrow \\ \\ \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 2{,}000{,}000 \\ 0 & 1 & 4 & \vdots & 2{,}600{,}000 \\ 0 & -1 & -3 & \vdots & -2{,}000{,}000 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 2{,}000{,}000 \\ 0 & 1 & 4 & \vdots & 2{,}600{,}000 \\ 0 & 0 & 1 & \vdots & 600{,}000 \end{array}\right]$$

The matrix is now in row-echelon form, and the corresponding system is shown.

$$\begin{cases} x + y + z = 2{,}000{,}000 \\ y + 4z = 2{,}600{,}000 \\ z = 600{,}000 \end{cases}$$

Using back-substitution, you can determine the solution.

$$\begin{aligned} y + 4(600{,}000) &= 2{,}600{,}000 \\ y &= 200{,}000 \\ x + 200{,}000 + 600{,}000 &= 2{,}000{,}000 \\ x &= 1{,}200{,}000 \end{aligned}$$

So, the government borrowed 1,200,000 at 8%, $200,000 at 9%, and $600,000 at 12%.

89. $\begin{cases} 12b + 66a = 831 \\ 66b + 506a = 5643 \end{cases}$

$$\left[\begin{array}{cc:c} 12 & 66 & 831 \\ 66 & 506 & 5643 \end{array}\right]$$

Using a graphing utility, this matrix reduces to

$$\left[\begin{array}{cc:c} 1 & 0 & 28 \\ 0 & 1 & 7.5 \end{array}\right].$$

The least squares regression line is $y = 7.5t + 28$.

For 2024, $t = 15$ and $y = 7.5(15) + 28 = 140.5$. So, there will be about 141 new cases in 2024. This seems reasonable because the pattern is linear.

91. False. It is a 2 × 4 matrix.

93. The matrix is in row-echelon form, not reduced row-echelon form.

95. $1(-2) + 0(1) + 3(-1) = -2 - 3 = -5$

97. $a + bi = -3 + 8i$

$a = -3, b = 8$

99. $(a - b) + (b + 1)i = 6 + 5i$

$a - 2 = 6 \Rightarrow a = 8$

$b + 1 = 5 \Rightarrow b = 4$

101. $3x + a = b$

$3x = b - a$

$x = \frac{1}{3}(b - a)$

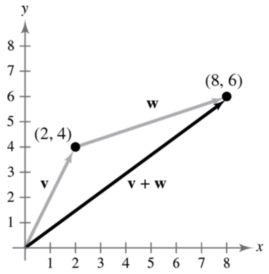
103. $6b - x = a$

$-x = a - 6b$

$x = 6b - a$

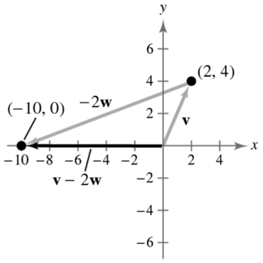
105. $\mathbf{v} = \langle 2, 4\rangle, \mathbf{w} = \langle 6, 2\rangle$

$\mathbf{u} = \mathbf{v} + \mathbf{w} = \langle 2 + 6, 4 + 2\rangle = \langle 8, 6\rangle$



107. $\mathbf{v} = \langle 2, 4\rangle, \mathbf{w} = \langle 6, 2\rangle$

$$\begin{aligned} \mathbf{u} = \mathbf{v} - 2\mathbf{w} &= \langle 2, 4\rangle - 2\langle 6, 2\rangle \\ &= \langle 2, 4\rangle + \langle -12, -4\rangle \\ &= \langle 2 - 12, 4 - 4\rangle \\ &= \langle -10, 0\rangle \end{aligned}$$



Section 8.2 Operations with Matrices

1. scalars

3. BA is defined, not AB.

5. $\begin{bmatrix} x & -2 \\ 7 & 23 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & y \end{bmatrix}$

$x = -4$

$y = 23$

7. $\begin{bmatrix} 16 & 4 & x & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x + 1 & 4 \\ 0 & 2 & 3y - 5 & 0 \end{bmatrix}$

$x = 2x + 1 \Rightarrow x = -1$

$4 = 3y - 5 \Rightarrow y = 3$

9. (a) $A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1 + 2 & -1 - 1 \\ 2 - 1 & -1 + 8 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1 - 2 & -1 + 1 \\ 2 + 1 & -1 - 8 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$

(c) $3A = 3\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(-1) \\ 3(2) & 3(-1) \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix}$

(d) $3A - 2B = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 2 & -16 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 8 & -19 \end{bmatrix}$

11. $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

(a) $A + B$ is not possible. A and B do not have the same order.

(b) $A - B$ is not possible. A and B do not have the same order.

(c) $3A = \begin{bmatrix} 18 & 0 & 9 \\ -3 & -12 & 0 \end{bmatrix}$

(d) $3A - 2B$ is not possible. A and B do not have the same order.

13. $A = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$

(a) $$A + B = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 8+1 & -1+6 \\ 2-1 & 3-5 \\ -4+1 & 5+10 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 1 & -2 \\ -3 & 15 \end{bmatrix}$$

(b) $$A - B = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 8-1 & -1-6 \\ 2-(-1) & 3-(-5) \\ -4-1 & 5-10 \end{bmatrix} = \begin{bmatrix} 7 & -7 \\ 3 & 8 \\ -5 & -5 \end{bmatrix}$$

(c) $$3A = 3\begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 3(8) & 3(-1) \\ 3(2) & 3(3) \\ 3(-4) & 3(5) \end{bmatrix} = \begin{bmatrix} 24 & -3 \\ 6 & 9 \\ -12 & 15 \end{bmatrix}$$

(d) $$3A - 2B = \begin{bmatrix} 24 & -3 \\ 6 & 9 \\ -12 & 15 \end{bmatrix} - 2\begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 24-2 & -3-12 \\ 6+2 & 9+10 \\ -12-2 & 15-20 \end{bmatrix} = \begin{bmatrix} 22 & -15 \\ 8 & 19 \\ -14 & -5 \end{bmatrix}$$

15. $A = \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix}$

(a) $$\begin{aligned} A + B &= \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 4+1 & 5+0 & -1-1 & 3+1 & 4+0 \\ 1-6 & 2+8 & -2+2 & -1-3 & 0-7 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 5 & -2 & 4 & 4 \\ -5 & 10 & 0 & -4 & -7 \end{bmatrix} \end{aligned}$$

(b) $$\begin{aligned} A - B &= \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 4-1 & 5-0 & -1-(-1) & 3-1 & 4-0 \\ 1-(-6) & 2-8 & -2-2 & -1-(-3) & 0-(-7) \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 & 0 & 2 & 4 \\ 7 & -6 & -4 & 2 & 7 \end{bmatrix} \end{aligned}$$

(c) $$3A = 3\begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3(4) & 3(5) & 3(-1) & 3(3) & 3(4) \\ 3(1) & 3(2) & 3(-2) & 3(-1) & 3(0) \end{bmatrix} = \begin{bmatrix} 12 & 15 & -3 & 9 & 12 \\ 3 & 6 & -6 & -3 & 0 \end{bmatrix}$$

(d) $$\begin{aligned} 3A - 2B &= \begin{bmatrix} 12 & 15 & -3 & 9 & 12 \\ 3 & 6 & -6 & -3 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 12-2 & 15+0 & -3+2 & 9-2 & 12-0 \\ 3+12 & 6-16 & -6-4 & -3+6 & 0+14 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 15 & -1 & 7 & 12 \\ 15 & -10 & -10 & 3 & 14 \end{bmatrix} \end{aligned}$$

17. $\begin{bmatrix}-5 & 0\\ 3 & -6\end{bmatrix} + \begin{bmatrix}7 & 1\\ -2 & -1\end{bmatrix} + \begin{bmatrix}-10 & -8\\ 14 & 6\end{bmatrix} = \begin{bmatrix}-5+7+(-10) & 0+1+(-8)\\ 3+(-2)+14 & -6+(-1)+6\end{bmatrix} = \begin{bmatrix}-8 & -7\\ 15 & -1\end{bmatrix}$

19. $4\left(\begin{bmatrix}-4 & 0 & 1\\ 0 & 2 & 3\end{bmatrix} - \begin{bmatrix}2 & 1 & -2\\ 3 & -6 & 0\end{bmatrix}\right) = 4\begin{bmatrix}-6 & -1 & 3\\ -3 & 8 & 3\end{bmatrix} = \begin{bmatrix}-24 & -4 & 12\\ -12 & 32 & 12\end{bmatrix}$

21. $-3\left(\begin{bmatrix}0 & -3\\ 7 & 2\end{bmatrix} + \begin{bmatrix}-6 & 3\\ 8 & 1\end{bmatrix}\right) - 2\begin{bmatrix}4 & -4\\ 7 & -9\end{bmatrix} = -3\begin{bmatrix}-6 & 0\\ 15 & 3\end{bmatrix} - \begin{bmatrix}8 & -8\\ 14 & -18\end{bmatrix} = \begin{bmatrix}18 & 0\\ -45 & -9\end{bmatrix} - \begin{bmatrix}8 & -8\\ 14 & -18\end{bmatrix} = \begin{bmatrix}10 & 8\\ -59 & 9\end{bmatrix}$

In Exercises 23–29, $A = \begin{bmatrix}-2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix}$ **and** $B = \begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix}$

23. $X = 2A + 2B = 2\begin{bmatrix}-2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix} + 2\begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix} = \begin{bmatrix}-4 & 2 & 6\\ -2 & 0 & 8\end{bmatrix} + \begin{bmatrix}0 & 4 & -8\\ 6 & 0 & 2\end{bmatrix} = \begin{bmatrix}-4 & 6 & -2\\ 4 & 0 & 10\end{bmatrix}$

25.
$$\begin{aligned}
2X &= 2A - B\\
X &= \tfrac{1}{2}(2A - B)\\
&= \tfrac{1}{2}\left(2\begin{bmatrix}-2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix} - \begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix}\right)\\
&= \tfrac{1}{2}\left(\begin{bmatrix}-4 & 2 & 6\\ -2 & 0 & 8\end{bmatrix} - \begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix}\right)\\
&= \tfrac{1}{2}\begin{bmatrix}-4 & 0 & 10\\ -5 & 0 & 7\end{bmatrix}\\
&= \begin{bmatrix}-2 & 0 & 5\\ -\frac{5}{2} & 0 & \frac{7}{2}\end{bmatrix}
\end{aligned}$$

27.
$$\begin{aligned}
2X + 3A &= B\\
2X &= B - 3A\\
X &= \tfrac{1}{2}(B - 3A)\\
&= \tfrac{1}{2}\left(\begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix} - 3\begin{bmatrix}-2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix}\right)\\
&= \tfrac{1}{2}\left(\begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix} - \begin{bmatrix}-6 & 3 & 9\\ -3 & 0 & 12\end{bmatrix}\right)\\
&= \tfrac{1}{2}\begin{bmatrix}6 & -1 & -13\\ 6 & 0 & -11\end{bmatrix}\\
&= \begin{bmatrix}3 & -\frac{1}{2} & -\frac{13}{2}\\ 3 & 0 & -\frac{11}{2}\end{bmatrix}
\end{aligned}$$

29.
$$\begin{aligned}
4B &= -2X - 2A\\
2X &= -2A - 4B\\
X &= \tfrac{1}{2}(-2A - 4B)\\
&= \tfrac{1}{2}\left(-2\begin{bmatrix}-2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix} - 4\begin{bmatrix}0 & 2 & -4\\ 3 & 0 & 1\end{bmatrix}\right)\\
&= \tfrac{1}{2}\left(\begin{bmatrix}4 & -2 & -6\\ 2 & 0 & -8\end{bmatrix} - \begin{bmatrix}0 & 8 & -16\\ 12 & 0 & 4\end{bmatrix}\right)\\
&= \tfrac{1}{2}\begin{bmatrix}4 & -10 & 10\\ -10 & 0 & -12\end{bmatrix}\\
&= \begin{bmatrix}2 & -5 & 5\\ -5 & 0 & -6\end{bmatrix}
\end{aligned}$$

31. A is 3×2, B is $2 \times 2 \Rightarrow AB$ is 3×2.

$$A = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} (-1)(2) + (6)(0) & (-1)(3) + (6)(9) \\ (-4)(2) + (5)(0) & (-4)(3) + (5)(9) \\ (0)(2) + (3)(0) & (0)(3) + (3)(9) \end{bmatrix} = \begin{bmatrix} -2 & 51 \\ -8 & 33 \\ 0 & 27 \end{bmatrix}$$

33. A is 3×2 and B is 3×3. AB is not possible.

35. A is 3×3, B is $3 \times 3 \Rightarrow AB$ is 3×3.

$$AB = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$$

37. (a) $AB = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} (1)(2) + (2)(-1) & (1)(-1) + (2)(8) \\ (4)(2) + (2)(-1) & (4)(-1) + (2)(8) \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-1)(4) & (2)(2) + (-1)(2) \\ (-1)(1) + (8)(4) & (-1)(2) + (8)(2) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(4) & (1)(2) + (2)(2) \\ (4)(1) + (2)(4) & (4)(2) + (2)(2) \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 12 & 12 \end{bmatrix}$

39. (a) $AB = \begin{bmatrix} 2 & -2 \\ -3 & 0 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-2)(0) & (2)(0) + (-2)(1) \\ (-3)(1) + (0)(0) & (-3)(0) + (0)(1) \\ (7)(1) + (6)(0) & (7)(0) + (6)(1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -3 & 0 \\ 7 & 6 \end{bmatrix}$

(b) BA is not possible, B is 2×2 and A is 3×2.

(c) AA is not possible, A is 3×2.

41. (a) $AB = \begin{bmatrix} -4 & -1 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} (-4)(-6) + (-1)(5) \\ (2)(-6) + (12)(5) \end{bmatrix} = \begin{bmatrix} 19 \\ 48 \end{bmatrix}$

(b) BA is not possible, B is 2×1 and A is 2×2.

(c) $A^2 = \begin{bmatrix} -4 & -1 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ 2 & 12 \end{bmatrix} = \begin{bmatrix} (-4)(-4) + (-1)(2) & (-4)(-1) + (-1)(12) \\ (2)(-4) + (12)(2) & (2)(-1) + (12)(12) \end{bmatrix} = \begin{bmatrix} 14 & -8 \\ 16 & 142 \end{bmatrix}$

43. (a) $AB = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 7(1) & 7(1) & 7(2) \\ 8(1) & 8(1) & 8(2) \\ -1(1) & -1(1) & -1(2) \end{bmatrix} = \begin{bmatrix} 7 & 7 & 14 \\ 8 & 8 & 16 \\ -1 & -1 & -2 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix} = \begin{bmatrix} (1)(7) + (1)(8) + (2)(-1) \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix}$

(c) A^2 is not possible.

45. $$\begin{bmatrix}7&5&-4\\-2&5&1\\10&-4&-7\end{bmatrix}\begin{bmatrix}2&-2&3\\8&1&4\\-4&2&-8\end{bmatrix}=\begin{bmatrix}14+40-16&-14+5-8&21+20+32\\-4+40-4&4+5+2&-6+20-8\\20-32+28&-20-4-14&30-16+56\end{bmatrix}=\begin{bmatrix}70&-17&73\\32&11&6\\16&-38&70\end{bmatrix}$$

47. $$\begin{bmatrix}3&1\\0&-2\end{bmatrix}\begin{bmatrix}1&0\\-2&2\end{bmatrix}\begin{bmatrix}1&0\\2&4\end{bmatrix}=\begin{bmatrix}1&2\\4&-4\end{bmatrix}\begin{bmatrix}1&0\\2&4\end{bmatrix}=\begin{bmatrix}5&8\\-4&-16\end{bmatrix}$$

49. $$\begin{bmatrix}0&2&-2\\4&1&2\end{bmatrix}\left(\begin{bmatrix}4&0\\0&-1\\-1&2\end{bmatrix}+\begin{bmatrix}-2&3\\-3&5\\0&-5\end{bmatrix}\right)=\begin{bmatrix}0&2&-2\\4&1&2\end{bmatrix}\begin{bmatrix}2&3\\-3&4\\-1&-1\end{bmatrix}=\begin{bmatrix}-4&10\\3&14\end{bmatrix}$$

51. $\mathbf{u}=\langle 1,5\rangle,\ \mathbf{v}=\langle 3,2\rangle$

(a) $\mathbf{u}+\mathbf{v}=\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}=\langle 4,7\rangle$

(b) $\mathbf{u}-\mathbf{v}=\begin{bmatrix}1\\5\end{bmatrix}-\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}=\langle -2,3\rangle$

(c) $3\mathbf{v}-\mathbf{u}=3\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}8\\1\end{bmatrix}=\langle 8,1\rangle$

53. $\mathbf{u}=\langle -2,2\rangle,\ \mathbf{v}=\langle 5,4\rangle$

(a) $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-2\\2\end{bmatrix}+\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}=\langle 3,6\rangle$

(b) $\mathbf{u}-\mathbf{v}=\begin{bmatrix}-2\\2\end{bmatrix}-\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}-7\\-2\end{bmatrix}=\langle -7,-2\rangle$

(c) $3\mathbf{v}-\mathbf{u}=3\begin{bmatrix}5\\4\end{bmatrix}-\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}17\\10\end{bmatrix}=\langle 17,10\rangle$

In Exercises 55–59, $\mathbf{v}=\langle 4,2\rangle=\begin{bmatrix}4\\2\end{bmatrix}$

55. $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ A\mathbf{v}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}=\langle 4,-2\rangle$ is a reflection in the x-axis.

57. $A=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ A\mathbf{v}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}=\langle 2,4\rangle$ is a reflection in the line $y=x$.

59. $A=\begin{bmatrix}2&0\\0&1\end{bmatrix},\ A\mathbf{v}=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}8\\2\end{bmatrix}=\langle 8,2\rangle$ is a horizontal stretch.

61. (a) $\begin{bmatrix}2&3\\1&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\10\end{bmatrix}$

(b)
$$\circlearrowleft\begin{array}{r}R_2\\R_1\end{array}\begin{bmatrix}1&4&\vdots&10\\2&3&\vdots&5\end{bmatrix}$$
$$\begin{array}{r}\\-2R_1+R_2\rightarrow\end{array}\begin{bmatrix}1&4&\vdots&10\\0&-5&\vdots&-15\end{bmatrix}$$
$$\begin{array}{r}\\-\frac{1}{5}R_2\rightarrow\end{array}\begin{bmatrix}1&4&\vdots&10\\0&1&\vdots&3\end{bmatrix}$$
$$\begin{array}{r}-4R_2+R_1\rightarrow\\-\frac{1}{5}R_2\rightarrow\end{array}\begin{bmatrix}1&0&\vdots&-2\\0&1&\vdots&3\end{bmatrix}$$
$$X=\begin{bmatrix}-2\\3\end{bmatrix}$$

63. (a) $\begin{bmatrix}1&-2&3\\-1&3&-1\\2&-5&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}9\\-6\\17\end{bmatrix}$

(b)
$$\begin{bmatrix}1&-2&3&\vdots&9\\-1&3&-1&\vdots&-6\\2&-5&5&\vdots&17\end{bmatrix}$$
$$\begin{array}{r}\\R_1+R_2\rightarrow\\-2R_1+R_3\rightarrow\end{array}\begin{bmatrix}1&-2&3&\vdots&9\\0&1&2&\vdots&3\\0&-1&-1&\vdots&-1\end{bmatrix}$$
$$\begin{array}{r}2R_2+R_1\rightarrow\\ \\R_2+R_3\rightarrow\end{array}\begin{bmatrix}1&0&7&\vdots&15\\0&1&2&\vdots&3\\0&0&1&\vdots&2\end{bmatrix}$$
$$\begin{array}{r}-7R_3+R_1\rightarrow\\-2R_3+R_2\rightarrow\\ \\\end{array}\begin{bmatrix}1&0&0&\vdots&1\\0&1&0&\vdots&-1\\0&0&1&\vdots&2\end{bmatrix}$$
$$X=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

65. (a) $$\begin{bmatrix}1&-5&2\\-3&1&-1\\0&-2&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-20\\8\\-16\end{bmatrix}$$

(b) $$\left[\begin{array}{rrrcr}1&-5&2&\vdots&-20\\-3&1&-1&\vdots&8\\0&-2&5&\vdots&-16\end{array}\right]$$

$$3R_1+R_2\rightarrow\left[\begin{array}{rrrcr}1&-5&2&\vdots&-20\\0&-14&5&\vdots&-52\\0&-2&5&\vdots&-16\end{array}\right]$$

$$-R_3+R_2\rightarrow\left[\begin{array}{rrrcr}1&-5&2&\vdots&-20\\0&-12&0&\vdots&-36\\0&-2&5&\vdots&-16\end{array}\right]$$

$$-\tfrac{1}{12}R_2\rightarrow\left[\begin{array}{rrrcr}1&-5&2&\vdots&-20\\0&1&0&\vdots&3\\0&-2&5&\vdots&-16\end{array}\right]$$

$$\begin{array}{r}5R_2+R_1\rightarrow\\ \\2R_2+R_3\rightarrow\end{array}\left[\begin{array}{rrrcr}1&0&2&\vdots&-5\\0&1&0&\vdots&3\\0&0&5&\vdots&-10\end{array}\right]$$

$$\begin{array}{r}\\ \\ \tfrac{1}{5}R_3\rightarrow\end{array}\left[\begin{array}{rrrcr}1&0&2&\vdots&-5\\0&1&0&\vdots&3\\0&0&1&\vdots&-2\end{array}\right]$$

$$-2R_3+R_1\rightarrow\left[\begin{array}{rrrcr}1&0&0&\vdots&-1\\0&1&0&\vdots&3\\0&0&1&\vdots&-2\end{array}\right]$$

$$X=\begin{bmatrix}-1\\3\\-2\end{bmatrix}$$

67. $$1.10\begin{bmatrix}100&90&70&30\\40&20&60&60\end{bmatrix}=\begin{bmatrix}110&99&77&33\\44&22&66&66\end{bmatrix}$$

69. $$BA=\begin{bmatrix}3.50&6.00\end{bmatrix}\begin{bmatrix}125&100&75\\100&175&125\end{bmatrix}$$
$$=\begin{bmatrix}\$1037.50&\$1400&\$1012.50\end{bmatrix}$$

The entries represent the profits from both crops at each of the three outlets.

71. $$ST=\begin{bmatrix}1.0&0.5&0.2\\1.6&1.0&0.2\\2.5&2.0&1.4\end{bmatrix}\begin{bmatrix}15&13\\12&11\\11&10\end{bmatrix}=\begin{bmatrix}\$23.20&\$20.50\\\$38.20&\$33.80\\\$76.90&\$68.50\end{bmatrix}$$

The entries represent the labor costs at each plant for each size of boat.

73. (a) $$B=\begin{array}{c}\begin{array}{ccc}\text{basketball}&\text{jumping rope}&\text{weight lifting}\end{array}\\ \begin{bmatrix}2&0.25&0.5\end{bmatrix}\end{array}$$

(b) $$BA=\begin{bmatrix}2&0.25&0.5\end{bmatrix}\begin{bmatrix}472&563\\590&704\\177&211\end{bmatrix}=\begin{bmatrix}1180&1407.5\end{bmatrix}$$

The resulting matrix BA represents the total calories burned playing basketball for 2 hours, jumping rope for 15 minutes and lifting weights for 30 minutes by each person. So, the 130-lb person burned 1180 calories, and the 155-lb person burned 1407.5 calories.

(c) $$\frac{18\text{ min}}{1\text{ hr}}=\frac{18\text{ min}}{60\text{ min}}=\frac{3}{10},\frac{45\text{ min}}{1\text{ hr}}=\frac{45\text{ min}}{60\text{ min}}=\frac{3}{4}$$

$$BA=\begin{bmatrix}1&\frac{3}{10}&\frac{3}{4}\end{bmatrix}\begin{bmatrix}472&563\\590&704\\177&211\end{bmatrix}=\begin{bmatrix}781.75&932.45\end{bmatrix}$$

The 130-pound person burned $1180-781.75=398.25$ fewer calories.

The 155-pound person burned $1407.5-932.45=475.05$ fewer calories.

75. False. For most matrices, $AB \neq BA$.

Consider

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

So, $AB \neq BA$.

77. Answers will vary. *Sample answer:*

$$(A + B)^2 = \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right)^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\neq A^2 + 2AB + B^2 = \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right)^2 + 2\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + \left(\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$$

79. Answers will vary. *Sample answer:*

$$(A + B)(A - B) = \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right)\left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ 4 & 3 \end{bmatrix}$$

$$\neq A^2 - B^2 = \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right)^2 - \left(\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right)^2 = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$$

81. $AC = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

$$BC = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

So, $AC = BC$ even though $A \neq B$.

83. The product of two diagonal matrices of the same order is a diagonal matrix whose entries are the products of the corresponding diagonal entries of A and B.

85. Answer will vary. *Sample answer:*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(0) + (0)(1) & (1)(1) + (0)(0) \\ (0)(0) + (1)(1) & (0)(1) + (1)(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (0)(1) + (1)(0) & (0)(0) + (1)(1) \\ (1)(1) + (0)(0) & (1)(0) + (0)(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So, $AB = BA$

87. (a) $$A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} (i)(i) + (0)(0) & (i)(0) + (0)(i) \\ (0)(i) + (i)(0) & (0)(0) + (i)(i) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } i^2 = -1$$

$$A^3 = A^2A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} (-1)(i) + (0)(0) & (-1)(0) + (0)(i) \\ (0)(i) + (-1)(0) & (0)(0) + (-1)(i) \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \text{ and } i^3 = -i$$

$$A^4 = A^3A = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} (-i)(i) + (0)(0) & (i)(0) + (0)(i) \\ (0)(i) + (-i)(0) & (0)(0) + (-i)(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } i^4 = -1$$

(b) $$B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} B^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} (0)(0) + (-i)(i) & (0)(-i) + (-i)(0) \\ (i)(0) + (0)(i) & (i)(-i) + (0)(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \text{ the identity matrix}$$

89. $$\begin{aligned} \log_3(15x) &= \log_3(3 \cdot 5 \cdot x) \\ &= \log_3 3 + \log_3 5 + \log_3 x \\ &= 1 + \log_3 5 + \log_3 x \end{aligned}$$

91. $\ln\left(\dfrac{a}{b^3}\right) = \ln a - \ln b^3 = \ln a - 3\ln b$

93. $$\begin{cases} 5x - 2y = 2 & \text{Equation 1} \\ 5x + 4y = 8 & \text{Equation 2} \end{cases}$$

Subtracting the equations you get $-6y = -6 \Rightarrow y = 1$.

Back-substitute $y = 1$: $5x - 2(1) = 2 \Rightarrow x = \frac{4}{5}$

Solution: $\left(\frac{4}{5}, 1\right)$

95. $$\begin{cases} 4x + y - z = -5 \\ -x - 6y + 2z = -16 \\ x + 3y + z = 7 \end{cases}$$

$$\begin{bmatrix} 4 & 1 & -1 & \vdots & -5 \\ -1 & -6 & 2 & \vdots & -16 \\ 1 & 3 & 1 & \vdots & 7 \end{bmatrix}$$

$$\begin{array}{r} R_3 \\ \circlearrowleft \\ R_1 \end{array}\begin{bmatrix} 1 & 3 & 1 & \vdots & 7 \\ -1 & -6 & 2 & \vdots & -16 \\ 4 & 1 & -1 & \vdots & -5 \end{bmatrix}$$

$$\begin{array}{r} \\ R_1 + R_2 \rightarrow \\ -4R_1 + R_2 \rightarrow \end{array}\begin{bmatrix} 1 & 3 & 1 & \vdots & 7 \\ 0 & -3 & 3 & \vdots & -9 \\ 0 & -11 & -5 & \vdots & -33 \end{bmatrix}$$

$$\begin{array}{r} \\ -\frac{1}{3}R_2 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & 3 & 1 & \vdots & 7 \\ 0 & 1 & -1 & \vdots & 3 \\ 0 & -11 & -5 & \vdots & -33 \end{bmatrix}$$

$$\begin{array}{r} \\ 11R_2 + R_3 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & 3 & 1 & \vdots & 7 \\ 0 & 1 & -1 & \vdots & 3 \\ 0 & 0 & -16 & \vdots & 0 \end{bmatrix}$$

$$\begin{cases} x + 3y + z = 7 \\ y - z = 3 \\ -16z = 0 \end{cases}$$

$-16z = 0 \Rightarrow z = 0$

$y - 0 = 3 \Rightarrow y = 3$

$x + 3(3) + 0 = 7 \Rightarrow x = -2$

Solution: $(-2, 3, 0)$

Section 8.3 The Inverse of a Square Matrix

1. inverse

3. A square matrix that is singular does not have an inverse.

5. $$AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & -2 + 2 \\ 15 - 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 3 - 3 \\ -10 + 10 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. $$AB = \frac{1}{10}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 12 - 2 & -6 + 6 \\ 4 - 4 & -2 + 12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 12 - 2 & 8 - 8 \\ -3 + 3 & -2 + 12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9. $$AB = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 2 - 34 + 33 & 2 - 68 + 66 & 4 + 51 - 55 \\ -1 + 22 - 21 & -1 + 44 - 42 & -2 - 33 + 35 \\ 6 - 6 & 12 - 12 & -9 + 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 - 1 & -17 + 11 + 6 & 11 - 7 - 4 \\ 4 - 4 & -34 + 44 - 9 & 22 - 28 + 6 \\ 6 - 6 & -51 + 66 - 15 & 33 - 42 + 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11. $$AB = \frac{1}{3}\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 + 0 + 2 + 3 & 6 + 0 + 0 - 6 & -4 + 0 - 2 + 6 & -4 + 0 - 2 + 6 \\ -3 + 0 + 0 + 3 & 9 + 0 + 0 - 6 & -6 + 0 + 0 + 6 & -6 + 0 + 0 + 6 \\ 1 - 2 - 2 + 3 & -3 + 9 + 0 - 6 & 2 - 7 + 2 + 6 & 2 - 10 + 2 + 6 \\ -3 + 2 + 1 + 0 & 9 - 9 + 0 + 0 & -6 + 7 - 1 + 0 & -6 + 10 - 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$BA = \frac{1}{3}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 + 9 + 2 - 6 & 0 + 0 - 2 + 2 & -2 + 0 + 4 - 2 & -1 + 3 - 2 + 0 \\ -4 + 27 + 7 - 30 & 0 + 0 - 7 + 10 & -4 + 0 + 14 - 10 & -2 + 9 - 7 + 0 \\ 2 + 0 + 1 - 3 & 0 + 0 - 1 + 1 & 2 + 0 + 2 - 1 & 1 + 0 - 1 + 0 \\ 6 - 18 - 6 + 18 & 0 + 0 + 6 - 6 & 6 + 0 - 12 + 6 & 3 - 6 + 6 + 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13. $$\begin{bmatrix} A & \vdots & I \end{bmatrix} = \begin{bmatrix} 1 & -2 & \vdots & 1 & 0 \\ 2 & -3 & \vdots & 0 & 1 \end{bmatrix}$$

$$-2R_1 + R_2 \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix}$$

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \vdots & -3 & 2 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & \vdots & A^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

15. $$\begin{bmatrix} A & \vdots & I \end{bmatrix} = \begin{bmatrix} 3 & 1 & \vdots & 1 & 0 \\ 4 & 2 & \vdots & 0 & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_2 \rightarrow \begin{bmatrix} 3 & 1 & \vdots & 1 & 0 \\ 2 & 1 & \vdots & 0 & \tfrac{1}{2} \end{bmatrix}$$

$$-R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \vdots & 1 & -\tfrac{1}{2} \\ 2 & 1 & \vdots & 0 & \tfrac{1}{2} \end{bmatrix}$$

$$-2R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & \vdots & 1 & -\tfrac{1}{2} \\ 0 & 1 & \vdots & -2 & \tfrac{3}{2} \end{bmatrix} = \begin{bmatrix} I & \vdots & A^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -\tfrac{1}{2} \\ -2 & \tfrac{3}{2} \end{bmatrix}$$

17. $$\left[A \; \vdots \; I\right] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 3 & 5 & 4 & \vdots & 0 & 1 & 0 \\ 3 & 6 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ -3R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 2 & 1 & \vdots & -3 & 1 & 0 \\ 0 & 3 & 2 & \vdots & -3 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ \frac{1}{2}R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \vdots & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 2 & \vdots & -3 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} -R_2 + R_1 \rightarrow \\ \\ -3R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \vdots & \frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \vdots & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \vdots & \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$$

$$\begin{array}{r} -R_3 + R_1 \rightarrow \\ -R_3 + R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & -1 \\ 0 & 1 & 0 & \vdots & -3 & 2 & -1 \\ 0 & 0 & \frac{1}{2} & \vdots & \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 2R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & -1 \\ 0 & 1 & 0 & \vdots & -3 & 2 & -1 \\ 0 & 0 & 1 & \vdots & 3 & -3 & 2 \end{bmatrix}$$

$$= \left[I \; \vdots \; A^{-1}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$$

19. $$\left[A \; \vdots \; I\right] = \begin{bmatrix} -5 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 2 & 0 & 0 & \vdots & 0 & 1 & 0 \\ -1 & 5 & 7 & \vdots & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_2 + 2R_3 \rightarrow \end{array} \begin{bmatrix} -5 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 2 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 10 & 14 & \vdots & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ 2R_1 + 5R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} -5 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 2 & 5 & 0 \\ 0 & 10 & 14 & \vdots & 0 & 1 & 2 \end{bmatrix}$$

Because the first three entries of row 2 are all zeros, the inverse of A does not exist.

21. $$\left[A \; \vdots \; I\right] = \begin{bmatrix} -8 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} -\frac{1}{8}R_1 \rightarrow \\ \\ \frac{1}{4}R_3 \rightarrow \\ -\frac{1}{5}R_4 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix} = \left[I \; \vdots \; A^{-1}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$$

23. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -175 & 37 & -13 \\ 95 & -20 & 7 \\ 14 & -3 & 1 \end{bmatrix}$$

25. $A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix}$$

27. $A = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 0 & -1.\overline{81} & 0.\overline{90} \\ -10 & 5 & 5 \\ 10 & -2.\overline{72} & -3.\overline{63} \end{bmatrix}$$

29. $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

31. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

$$ad - bc = (2)(5) - (3)(-1) = 13$$

$$A^{-1} = \tfrac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

33. $A = \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

$$ad - bc = (-4)(3) - (-2)(-6) = 0$$

Because $ad - bc = 0$, A^{-1} does not exist.

35. $A = \begin{bmatrix} 0.5 & 0.3 \\ 1.5 & 0.6 \end{bmatrix}$

$$\begin{aligned} ad - bc &= (0.5)(0.6) - (0.3)(1.5) \\ &= 0.3 - 0.45 = -0.15 \end{aligned}$$

$$A^{-1} = -\tfrac{1}{0.15}\begin{bmatrix} 0.6 & -0.3 \\ -1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 10 & -\frac{10}{3} \end{bmatrix}$$

37. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Solution: $(5, 0)$

39. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \end{bmatrix}$

Solution: $(-8, -6)$

41. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ -11 \end{bmatrix}$

Solution: $(3, 8, -11)$

43. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -24 & 7 & 1 & -2 \\ -10 & 3 & 0 & -1 \\ -29 & 7 & 3 & -2 \\ 12 & -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Solution: $(2, 1, 0, 0)$

45. $A = \begin{bmatrix} 5 & 4 \\ 2 & 5 \end{bmatrix}$

$$A^{-1} = \frac{1}{25 - 8}\begin{bmatrix} 5 & -4 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} 5 & -4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution: $(-1, 1)$

47. $A = \begin{bmatrix} -0.4 & 0.8 \\ 2 & -4 \end{bmatrix}$

$$A^{-1} = \frac{1}{1.6 - 1.6}\begin{bmatrix} -4 & -0.8 \\ -2 & -0.4 \end{bmatrix}$$

A^{-1} does not exist.

This implies that there is no unique solution; that is, either the system is inconsistent *or* there are infinitely many solutions.

Find the reduced row-echelon form of the matrix corresponding to the system.

$$\begin{bmatrix} -0.4 & 0.8 & \vdots & 1.6 \\ 2 & -4 & \vdots & 5 \end{bmatrix}$$

$$-2.5R_1 \rightarrow \begin{bmatrix} 1 & -2 & \vdots & -4 \\ 2 & -4 & \vdots & 5 \end{bmatrix}$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \end{array} \begin{bmatrix} 1 & -2 & \vdots & -4 \\ 0 & 0 & \vdots & 13 \end{bmatrix}$$

The given system is inconsistent and there is no solution.

49. $A = \begin{bmatrix} 2.3 & -1.9 \\ 1.5 & 0.75 \end{bmatrix}$

$$A^{-1} = \frac{1}{1.725 + 2.85}\begin{bmatrix} 0.75 & 1.9 \\ -1.5 & 2.3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4.575}\begin{bmatrix} 0.75 & 1.9 \\ -1.5 & 2.3 \end{bmatrix}\begin{bmatrix} 6 \\ -12 \end{bmatrix} = \frac{1}{4.575}\begin{bmatrix} -18.3 \\ -36.6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -8 \end{bmatrix}$$

Solution: $(-4, -8)$

51. $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{bmatrix}$

Find A^{-1}.

$$[A \ \vdots \ I] = \begin{bmatrix} 4 & -1 & 1 & \vdots & 1 & 0 & 0 \\ 2 & 2 & 3 & \vdots & 0 & 1 & 0 \\ 5 & -2 & 6 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} R_1 \\ \\ R_3 \end{array} \begin{bmatrix} 5 & -2 & 6 & \vdots & 0 & 0 & 1 \\ 2 & 2 & 3 & \vdots & 0 & 1 & 0 \\ 4 & -1 & 1 & \vdots & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r} -R_3 + R_1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 5 & \vdots & -1 & 0 & 1 \\ 2 & 2 & 3 & \vdots & 0 & 1 & 0 \\ 4 & -1 & 1 & \vdots & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -4R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 5 & \vdots & -1 & 0 & 1 \\ 0 & 4 & -7 & \vdots & 2 & 1 & -2 \\ 0 & 3 & -19 & \vdots & 5 & 0 & -4 \end{bmatrix}$$

$$\begin{array}{r} \\ -R_3 + R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 5 & \vdots & -1 & 0 & 1 \\ 0 & 1 & 12 & \vdots & -3 & 1 & 2 \\ 0 & 3 & -19 & \vdots & 5 & 0 & -4 \end{bmatrix}$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -3R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 17 & \vdots & -4 & 1 & 3 \\ 0 & 1 & 12 & \vdots & -3 & 1 & 2 \\ 0 & 0 & -55 & \vdots & 14 & -3 & -10 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -\frac{1}{55}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 17 & \vdots & -4 & 1 & 3 \\ 0 & 1 & 12 & \vdots & -3 & 1 & 2 \\ 0 & 0 & 1 & \vdots & -\frac{14}{55} & \frac{3}{55} & \frac{2}{11} \end{bmatrix}$$

$$\begin{array}{r} -17R_3 + R_1 \rightarrow \\ -12R_3 + R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{18}{55} & \frac{4}{55} & -\frac{1}{11} \\ 0 & 1 & 0 & \vdots & \frac{3}{55} & \frac{19}{55} & -\frac{2}{11} \\ 0 & 0 & 1 & \vdots & -\frac{14}{55} & \frac{3}{55} & \frac{2}{11} \end{bmatrix}$$

$$= [I \ \vdots \ A^{-1}]$$

$$A^{-1} = \frac{1}{55}\begin{bmatrix} 18 & 4 & -5 \\ 3 & 19 & -10 \\ -14 & 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{55}\begin{bmatrix} 18 & 4 & -5 \\ 3 & 19 & -10 \\ -14 & 3 & 10 \end{bmatrix}\begin{bmatrix} -5 \\ 10 \\ 1 \end{bmatrix} = \frac{1}{55}\begin{bmatrix} -55 \\ 165 \\ 110 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Solution: $(-1, 3, 2)$

53. $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 7z = -4 \end{cases}$

Using a graphing utility $(0.8125, 0.6875, 0) = \left(\frac{13}{16}, \frac{11}{16}, 0\right)$

55. $\begin{cases} 2I_1 + 4I_3 = 15 \\ I_2 + 4I_3 = 17 \\ I_1 + I_2 - I_3 = 0 \end{cases}$

$$\underset{A}{\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 15 \\ 17 \\ 0 \end{bmatrix}}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{5}{14} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{4}{7} \\ \frac{1}{14} & \frac{1}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 15 \\ 17 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 3 \\ \frac{7}{2} \end{bmatrix}$$

So, $I_1 = 0.5$ ampere, $I_2 = 3.0$ ampere, and $I_3 = 3.5$ ampere.

57. $\begin{cases} 2I_1 + 4I_3 = 28 \\ I_2 + 4I_3 = 21 \\ I_1 + I_2 - I_3 = 0 \end{cases}$

$$\underset{A}{\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 28 \\ 21 \\ 0 \end{bmatrix}}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{5}{14} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{4}{7} \\ \frac{1}{14} & \frac{1}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 28 \\ 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$$

So, $I_1 = 4$ ampere, $I_2 = 1$ ampere, and $I_3 = 5$ ampere.

In Exercise 59, use the following:
Let x = bags of potting soil for seedlings,
y = bags of potting soil for general potting, and
z = bags of potting soil for hardwood plants.

$$\mathbf{AX = B} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \textbf{Sand} \\ \textbf{Loam} \\ \textbf{Peat Moss} \end{bmatrix}$$

$$\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ -\frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{3}}{\mathbf{2}} \end{bmatrix}$$

59. $A^{-1}\begin{bmatrix} 500 \\ 500 \\ 400 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$

Solution:

x = 100 bags of potting soil for seedlings,

y = 100 bags of potting soil for general potting,

z = 100 bags of potting soil for hardwood plants.

61. (16, 82,700), (17, 86,918), (18, 89,400)

(a) $\begin{cases} 256a + 16b + c = 82{,}700 \\ 289a + 17b + c = 86{,}918 \\ 324a + 18b + c = 89{,}400 \end{cases}$

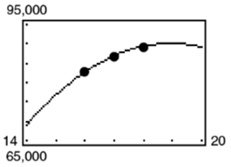
(b) $AX = B = \begin{bmatrix} 256 & 16 & 1 \\ 289 & 17 & 1 \\ 324 & 18 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 82{,}700 \\ 86{,}918 \\ 89{,}400 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 256 & 16 & 1 \\ 289 & 17 & 1 \\ 324 & 18 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -17.5 & 34 & -16.5 \\ 153 & -288 & 136 \end{bmatrix}$$

(c) $X = A^{-1}B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1}\begin{bmatrix} 82{,}700 \\ 86{,}918 \\ 89{,}400 \end{bmatrix} = \begin{bmatrix} -868 \\ 32{,}862 \\ -220{,}884 \end{bmatrix}$

$y = -868t^2 + 32{,}862t - 220{,}884$

(d)



63. True. If B is the inverse of A, then $AB = I = BA$.

65. If the determinant of a 2×2 matrix is not equal to 0, then the inverse exists.

To find the inverse, take 1 divided by the determinant and multiply it by the matrix which has a diagonal from top left to bottom right that has the terms from the original matrix flipped and the other diagonal is the negative of the terms from the original matrix.

67. Answers will vary. *Sample Answer.* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1} \cdot A = \left(\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

69. If A^{-1} does not exist, it is singular.

$\begin{bmatrix} 4 & 3 \\ -2 & k \end{bmatrix}$, $(4)(k) - (-2)(3) = 0 \Rightarrow 4k + 6 = 0 \Rightarrow k = -\frac{3}{2}$.

When $k \neq -\frac{3}{2}$, A^{-1} exists because the determinant does not equal zero.

71. $\mathbf{u} \cdot \mathbf{v} = \langle -5, -7\rangle \cdot \langle -6, 1\rangle = -5(-6) + (-7)(1) = 23$

73. $\mathbf{u} \cdot \mathbf{v} = \langle 7, -12\rangle \cdot \langle -8, 8\rangle = 7(-8) + (-12)(8) = -152$

75. $\mathbf{u} \cdot \mathbf{v} = (2\mathbf{i} - \mathbf{j}) \cdot (7\mathbf{i} + 6\mathbf{j}) = 2(7) + (-1)(6) = 8$

77. $-\frac{1}{3}\begin{bmatrix} 3 & 6 \\ 0 & -3 \end{bmatrix} + 2\begin{bmatrix} 6 & 8 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 12 & 16 \\ 10 & -10 \end{bmatrix}$

$= \begin{bmatrix} 11 & 14 \\ 10 & -9 \end{bmatrix}$

79. $\frac{2}{3}\begin{bmatrix} 3 & -21 \\ -9 & 12 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 15 & 24 \\ 18 & 0 \end{bmatrix} = -2X$

$\begin{bmatrix} 2 & -14 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 6 & 0 \end{bmatrix} = -2X$

$\begin{bmatrix} 7 & -6 \\ 0 & 8 \end{bmatrix} = -2X$

$X = -\frac{1}{2}\begin{bmatrix} 7 & -6 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & 3 \\ 0 & -4 \end{bmatrix}$

Section 8.4 The Determinant of a Square Matrix

1. determinant

3. –5

5. 4

7. $\begin{vmatrix} 8 & 4 \\ 2 & 3 \end{vmatrix} = (8)(3) - (4)(2) = 16$

9. $\begin{vmatrix} 6 & -3 \\ -5 & 2 \end{vmatrix} = (6)(2) - (-3)(-5) = -3$

11. $\begin{vmatrix} -7 & 0 \\ 3 & 0 \end{vmatrix} = -7(0) - 0(3) = 0$

13. $\begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} = 3(1) - 4(-2) = 3 + 8 = 11$

15. $\begin{vmatrix} -3 & -2 \\ -6 & -4 \end{vmatrix} = (-3)(-4) - (-2)(-6) = 12 - 12 = 0$

17. $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{vmatrix} = -\frac{1}{2}\left(\frac{1}{3}\right) - \frac{1}{3}(-6) = -\frac{1}{2} + 2 = \frac{11}{6}$

19. $\begin{vmatrix} 19 & 20 \\ 43 & -56 \end{vmatrix} = -1924$

21. $\begin{vmatrix} \frac{1}{10} & \frac{1}{5} \\ -\frac{3}{10} & \frac{1}{5} \end{vmatrix} = 0.08$

23. $\begin{bmatrix} 4 & 5 \\ 3 & -6 \end{bmatrix}$

(a) $M_{11} = -6$
$M_{12} = 3$
$M_{21} = 5$
$M_{22} = 4$

(b) $C_{11} = M_{11} = -6$
$C_{12} = -M_{12} = -3$
$C_{21} = -M_{21} = -5$
$C_{22} = M_{22} = 4$

25. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(a) $M_{11} = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 2 - (-1) = 3$

$M_{12} = \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = -3 - 1 = -4$

$M_{13} = \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} = 3 - 2 = 1$

$M_{21} = \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = 0 - (-2) = 2$

$M_{22} = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 4 - 2 = 2$

$M_{23} = \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix} = -4 - 0 = -4$

$M_{31} = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 0 - 4 = -4$

$M_{32} = \begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} = 4 - (-6) = 10$

$M_{33} = \begin{vmatrix} 4 & 0 \\ -3 & 2 \end{vmatrix} = 8 - 0 = 8$

(b) $C_{11} = (-1)^2 M_{11} = 3$

$C_{12} = (-1)^3 M_{12} = 4$

$C_{13} = (-1)^4 M_{13} = 1$

$C_{21} = (-1)^3 M_{21} = -2$

$C_{22} = (-1)^4 M_{22} = 2$

$C_{23} = (-1)^5 M_{23} = 4$

$C_{31} = (-1)^4 M_{31} = -4$

$C_{32} = (-1)^5 M_{32} = -10$

$C_{33} = (-1)^6 M_{33} = 8$

27. $\begin{bmatrix} -4 & 6 & 3 \\ 7 & -2 & 8 \\ 1 & 0 & -5 \end{bmatrix}$

(a) $M_{11} = \begin{vmatrix} -2 & 8 \\ 0 & -5 \end{vmatrix} = (-2)(-5) - (8)(0) = 10$

$M_{12} = \begin{vmatrix} 7 & 8 \\ 1 & -5 \end{vmatrix} = (7)(-5) - (8)(1) = -43$

$M_{13} = \begin{vmatrix} 7 & -2 \\ 1 & 0 \end{vmatrix} = (7)(0) - (-2)(1) = 2$

$M_{21} = \begin{vmatrix} 6 & 3 \\ 0 & -5 \end{vmatrix} = (6)(-5) - (3)(0) = -30$

$M_{22} = \begin{vmatrix} -4 & 3 \\ 1 & -5 \end{vmatrix} = (-4)(-5) - (3)(1) = 17$

$M_{23} = \begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} = (-4)(0) - (6)(1) = -6$

$M_{31} = \begin{vmatrix} 6 & 3 \\ -2 & 8 \end{vmatrix} = (6)(8) - (3)(-2) = 54$

$M_{32} = \begin{vmatrix} -4 & 3 \\ 7 & 8 \end{vmatrix} = (-4)(8) - (3)(7) = -53$

$M_{33} = \begin{vmatrix} -4 & 6 \\ 7 & -2 \end{vmatrix} = (-4)(-2) - (6)(7) = -34$

(b) $C_{11} = (-1)^2 M_{11} = 10$

$C_{12} = (-1)^3 M_{12} = 43$

$C_{13} = (-1)^4 M_{13} = 2$

$C_{21} = (-1)^3 M_{21} = 30$

$C_{22} = (-1)^4 M_{22} = 17$

$C_{23} = (-1)^5 M_{23} = 6$

$C_{31} = (-1)^4 M_{31} = 54$

$C_{32} = (-1)^5 M_{32} = 53$

$C_{33} = (-1)^6 M_{33} = -34$

29. (a) $\begin{vmatrix} 2 & 5 \\ 6 & -3 \end{vmatrix} = 2(-3) - 5(6) = -36$

(b) $\begin{vmatrix} 2 & 5 \\ 6 & -3 \end{vmatrix} = 2(-3) - 6(5) = -36$

31. (a) $\begin{vmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{vmatrix} = 0\begin{vmatrix} 0 & -3 \\ 6 & 3 \end{vmatrix} + 12\begin{vmatrix} 5 & -3 \\ 1 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 1 & 6 \end{vmatrix} = 0(18) + 12(18) - 4(30) = 96$

(b) $\begin{vmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{vmatrix} = 0\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} + 12\begin{vmatrix} 5 & -3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 5 & -3 \\ 0 & 4 \end{vmatrix} = 0(-4) + 12(18) - 6(20) = 96$

33. (a) $$\begin{vmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{vmatrix} = -3\begin{vmatrix} 5 & 6 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix} = -3(23) - 2(-8) - 22 = -75$$

(b) $$\begin{vmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} -3 & 1 \\ 4 & 6 \end{vmatrix} = -2(-8) + 5(-5) + 3(-22) = -75$$

35. (a) $$\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 0 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 0 & 0 & 2 \end{vmatrix} = -8\begin{vmatrix} 0 & -3 & 5 \\ 0 & 6 & -8 \\ 0 & 7 & 4 \end{vmatrix} + 0\begin{vmatrix} 6 & -3 & 5 \\ 4 & 6 & -8 \\ -1 & 7 & 4 \end{vmatrix} - 0\begin{vmatrix} 6 & 0 & 5 \\ 4 & 0 & -8 \\ -1 & 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 6 & 0 & -3 \\ 4 & 0 & 6 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= -8(0) + 2(0) = 0$$

(b) $$\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 0 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 0 & 0 & 2 \end{vmatrix} = -0\begin{vmatrix} 4 & 6 & -8 \\ -1 & 7 & 4 \\ 8 & 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 6 & -3 & 5 \\ -1 & 7 & 4 \\ 8 & 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 6 & -3 & 5 \\ 4 & 6 & -8 \\ 8 & 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 6 & -3 & 5 \\ 4 & 6 & -8 \\ -1 & 7 & 4 \end{vmatrix}$$

$$= 0$$

37. (a) $$\begin{vmatrix} -2 & 4 & 7 & 1 \\ 3 & 0 & 0 & 0 \\ 8 & 5 & 10 & 5 \\ 6 & 0 & 5 & 0 \end{vmatrix} = -3\begin{vmatrix} 4 & 7 & 1 \\ 5 & 10 & 5 \\ 0 & 5 & 0 \end{vmatrix} + 0\begin{vmatrix} -2 & 7 & 1 \\ 8 & 10 & 5 \\ 6 & 5 & 0 \end{vmatrix} - 0\begin{vmatrix} -2 & 4 & 1 \\ 8 & 5 & 5 \\ 6 & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} -2 & 4 & 7 \\ 8 & 5 & 10 \\ 6 & 0 & 5 \end{vmatrix}$$

$$= -3(-75) = 225$$

(b) $$\begin{vmatrix} -2 & 4 & 7 & 1 \\ 3 & 0 & 0 & 0 \\ 8 & 5 & 10 & 5 \\ 6 & 0 & 5 & 0 \end{vmatrix} = -1\begin{vmatrix} 3 & 0 & 0 \\ 8 & 5 & 10 \\ 6 & 0 & 5 \end{vmatrix} + 0\begin{vmatrix} -2 & 4 & 7 \\ 8 & 5 & 10 \\ 6 & 0 & 5 \end{vmatrix} - 5\begin{vmatrix} -2 & 4 & 7 \\ 3 & 0 & 0 \\ 6 & 0 & 5 \end{vmatrix} + 0\begin{vmatrix} -2 & 4 & 7 \\ 3 & 0 & 0 \\ 8 & 5 & 10 \end{vmatrix}$$

$$= (-1)(75) - 5(-60) = 225$$

39. Expand along Column 1.

$$\begin{vmatrix} -1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{vmatrix} = -1\begin{vmatrix} 3 & 4 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -5 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 \\ 3 & 4 \end{vmatrix}$$

$$= -1(9) - 0(6) + 0(23) = -9$$

41. Expand along Row 2.

$$\begin{vmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{vmatrix} = 0\begin{vmatrix} 3 & -7 \\ -6 & 3 \end{vmatrix} - 0\begin{vmatrix} 6 & -7 \\ 4 & 3 \end{vmatrix} + 0\begin{vmatrix} 6 & 3 \\ 4 & -6 \end{vmatrix} = 0$$

43. Expand along Column 1.

$$\begin{vmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 2(0) - 4(-1) + 4(-1) = 0$$

45. Expand along Column 3.

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= -2(14) + 3(-10) = -58$$

47. Expand along Column 3.

$$\begin{vmatrix} 2 & 6 & 0 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 0 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{vmatrix} = 0\begin{vmatrix} 2 & 7 & 6 \\ 1 & 0 & 1 \\ 3 & 7 & 7 \end{vmatrix} - 3\begin{vmatrix} 2 & 6 & 2 \\ 1 & 0 & 1 \\ 3 & 7 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 & 2 \\ 2 & 7 & 6 \\ 3 & 7 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 6 & 2 \\ 2 & 7 & 6 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -3(-24) = 72$$

49. Expand along Column 1.

$$\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5\begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= 5(0) - 4(0) = 0$$

51. Expand along Column 2, then along Column 4.

$$\begin{vmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} -2 & 1 & 3 & 2 \\ 1 & 0 & 4 & 0 \\ 6 & 2 & -1 & 0 \\ 3 & 5 & 1 & 0 \end{vmatrix} = (-2)(-2)\begin{vmatrix} 1 & 0 & 4 \\ 6 & 2 & -1 \\ 3 & 5 & 1 \end{vmatrix} = 4(103) = 412$$

53. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix} = -126$

55. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix} = -336$

57. (a) $\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} = -3$

(b) $\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

(d) $\begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} = 6$

59. (a) $\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix} = -8$

(b) $\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$

(c) $\begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{vmatrix} -4 & 4 \\ 1 & -1 \end{vmatrix} = 0$

61. (a) $\begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2$

(b) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -6$

(c) $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$

(d) $\begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} = -12$

63. Answers will vary. *Sample answer*: $|A| = \begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} = 9 - 6 = 3$

65. Answers will vary. *Sample Answer*: $|A| = \begin{vmatrix} 4 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} = -2(7) + 13 = -1$

67. Answers will vary. *Sample Answer*: $|A| = \begin{vmatrix} 2 & 3 \\ 8 & 12 \end{vmatrix} = 24 - 24 = 0$

69. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = wz - xy$

$$-\begin{vmatrix} y & z \\ w & x \end{vmatrix} = -(xy - wz) = wz - xy$$

So, $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$.

71. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = wz - xy$

$$\begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix} = w(z + cy) - y(x + cw) = wz - xy$$

So, $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$.

73. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} y & y^2 \\ z & z^2 \end{vmatrix} - \begin{vmatrix} x & x^2 \\ z & z^2 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ y & y^2 \end{vmatrix}$

$$\begin{aligned}
&= \left(yz^2 - y^2z\right) - \left(xz^2 - x^2z\right) + \left(xy^2 - x^2y\right) \\
&= yz^2 - xz^2 - y^2z + x^2z + xy(y - x) \\
&= z^2(y - x) - z\left(y^2 - x^2\right) + xy(y - x) \\
&= z^2(y - x) - z(y - x)(y + x) + xy(y - x) \\
&= (y - x)\left[z^2 - z(y + x) + xy\right] \\
&= (y - x)\left[z^2 - zy - zx + xy\right] \\
&= (y - x)\left[z^2 - zx - zy + xy\right] \\
&= (y - x)\left[z(z - x) - y(z - x)\right] \\
&= (y - x)(z - x)(z - y)
\end{aligned}$$

75. $\begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2$

$$\begin{aligned}
x^2 - 2 &= 2 \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

77. $\begin{vmatrix} x + 1 & 2 \\ -1 & x \end{vmatrix} = 4$

$$\begin{aligned}
(x + 1)(x) - (2)(-1) &= 4 \\
x^2 + x - 2 &= 0 \\
(x + 2)(x - 1) &= 0 \\
x &= -2 \text{ or } x = 1
\end{aligned}$$

79. $\begin{vmatrix} x + 3 & 2 \\ 1 & x + 2 \end{vmatrix} = 0$

$$\begin{aligned}
(x + 3)(x + 2) - 2 &= 0 \\
x^2 + 5x + 4 &= 0 \\
(x + 1)(x + 4) &= 0 \\
x &= -1 \text{ or } x = -4
\end{aligned}$$

81. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix} = 8uv - 1$

83. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x}$

85. $\begin{vmatrix} x & \ln x \\ 1 & \dfrac{1}{x} \end{vmatrix} = 1 - \ln x$

87. True. If an entire row is zero, then each cofactor in the expansion is multiplied by zero.

89. *Sample answer:* Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 0 \\ 3 & 5 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} = 10,\ |B| = \begin{vmatrix} -4 & 0 \\ 3 & 5 \end{vmatrix} = -20,\ |A| + |B| = -10$$

$$A + B = \begin{bmatrix} -3 & 3 \\ 1 & 9 \end{bmatrix},\ |A + B| = \begin{vmatrix} -3 & 3 \\ 1 & 9 \end{vmatrix} = -30$$

So, $|A + B| \neq |A| + |B|$.

91. (a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -115$

$$-\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix} = -115$$

Column 2 and Column 3 were interchanged.

(b) Multiplying Row 1 of the matrix $\begin{bmatrix} 1 & -3 \\ 5 & 2 \end{bmatrix}$ by –5 and adding it to Row 2 gives the matrix $\begin{bmatrix} 1 & -3 \\ 0 & 17 \end{bmatrix}$.

$$\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = 17 = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$$

(c) $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 5 & 10 \\ 2 & -3 \end{bmatrix}$

$$|B| = \begin{vmatrix} 5 & 10 \\ 2 & -3 \end{vmatrix} = -35$$

$$5|A| = 5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -35$$

Row 1 was multiplied by 5.

$|B| = 5|A|$

93. (a) $\begin{vmatrix} 7 & 0 \\ 0 & 4 \end{vmatrix} = 7(4) - 0 = 28$

(b) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 5 \\ 0 & 0 \end{vmatrix} = (-1)(10) = -10$

(c) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (-2)\begin{vmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

$$= (-2)\left((2)\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right)$$

$$= (-2)(2)(3) = -12$$

The determinant of a diagonal matrix is the product of the entries on the main diagonal.

95. $\dfrac{1(3) - 9(4)}{3(3) - 5(4)} = \dfrac{3 - 36}{9 - 20} = \dfrac{-33}{-11} = 3$

97. $4(2)(6) + (-1)(3)(5) + 1(2)(-2) = 48 - 15 - 4 = 29$

99. $4(-1)^{1+1}\left[2(6) - (-2)(3)\right] = 4[12 + 6] = 4(18) = 72$

101. $\begin{cases} 4x - 2y = 10 & \text{Equation 1} \\ 3x - 5y = 11 & \text{Equation 2} \end{cases}$

Solve for x in Equation 1: $4x = 10 + 2y \Rightarrow x = \frac{1}{4}(10 + 2y)$

Substitute for x in Equation 2: $3\left[\frac{1}{4}(10 + 2y)\right] - 5y = 11 \Rightarrow \frac{3}{2}y - 5y = 11 - \frac{15}{2}$

$\Rightarrow -\frac{7}{2}y = \frac{7}{2}$

$\Rightarrow y = -1$

Back-substitute $y = -1$: $x = \frac{1}{4}\left[10 + 2(-1)\right] = \frac{1}{4}(8) = 2$

Solution: $(2, -1)$

103. $\begin{cases} 4x - y + z = 12 \\ 2x + 2y + 3z = 1 \\ 5x - 2y + 6z = 22 \end{cases}$

$$\left[\begin{array}{rrr:r} 4 & -1 & 1 & 12 \\ 2 & 2 & 3 & 1 \\ 5 & -2 & 6 & 22 \end{array}\right]$$

$$\begin{array}{r} R_2 \\ R_1 \\ \\ \end{array}\left[\begin{array}{rrr:r} 2 & 2 & 3 & 1 \\ 4 & -1 & 1 & 12 \\ 5 & -2 & 6 & 22 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -5R_1 + 2R_3 \rightarrow \end{array}\left[\begin{array}{rrr:r} 2 & 2 & 3 & 1 \\ 0 & -5 & -5 & 10 \\ 0 & -14 & -3 & 39 \end{array}\right]$$

$$\begin{array}{r} \\ -\frac{1}{5}R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrr:r} 2 & 2 & 3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -14 & -3 & 39 \end{array}\right]$$

$$\begin{array}{r} \\ 14R_2 + R_3 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrr:r} 2 & 2 & 3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 11 & 11 \end{array}\right]$$

$\begin{cases} 2x + 2y + 3z = 1 \\ \quad\;\; y + z = -2 \\ \qquad\quad 11z = 11 \end{cases}$

$11z = 11 \Rightarrow z = 1$

$y = -2 - 1 = -3$

$2x = 1 - 2(-3) - 3(1) = 4 \Rightarrow x = 2$

Solution: $(2, -3, 1)$

105. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1(2) + 0(0) \\ 0(2) + 1(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

Section 8.5 Applications of Matrices and Determinants

1. Cramer's Rule

3. $\begin{cases} 2x + 3y = 8 \\ 3x + 2y = 7 \end{cases}$

$$\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 2(2) - 3(3) = 4 - 9 = -5$$

5. $\begin{cases} -5x + 9y = -14 \\ 3x - 7y = 10 \end{cases}$

$$x = \frac{\begin{vmatrix} -14 & 9 \\ 10 & -7 \end{vmatrix}}{\begin{vmatrix} -5 & 9 \\ 3 & -7 \end{vmatrix}} = \frac{8}{8} = 1$$

$$y = \frac{\begin{vmatrix} -5 & -14 \\ 3 & 10 \end{vmatrix}}{\begin{vmatrix} -5 & 9 \\ 3 & -7 \end{vmatrix}} = \frac{-8}{8} = -1$$

Solution: $(1, -1)$

7. $\begin{cases} 3x + 2y = -2 \\ 6x + 4y = 4 \end{cases}$

Because $\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$, Cramer's Rule does not apply.

The system is inconsistent in this case and has no solution.

9. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10, \\ 5x - 2y + 6z = 1 \end{cases}$ $D = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -1 & 6 \end{vmatrix} = 55$

$$x = \frac{\begin{vmatrix} -5 & -1 & 1 \\ 10 & 2 & 3 \\ 1 & -2 & 6 \end{vmatrix}}{55} = \frac{-55}{55} = -1,\ y = \frac{\begin{vmatrix} 4 & -5 & 1 \\ 2 & 10 & 3 \\ 5 & 1 & 6 \end{vmatrix}}{55} = \frac{165}{55} = 3,\ z = \frac{\begin{vmatrix} 4 & -1 & -5 \\ 2 & 2 & 10 \\ 5 & -2 & 1 \end{vmatrix}}{55} = \frac{110}{55} = 2$$

Solution: $(-1, 3, 2)$

11. $\begin{cases} x + 2y + 3z = -3 \\ -2x + y - z = 6, \\ 3x - 3y + 2z = -11 \end{cases}$ $D = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \\ 3 & -3 & 2 \end{vmatrix} = 10$

$$x = \frac{\begin{vmatrix} -3 & 2 & 3 \\ 6 & 1 & -1 \\ -11 & -3 & 2 \end{vmatrix}}{10} = \frac{-20}{10} = -2$$

$$y = \frac{\begin{vmatrix} 1 & -3 & 3 \\ -2 & 6 & -1 \\ 3 & -11 & 2 \end{vmatrix}}{10} = \frac{10}{10} = 1$$

$$z = \frac{\begin{vmatrix} 1 & 2 & -3 \\ -2 & 1 & 6 \\ 3 & -3 & -11 \end{vmatrix}}{10} = \frac{-10}{10} = -1$$

Solution: $(-2, 1, -1)$

13. Vertices: $(0, 0)\ (3, 1), (1, 5)$

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 5 & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 7 \text{ square units}$$

15. Vertices: $(-2, -3), (2, -3), (0, 4)$

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 4 & 1 \end{vmatrix} = \tfrac{1}{2}\left(-2\begin{vmatrix} -3 & 1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -3 & 1 \\ 4 & 1 \end{vmatrix}\right)$$

$$= \tfrac{1}{2}(14 + 14) = 14 \text{ square units}$$

17.
$$
\begin{aligned}
4 &= \pm\frac{1}{2}\begin{vmatrix} -5 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & y & 1 \end{vmatrix} \\
\pm 8 &= -5\begin{vmatrix} 2 & 1 \\ y & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\
\pm 8 &= -5(2 - y) - 2(-1) \\
\pm 8 &= 5y - 8 \\
y &= \frac{8 \pm 8}{5} \\
y &= \frac{16}{5} \text{ or } y = 0
\end{aligned}
$$

19. Vertices: $(0, 25), (10, 0), (28, 5)$

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 0 & 25 & 1 \\ 10 & 0 & 1 \\ 28 & 5 & 1 \end{vmatrix} = 250 \text{ square miles}$$

21. Points: $(2, -6), (0, -2), (3, -8)$

$$
\begin{aligned}
\begin{vmatrix} 2 & -6 & 1 \\ 0 & -2 & 1 \\ 3 & -8 & 1 \end{vmatrix} &= 2\begin{vmatrix} -2 & 1 \\ -8 & 1 \end{vmatrix} + 3\begin{vmatrix} -6 & 1 \\ -2 & 1 \end{vmatrix} \\
&= 2(6) + 3(-4) \\
&= 0
\end{aligned}
$$

The points are collinear.

23. Points: $\left(2, -\frac{1}{2}\right), (-4, 4), (6, -3)$

$$
\begin{aligned}
\begin{vmatrix} 2 & -\frac{1}{2} & 1 \\ -4 & 4 & 1 \\ 6 & -3 & 1 \end{vmatrix} &= \begin{vmatrix} -4 & 4 \\ 6 & -3 \end{vmatrix} - \begin{vmatrix} 2 & -\frac{1}{2} \\ 6 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -\frac{1}{2} \\ -4 & 4 \end{vmatrix} \\
&= -12 + 3 + 6 \\
&= -3 \neq 0
\end{aligned}
$$

The points are not collinear.

25. Points: $(0, 2), (1, 2.4), (-1, 1.6)$

$$\begin{vmatrix} 0 & 2 & 1 \\ 1 & 2.4 & 1 \\ -1 & 1.6 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2.4 \\ -1 & 1.6 \end{vmatrix} = -2(2) + 4 = 0$$

The points are collinear.

27.
$$
\begin{aligned}
\begin{vmatrix} 2 & -5 & 1 \\ 4 & y & 1 \\ 5 & -2 & 1 \end{vmatrix} &= 0 \\
2\begin{vmatrix} y & 1 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 4 & y \\ 5 & -2 \end{vmatrix} &= 0 \\
2(y + 2) + 5(-1) + (-8 - 5y) &= 0 \\
-3y - 9 &= 0 \\
y &= -3
\end{aligned}
$$

29. Points: $(0, 0), (5, 3)$

$$
\begin{aligned}
\text{Equation: } \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 5 & 3 & 1 \end{vmatrix} &= -\begin{vmatrix} x & y \\ 5 & 3 \end{vmatrix} = 5y - 3x \\
&= 0 \Rightarrow 3x - 5y = 0
\end{aligned}
$$

31. Points: $(-4, 3), (2, 1)$

$$\text{Equation: } \begin{vmatrix} x & y & 1 \\ -4 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = x\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - y\begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -4 & 3 \\ 2 & 1 \end{vmatrix} = 2x + 6y - 10 = 0 \Rightarrow x + 3y - 5 = 0$$

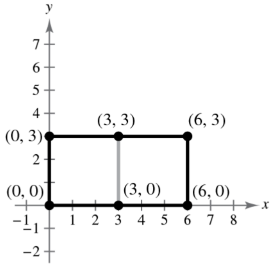
33. Points: $\left(-\frac{1}{2}, 3\right), \left(\frac{5}{2}, 1\right)$

$$\text{Equation: } \begin{vmatrix} x & y & 1 \\ -\frac{1}{2} & 3 & 1 \\ \frac{5}{2} & 1 & 1 \end{vmatrix} = x\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - y\begin{vmatrix} -\frac{1}{2} & 1 \\ \frac{5}{2} & 1 \end{vmatrix} + \begin{vmatrix} -\frac{1}{2} & 3 \\ \frac{5}{2} & 1 \end{vmatrix} = 2x + 3y - 8 = 0$$

35. A horizontal stretch, $k = 2$, of the square with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$ and $(3, 3)$.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

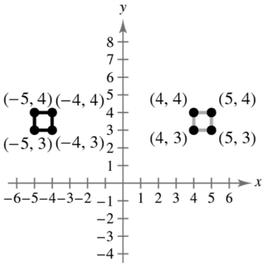
New Vertices: $(0, 0)$, $(0, 3)$, $(6, 0)$ and $(6, 3)$



37. A reflection in the y-axis of the square with vertices $(4, 3)$, $(5, 3)$, $(4, 4)$ and $(5, 4)$.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}.$$

New Vertices: $(-4, 3)$, $(-5, 3)$, $(-4, 4)$ and $(-5, 4)$



39. The area of the parallelogram with vertices: $(0, 0)$, $(1, 0)$, $(2, 2)$ and $(3, 2)$ $\Rightarrow a = 1, b = 2, c = 2$ and $d = 2$.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

Area $= \left|\det(A)\right| = \left|2 - 0\right| = 2$ square units.

41. The area of the parallelogram with vertices: $(0, 0)$, $(-2, 0)$, $(3, 5)$ and $(1, 5)$ $\Rightarrow a = -2,\ b = 0, c = 3$ and $d = 5$.

$$A = \begin{bmatrix} -2 & 0 \\ 3 & 5 \end{bmatrix}$$

Area $= \left|\det(A)\right| = \left|-10 - 0\right| = 10$ square units.

43. (a) Uncoded: C O M E _ H O M E _

[3 15] [13 5] [0 8] [15 13] [5 0]

S O O N

[19 15] [15 14]

(b) $\begin{bmatrix} 3 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 48 & 81 \end{bmatrix}$

$\begin{bmatrix} 13 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 28 & 51 \end{bmatrix}$

$\begin{bmatrix} 0 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 24 & 40 \end{bmatrix}$

$\begin{bmatrix} 15 & 13 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 54 & 95 \end{bmatrix}$

$\begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \end{bmatrix}$

$\begin{bmatrix} 19 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 64 & 113 \end{bmatrix}$

$\begin{bmatrix} 15 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 57 & 100 \end{bmatrix}$

Encoded: 48 81 28 51 24 40 54
95 5 10 64 113 57 100

45. (a) Uncoded: T E X T _ M E _ T

[20 5 24] [20 0 13] [5 0 20]

O M O R R O W _ _

[15 13 15] [18 18 15] [23 0 0]

(b) $\begin{bmatrix} 20 & 5 & 24 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -119 & 28 & 67 \end{bmatrix}$

$\begin{bmatrix} 20 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -58 & 6 & 39 \end{bmatrix}$

$\begin{bmatrix} 5 & 0 & 20 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -115 & 35 & 60 \end{bmatrix}$

$\begin{bmatrix} 15 & 13 & 15 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -62 & 15 & 32 \end{bmatrix}$

$\begin{bmatrix} 18 & 18 & 15 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -54 & 12 & 27 \end{bmatrix}$

$\begin{bmatrix} 23 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 23 & -23 & 0 \end{bmatrix}$

Encoded: −119 28 67 −58 6 39 −115 35 60
−62 15 32 −54 12 27 23 −23 0

In Exercises 47–49, use the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{3} & \mathbf{7} & \mathbf{9} \\ \mathbf{-1} & \mathbf{-4} & \mathbf{-7} \end{bmatrix}$**.**

47. D O W N L O A D _ S U C C E S S F U L _ _

$$[4\quad 15\quad 23]\,[14\quad 12\quad 15]\,[1\quad 4\quad 0]\,[19\quad 21\quad 3]\,[3\quad 5\quad 19]\,[19\quad 6\quad 21]\,[12\quad 0\quad 0]$$

$$[4\quad 15\quad 23]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [26\quad 21\quad -18]$$

$$[14\quad 12\quad 15]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [35\quad 52\quad 31]$$

$$[1\quad 4\quad 0]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [13\quad 30\quad 38]$$

$$[19\quad 21\quad 3]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [79\quad 173\quad 206]$$

$$[3\quad 5\quad 19]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [-1\quad -35\quad -82]$$

$$[19\quad 6\quad 21]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [16\quad -4\quad -55]$$

$$[12\quad 0\quad 0]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [12\quad 24\quad 24]$$

Cryptogram:
26 21 −18 35 52 31 13 30 38
79 173 206 −1 −35 −82 16 −4 −55
12 24 24

49. H A P P Y _ B I R T H D A Y _

$[8 \quad 1 \quad 16]\,[16 \quad 25 \quad 0]\,[2 \quad 9 \quad 18]\,[20 \quad 8 \quad 4]\,[1 \quad 25 \quad 0]$

$$[8 \quad 1 \quad 16]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [-5 \quad -41 \quad -87]$$

$$[16 \quad 25 \quad 0]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [91 \quad 207 \quad 257]$$

$$[2 \quad 9 \quad 18]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [11 \quad -5 \quad -41]$$

$$[20 \quad 8 \quad 4]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [40 \quad 80 \quad 84]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [76 \quad 177 \quad 227]$$

Cryptogram: –5 –41 –87 91 207 257 11 –5 –41 40 80 84 76 177 227

51. $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

$$\begin{bmatrix} 11 & 21 \\ 64 & 112 \\ 25 & 50 \\ 29 & 53 \\ 23 & 46 \\ 40 & 75 \\ 55 & 92 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{array}{c|cc} \begin{bmatrix} 8 & 1 \\ 16 & 16 \\ 25 & 0 \\ 14 & 5 \\ 23 & 0 \\ 25 & 5 \\ 1 & 18 \end{bmatrix} & \begin{matrix} \text{H} & \text{A} \\ \text{P} & \text{P} \\ \text{Y} & \_ \\ \text{N} & \text{E} \\ \text{W} & \_ \\ \text{Y} & \text{E} \\ \text{A} & \text{R} \end{matrix} \end{array}$$

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53. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

$$\begin{bmatrix} 9 & -1 & -9 \\ 38 & -19 & -19 \\ 28 & -9 & -19 \\ -80 & 25 & 41 \\ -64 & 21 & 31 \\ 9 & -5 & -4 \end{bmatrix}\begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} = \begin{array}{c|ccc} \begin{bmatrix} 3 & 12 & 1 \\ 19 & 19 & 0 \\ 9 & 19 & 0 \\ 3 & 1 & 14 \\ 3 & 5 & 12 \\ 5 & 4 & 0 \end{bmatrix} & \begin{matrix} \text{C} & \text{L} & \text{A} \\ \text{S} & \text{S} & \_ \\ \text{I} & \text{S} & \_ \\ \text{C} & \text{A} & \text{N} \\ \text{C} & \text{E} & \text{L} \\ \text{E} & \text{D} & \_ \end{matrix} \end{array}$$

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55. Let A be the 2×2 matrix needed to decode the message.

$$\begin{bmatrix}-18 & -18\\ 1 & 16\end{bmatrix}A = \begin{bmatrix}0 & 18\\ 15 & 14\end{bmatrix}\begin{array}{cc}\_ & \text{R}\\ \text{O} & \text{N}\end{array}$$

$$A = \begin{bmatrix}-18 & -18\\ 1 & 16\end{bmatrix}^{-1}\begin{bmatrix}0 & 18\\ 15 & 14\end{bmatrix} = \begin{bmatrix}-\frac{8}{135} & -\frac{1}{15}\\ \frac{1}{270} & \frac{1}{15}\end{bmatrix}\begin{bmatrix}0 & 18\\ 15 & 14\end{bmatrix} = \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}8 & 21\\ -15 & -10\\ -13 & -13\\ 5 & 10\\ 5 & 25\\ 5 & 19\\ -1 & 6\\ 20 & 40\\ -18 & -18\\ 1 & 16\end{bmatrix}\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix} = \begin{bmatrix}13 & 5\\ 5 & 20\\ 0 & 13\\ 5 & 0\\ 20 & 15\\ 14 & 9\\ 7 & 8\\ 20 & 0\\ 0 & 18\\ 15 & 14\end{bmatrix}\begin{array}{cc}\text{M} & \text{E}\\ \text{E} & \text{T}\\ \_ & \text{M}\\ \text{E} & \_\\ \text{T} & \text{O}\\ \text{N} & \text{I}\\ \text{G} & \text{H}\\ \text{T} & \_\\ \_ & \text{R}\\ \text{O} & \text{N}\end{array}$$

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57. $D = \begin{vmatrix}4 & 0 & 8\\ 0 & 2 & 8\\ 1 & 1 & -1\end{vmatrix} = -56$

$$I_1 = \frac{\begin{vmatrix}2 & 0 & 8\\ 6 & 2 & 8\\ 0 & 1 & -1\end{vmatrix}}{-56} = -\frac{28}{56} = -\frac{1}{2}$$

$$I_2 = \frac{\begin{vmatrix}4 & 2 & 8\\ 0 & 6 & 8\\ 1 & 0 & -1\end{vmatrix}}{-56} = \frac{-56}{-56} = 1$$

$$I_3 = \frac{\begin{vmatrix}4 & 0 & 2\\ 0 & 2 & 6\\ 1 & 1 & 0\end{vmatrix}}{-56} = \frac{-28}{-56} = \frac{1}{2}$$

So, the solution is $I_1 = -0.5$ ampere, $I_2 = 1$ ampere, and $I_3 = 0.5$ ampere.

59. True. The determinant is zero when the points are collinear.

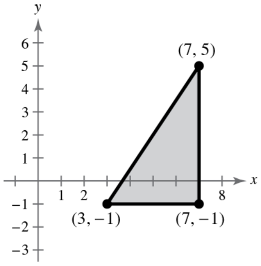
61. If the determinant of the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

63. $$\text{Area} = \frac{1}{2}\begin{vmatrix} 3 & -1 & 1 \\ 7 & -1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left(3\begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 7 & 1 \\ 7 & 1 \end{vmatrix} + 1\begin{vmatrix} 7 & -1 \\ 7 & 5 \end{vmatrix}\right)$$

$$= \frac{1}{2}(-18 + 0 + 42)$$

$$= 12 \text{ square units}$$



$$\text{Area} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(7-3)(5-(-1)) = \tfrac{1}{2}(4)(6) = 12 \text{ square units}$$

65. $3 + (-1)^n$

$n = 1: 3 + (-1)^1 = 3 - 1 = 2$

$n = 2: 3 + (-1)^2 = 3 + 1 = 4$

$n = 3: 3 + (-1)^3 = 3 - 1 = 2$

$n = 4: 3 + (-1)^4 = 3 + 1 = 4$

67. $(-1)^{n+1}(n^2 + 1)$

$n = 1: (-1)^{1+1}(1^2 + 1) = 1 + 1 = 2$

$n = 2: (-1)^{2+1}(2^2 + 1) = -(4 + 1) = -5$

$n = 3: (-1)^{3+1}(3^2 + 1) = 9 + 1 = 10$

$n = 4: (-1)^{4+1}(4^2 + 1) = -(16 + 1) = -17$

Review Exercises for Chapter 8

1. $\begin{bmatrix} -1 & 3 \end{bmatrix}$

Order: 1 × 2

3. $\begin{bmatrix} 2 & 1 & 0 & 4 & -1 \\ 6 & 2 & 1 & 8 & 0 \end{bmatrix}$

Order: 2 × 5

5. $\begin{cases} 3x - 10y = 15 \\ 5x + 4y = 22 \end{cases}$

$$\left[\begin{array}{cc|c} 3 & -10 & 15 \\ 5 & 4 & 22 \end{array}\right]$$

7. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -8 \\ 2 & -2 & 3 & 12 \\ 4 & 7 & 1 & 3 \end{array}\right]$

$$\begin{cases} x + 2z = -8 \\ 2x - 2y + 3z = 12 \\ 4x + 7y + z = 3 \end{cases}$$

9. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

$$\begin{array}{r} R_1 \\ R_2 \\ \\ \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 2R_2 + R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -\frac{1}{2}R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

11. $\left[\begin{array}{cc|c} 1 & -3 & 9 \\ 0 & 1 & -1 \end{array}\right] \begin{array}{c}\Rightarrow \\ \Rightarrow\end{array} \begin{cases} x - 3y = 9 \\ y = -1 \end{cases}$

$x - 3(-1) = 9 \Rightarrow x = 6$

Solution: $(6, -1)$

13. $$\begin{bmatrix}1&3&4&\vdots&1\\0&1&2&\vdots&3\\0&0&1&\vdots&4\end{bmatrix}\begin{matrix}\Rightarrow\\\Rightarrow\\\Rightarrow\end{matrix}\begin{cases}x+3y+4z=1\\ y+2z=3\\ z=4\end{cases}$$

$$y + 2(4) = 3 \Rightarrow y = -5$$

$$x + 3(-5) + 4(4) = 1 \Rightarrow x = 0$$

Solution: $(0, -5, 4)$

15. $$\begin{bmatrix}5&4&\vdots&2\\-1&1&\vdots&-22\end{bmatrix}$$

$$4R_2 + R_1 \rightarrow \begin{bmatrix}1&8&\vdots&-86\\-1&1&\vdots&-22\end{bmatrix}$$

$$\begin{matrix}\\R_1 + R_2 \rightarrow\end{matrix}\begin{bmatrix}1&8&\vdots&-86\\0&9&\vdots&-108\end{bmatrix}$$

$$\begin{matrix}\\\frac{1}{9}R_2 \rightarrow\end{matrix}\begin{bmatrix}1&8&\vdots&-86\\0&1&\vdots&-12\end{bmatrix}$$

$$\begin{cases}x+8y=-86\\ y=-12\end{cases}$$

$$y = -12$$

$$x + 8(-12) = -86 \Rightarrow x = 10$$

Solution: $(10, -12)$

17. $$\begin{cases}-x+2y=3\\2x-4y=6\end{cases}$$

$$\begin{bmatrix}-1&2&\vdots&3\\2&-4&\vdots&6\end{bmatrix}$$

$$\begin{matrix}\\2R_1 + R_2 \rightarrow\end{matrix}\begin{bmatrix}-1&2&\vdots&3\\0&0&\vdots&12\end{bmatrix}$$

Because the last row consists of all zeros except for the last entry, the system is inconsistent and there is no solution.

19. $$\begin{bmatrix}2&1&2&\vdots&4\\2&2&0&\vdots&5\\2&-1&6&\vdots&2\end{bmatrix}$$

$$\begin{matrix}\\-R_1+R_2\rightarrow\\-R_1+R_3\rightarrow\end{matrix}\begin{bmatrix}2&1&2&\vdots&4\\0&1&-2&\vdots&1\\0&-2&4&\vdots&-2\end{bmatrix}$$

$$\begin{matrix}-R_2+R_1\rightarrow\\\\2R_2+R_3\rightarrow\end{matrix}\begin{bmatrix}2&0&4&\vdots&3\\0&1&-2&\vdots&1\\0&0&0&\vdots&0\end{bmatrix}$$

$$\begin{matrix}\frac{1}{2}R_1\rightarrow\\\\\\\end{matrix}\begin{bmatrix}1&0&2&\vdots&\frac{3}{2}\\0&1&-2&\vdots&1\\0&0&0&\vdots&0\end{bmatrix}$$

Let $z = a$, then:

$$y - 2a = 1 \Rightarrow y = 2a + 1$$

$$x + 2a = \tfrac{3}{2} \Rightarrow x = -2a + \tfrac{3}{2}$$

Solution: $\left(-2a + \frac{3}{2}, 2a + 1, a\right)$ where a is any real number

21. $$\begin{bmatrix}2&3&1&\vdots&10\\2&-3&-3&\vdots&22\\4&-2&3&\vdots&-2\end{bmatrix}$$

$$\begin{matrix}\\-R_1+R_2\rightarrow\\-2R_1+R_3\rightarrow\end{matrix}\begin{bmatrix}2&3&1&\vdots&10\\0&-6&-4&\vdots&12\\0&-8&1&\vdots&-22\end{bmatrix}$$

$$\begin{matrix}\frac{1}{2}R_1\rightarrow\\-\frac{1}{6}R_2\rightarrow\\\\\end{matrix}\begin{bmatrix}1&\frac{3}{2}&\frac{1}{2}&\vdots&5\\0&1&\frac{2}{3}&\vdots&-2\\0&-8&1&\vdots&-22\end{bmatrix}$$

$$\begin{matrix}\\\\8R_2+R_3\rightarrow\end{matrix}\begin{bmatrix}1&\frac{3}{2}&\frac{1}{2}&\vdots&5\\0&1&\frac{2}{3}&\vdots&-2\\0&0&\frac{19}{3}&\vdots&-38\end{bmatrix}$$

$$\begin{matrix}\\\\\frac{3}{19}R_3\rightarrow\end{matrix}\begin{bmatrix}1&\frac{3}{2}&\frac{1}{2}&\vdots&5\\0&1&\frac{2}{3}&\vdots&-2\\0&0&1&\vdots&-6\end{bmatrix}$$

$$z = -6$$

$$y + \tfrac{2}{3}(-6) = -2 \Rightarrow y = 2$$

$$x + \tfrac{3}{2}(2) + \tfrac{1}{2}(-6) = 5 \Rightarrow x = 5$$

Solution: $(5, 2, -6)$

23. $$\begin{cases} x - 2y + z - w = 11 \\ 2x + y - 2z + 3w = -16 \\ -x + 3y + 2z - w = 1 \\ \quad - y + 5z + w = 8 \end{cases}$$

$$\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 2 & 1 & -2 & 3 & \vdots & -16 \\ -1 & 3 & 2 & -1 & \vdots & 1 \\ 0 & -1 & 5 & 1 & \vdots & 8 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ R_1 + R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 5 & -4 & 5 & \vdots & -38 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & -1 & 5 & 1 & \vdots & 8 \end{array}\right]$$

$$\begin{array}{r} \\ R_3 \rightarrow \\ R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & 5 & -4 & 5 & \vdots & -38 \\ 0 & -1 & 5 & 1 & \vdots & 8 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -5R_2 + R_3 \rightarrow \\ R_2 + R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & 0 & -19 & 15 & \vdots & -98 \\ 0 & 0 & 8 & -1 & \vdots & 20 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{19}R_3 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & 0 & 1 & -\frac{15}{19} & \vdots & \frac{98}{19} \\ 0 & 0 & 8 & -1 & \vdots & 20 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ -8R_3 + R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & 0 & 1 & -\frac{15}{19} & \vdots & \frac{98}{19} \\ 0 & 0 & 0 & \frac{101}{19} & \vdots & -\frac{404}{19} \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \frac{19}{101}R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & -2 & 1 & -1 & \vdots & 11 \\ 0 & 1 & 3 & -2 & \vdots & 12 \\ 0 & 0 & 1 & -\frac{15}{19} & \vdots & \frac{98}{19} \\ 0 & 0 & 0 & 1 & \vdots & -4 \end{array}\right]$$

$$\begin{cases} x - 2y + z - w = 11 \\ \quad y + 3z - 2w = 12 \\ \quad\quad z - \frac{15}{19}w = \frac{98}{19} \\ \quad\quad\quad w = -4 \end{cases}$$

$w = -4$

$z - \frac{15}{19}(-4) = \frac{98}{19} \Rightarrow z = 2$

$y + 3(2) - 2(-4) = 12 \Rightarrow y = -2$

$x - 2(-2) + 2 - (-4) \Rightarrow x = 1$

Solution: $(1, -2, 2, -4)$

25. $$\begin{cases} x + 2y - z = 3 \\ x - y - z = -3 \\ 2x + y + 3z = 10 \end{cases}$$

$$\left[\begin{array}{rrrcr} 1 & 2 & -1 & \vdots & 3 \\ 1 & -1 & -1 & \vdots & -3 \\ 2 & 1 & 3 & \vdots & 10 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -2R_2 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & 2 & -1 & \vdots & 3 \\ 0 & -3 & 0 & \vdots & -6 \\ 0 & 3 & 5 & \vdots & 16 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & 2 & -1 & \vdots & 3 \\ 0 & -3 & 0 & \vdots & -6 \\ 0 & 0 & 5 & \vdots & 10 \end{array}\right]$$

$$\begin{array}{r} 3R_1 + 2R_2 \rightarrow \\ \\ \\ \end{array}\left[\begin{array}{rrrcr} 3 & 0 & -3 & \vdots & -3 \\ 0 & -3 & 0 & \vdots & -6 \\ 0 & 0 & 5 & \vdots & 10 \end{array}\right]$$

$$\begin{array}{r} 5R_1 + 3R_3 \rightarrow \\ \\ \\ \end{array}\left[\begin{array}{rrrcr} 15 & 0 & 0 & \vdots & 15 \\ 0 & -3 & 0 & \vdots & -6 \\ 0 & 0 & 5 & \vdots & 10 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{15}R_1 \rightarrow \\ \frac{1}{3}R_2 \rightarrow \\ \frac{1}{5}R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

$x = 1$

$y = 2$

$z = 2$

Solution: $(1, 2, 2)$

27. $$\begin{bmatrix} -1 & 1 & 2 & \vdots & 1 \\ 2 & 3 & 1 & \vdots & -2 \\ 5 & 4 & 2 & \vdots & 4 \end{bmatrix}$$

$$\begin{array}{r} -R_1 \rightarrow \\ \\ \\ \end{array}\begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 2 & 3 & 1 & \vdots & -2 \\ 5 & 4 & 2 & \vdots & 4 \end{bmatrix}$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -5R_1 + R_3 \rightarrow \end{array}\begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 0 & 5 & 5 & \vdots & 0 \\ 0 & 9 & 12 & \vdots & 9 \end{bmatrix}$$

$$\begin{array}{r} \\ \frac{1}{5}R_2 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & -1 & -2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 9 & 12 & \vdots & 9 \end{bmatrix}$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -9R_2 + R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 0 & -1 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 3 & \vdots & 9 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 0 & -1 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ -R_3 + R_2 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

$x = 2, y = -3, z = 3$

Solution: $(2, -3, 3)$

29. Use the reduced row-echelon form feature of a graphing utility.

$$\begin{bmatrix} 3 & -1 & 5 & -2 & \vdots & -44 \\ 1 & 6 & 4 & -1 & \vdots & 1 \\ 5 & -1 & 1 & 3 & \vdots & -15 \\ 0 & 4 & -1 & -8 & \vdots & 58 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 6 \\ 0 & 0 & 1 & 0 & \vdots & -10 \\ 0 & 0 & 0 & 1 & \vdots & -3 \end{bmatrix}$$

$x = 2, y = 6, z = -10, w = -3$

Solution: $(2, 6, -10, -3)$

31. $\begin{bmatrix} -1 & x \\ y & 9 \end{bmatrix} = \begin{bmatrix} -1 & 12 \\ 11 & 9 \end{bmatrix} \Rightarrow x = 12$ and $y = 11$

33. $$\begin{bmatrix} x+3 & -4 & 44 \\ 0 & -3 & 2 \\ -2 & y+5 & 6 \end{bmatrix} = \begin{bmatrix} 5x-1 & -4 & 44 \\ 0 & -3 & 2 \\ -2 & 16 & 6 \end{bmatrix}$$

$$\left.\begin{array}{rcl} x + 3 & = & 5x - 1 \\ 4 & = & 4x \\ y + 5 & = & 16 \\ y & = & 11 \end{array}\right\} x = 1 \text{ and } y = 11$$

35. (a) $A + B = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 15 & 13 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix} = \begin{bmatrix} 5 & -12 \\ -9 & -3 \end{bmatrix}$

(c) $4A = 4\begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ 12 & 20 \end{bmatrix}$

(d) $2A + 2B = 2\begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} + 2\begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 6 & 10 \end{bmatrix} + \begin{bmatrix} -6 & 20 \\ 24 & 16 \end{bmatrix} = \begin{bmatrix} -2 & 16 \\ 30 & 26 \end{bmatrix}$

37. (a) $A + B = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 4 & 12 \\ 20 & 40 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ -3 & 14 \\ 31 & 42 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & 12 \\ 20 & 40 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -11 & -10 \\ -9 & -38 \end{bmatrix}$

(c) $4A = 4\begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 16 \\ -28 & 8 \\ 44 & 8 \end{bmatrix}$

(d) $2A + 2B = 2\begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix} + 2\begin{bmatrix} 0 & 3 \\ 4 & 12 \\ 20 & 40 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ -14 & 4 \\ 22 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ 8 & 24 \\ 40 & 80 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ -6 & 28 \\ 62 & 84 \end{bmatrix}$

39. $\begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 10 & -20 \\ 14 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 7 + 10 + 5 & 3 - 20 + 0 \\ -1 + 14 + 1 & 5 - 3 + 9 \end{bmatrix} = \begin{bmatrix} 22 & -17 \\ 14 & 11 \end{bmatrix}$

41. $-2\left(\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\right) = -2\begin{bmatrix} 8 & 3 \\ 6 & -2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ -12 & 4 \\ -14 & -8 \end{bmatrix}$

43. $X = 2A - 3B = 2\begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ 2 & -10 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ 6 & -3 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 8 & -13 \\ -18 & -8 \end{bmatrix}$

45. $X = \frac{1}{3}[B - 2A] = \frac{1}{3}\left(\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix} - 2\begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 9 & 2 \\ -4 & 11 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} 3 & \frac{2}{3} \\ -\frac{4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{bmatrix}$

47. A and B are both 2×2, so AB exists and has dimensions 2×2.

$$AB = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix} = \begin{bmatrix} 2(-3) + (-2)(12) & 2(10) + (-2)(8) \\ 3(-3) + 5(12) & 3(10) + 5(8) \end{bmatrix} = \begin{bmatrix} -30 & 4 \\ 51 & 70 \end{bmatrix}$$

49. Because A is 3×2 and B is 2×2, AB exists and has dimensions 3×2.

$$AB = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}\begin{bmatrix} 4 & 12 \\ 20 & 40 \end{bmatrix} = \begin{bmatrix} 5(4) + 4(20) & 5(12) + 4(40) \\ -7(4) + 2(20) & -7(12) + 2(40) \\ 11(4) + 2(20) & 11(12) + 2(40) \end{bmatrix} = \begin{bmatrix} 100 & 220 \\ 12 & -4 \\ 84 & 212 \end{bmatrix}$$

51. (a) $AB = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} (1)(5) + (3)(-2) & (1)(-1) + (3)(0) \\ (4)(5) + (1)(-2) & (4)(-1) + (1)(0) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 18 & -4 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} (5)(1) + (-1)(4) & (5)(3) + (-1)(1) \\ (-2)(1) + (0)(4) & (-2)(3) + (0)(1) \end{bmatrix} = \begin{bmatrix} 1 & 14 \\ -2 & -6 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (3)(4) & (1)(3) + (3)(1) \\ (4)(1) + (1)(4) & (4)(3) + (1)(1) \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 8 & 13 \end{bmatrix}$

53. $\begin{bmatrix} 4 & 1 \\ 11 & -7 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 6 \\ 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 4(3) + 1(2) & 4(-5) + 1(-2) & 4(6) + 1(-2) \\ 11(3) - 7(2) & 11(-5) - 7(-2) & 11(6) - 7(-2) \\ 12(3) + 3(2) & 12(-5) + 3(-2) & 12(6) + 3(-2) \end{bmatrix} = \begin{bmatrix} 14 & -22 & 22 \\ 19 & -41 & 80 \\ 42 & -66 & 66 \end{bmatrix}$

55.
$$\begin{aligned} \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 & 1 \\ 4 & -5 & 2 \end{bmatrix} &= \begin{bmatrix} 3(2) + 5(1) & 3(0) + 5(1) \\ 0(2) - 1(1) & 0(0) - 1(1) \end{bmatrix}\begin{bmatrix} -2 & 4 & 1 \\ 4 & -5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 5 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -2 & 4 & 1 \\ 4 & -5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 11(-2) + 5(4) & 11(4) + 5(-5) & 11(1) + 5(2) \\ -1(-2) - 1(4) & -1(4) - 1(-5) & -1(1) - 9(2) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 19 & 21 \\ -2 & 1 & -3 \end{bmatrix} \end{aligned}$$

57. $A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \langle 2, -5 \rangle$ is a reflection in the x-axis.

59. $A\mathbf{v} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \langle 1, 5 \rangle$ is a horizontal shrink.

61. $0.95A = 0.95\begin{bmatrix} 80 & 120 & 140 \\ 40 & 100 & 80 \end{bmatrix} = \begin{bmatrix} 76 & 114 & 133 \\ 38 & 95 & 76 \end{bmatrix}$

63. $AB = \begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} -4(-2) + (-1)(7) & -4(-1) + (-1)(4) \\ 7(-2) + 2(7) & 7(-1) + 2(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$BA = \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} -2(-4) + (-1)(7) & -2(-1) + (-1)(2) \\ 7(-4) + 4(7) & 7(-1) + 4(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

65. $AB = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1(-2) + 1(3) + 0(2) & 1(-3) + 1(3) + 0(4) & 1(1) + 1(-1) + 0(-1) \\ 1(-2) + 0(3) + 1(2) & 1(-3) + 0(3) + 1(4) & 1(1) + 0(-1) + 1(-1) \\ 6(-2) + 2(3) + 3(2) & 6(-3) + 2(3) + 3(4) & 6(1) + 2(-1) + 3(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$BA = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -2(1) + (-3)(1) + 1(6) & -2(1) + (-3)(0) + 1(2) & -2(0) + (-3)(1) + 1(3) \\ 3(1) + 3(1) + (-1)(6) & 3(1) + 3(0) + (-1)(2) & 3(0) + 3(1) + (-1)(3) \\ 2(1) + 4(1) + (-1)(6) & 2(1) + 4(0) + (-1)(2) & 2(0) + 4(1) + (-1)(3) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

67. $$[A \ \vdots\ I] = \begin{bmatrix} -6 & 5 & \vdots & 1 & 0 \\ -5 & 4 & \vdots & 0 & 1 \end{bmatrix}$$

$$-\tfrac{1}{6}R_1 \rightarrow \begin{bmatrix} 1 & -\tfrac{5}{6} & \vdots & -\tfrac{1}{6} & 0 \\ -5 & 4 & \vdots & 0 & 1 \end{bmatrix}$$

$$5R_1 + R_2 \rightarrow \begin{bmatrix} 1 & -\tfrac{5}{6} & \vdots & -\tfrac{1}{6} & 0 \\ 0 & -\tfrac{1}{6} & \vdots & -\tfrac{5}{6} & 1 \end{bmatrix}$$

$$-6R_2 \rightarrow \begin{bmatrix} 1 & -\tfrac{5}{6} & \vdots & -\tfrac{1}{6} & 0 \\ 0 & 1 & \vdots & 5 & -6 \end{bmatrix}$$

$$\tfrac{5}{6}R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \vdots & 4 & -5 \\ 0 & 1 & \vdots & 5 & -6 \end{bmatrix} = [I \ \vdots\ A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$$

69. $$[A \ \vdots\ I] = \begin{bmatrix} 2 & 0 & 3 & \vdots & 1 & 0 & 0 \\ -1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 2 & -2 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 2R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 2 & 0 & 3 & \vdots & 1 & 0 & 0 \\ -1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 3 & \vdots & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{r} -R_3 + R_1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 2 & 0 & 0 & \vdots & 1 & -2 & -1 \\ -1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 3 & \vdots & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{r} \tfrac{1}{2}R_1 \rightarrow \\ \\ \tfrac{1}{3}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & \tfrac{1}{2} & -1 & -\tfrac{1}{2} \\ -1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$$

$$R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \tfrac{1}{2} & -1 & -\tfrac{1}{2} \\ 0 & 1 & 1 & \vdots & \tfrac{1}{2} & 0 & -\tfrac{1}{2} \\ 0 & 0 & 1 & \vdots & 0 & \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$$

$$-R_3 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \tfrac{1}{2} & -1 & -\tfrac{1}{2} \\ 0 & 1 & 0 & \vdots & \tfrac{1}{2} & -\tfrac{2}{3} & -\tfrac{5}{6} \\ 0 & 0 & 1 & \vdots & 0 & \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix} = [I \ \vdots\ A^{-1}]$$

$$A^{-1} = \begin{bmatrix} \tfrac{1}{2} & -1 & -\tfrac{1}{2} \\ \tfrac{1}{2} & -\tfrac{2}{3} & -\tfrac{5}{6} \\ 0 & \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$$

71. $$\begin{bmatrix} -1 & -2 & -2 \\ 3 & 7 & 9 \\ 1 & 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 13 & 6 & -4 \\ -12 & -5 & 3 \\ 5 & 2 & -1 \end{bmatrix}$$

73. $$A = \begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-7(2) - 2(-8)}\begin{bmatrix} 2 & -2 \\ 8 & -7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 8 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & -\dfrac{7}{2} \end{bmatrix}$$

75. $A = \begin{bmatrix} -12 & 6 \\ 10 & -5 \end{bmatrix}$

$ad - bc = (-12)(-5) - (6)(10) = 0$

A^{-1} does not exist.

77. $\begin{cases} -x + 4y = 8 \\ 2x - 7y = -5 \end{cases}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 7(8) + 4(-5) \\ 2(8) + 1(-5) \end{bmatrix} = \begin{bmatrix} 36 \\ 11 \end{bmatrix}$$

Solution: $(36, 11)$

79. $\begin{cases} -3x + 10y = 8 \\ 5x - 17y = -13 \end{cases}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ 5 & -17 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -13 \end{bmatrix} = \begin{bmatrix} -17 & -10 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} -17(8) + (-10)(-13) \\ -5(8) + (-3)(-13) \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$$

Solution: $(-6, -1)$

81. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ -6x - 4y = 0 \end{cases}$

Inverse of $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ -6 & -4 \end{bmatrix}$ does not exist because

$\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ -6 & -4 \end{vmatrix} = 0.$

System has no solution.

83. $\begin{cases} 0.3x + 0.7y = 10.2 \\ 0.4x + 0.6y = 7.6 \end{cases}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 10.2 \\ 7.6 \end{bmatrix} = \begin{bmatrix} -6 & 7 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 10.2 \\ 7.6 \end{bmatrix} = \begin{bmatrix} -8 \\ 18 \end{bmatrix}$$

Solution: $(-8, 18)$

85. $\begin{cases} 3x + 2y - z = 6 \\ x - y + 2z = -1 \\ 5x + y + z = 7 \end{cases}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 3 & \frac{8}{3} & -\frac{7}{3} \\ 2 & \frac{7}{3} & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1(6) - 1(-1) + 1(7) \\ 3(6) + \frac{8}{3}(-1) - \frac{7}{3}(7) \\ 2(6) + \frac{7}{3}(-1) - \frac{5}{3}(7) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Solution: $(2, -1, -2)$

87. $\begin{cases} 5x + 10y = 7 \\ 2x + y = -98 \end{cases}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -98 \end{bmatrix} = \begin{bmatrix} -\frac{1}{15} & \frac{2}{3} \\ \frac{2}{15} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 \\ -98 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{329}{5} \\ \frac{168}{5} \end{bmatrix} = \begin{bmatrix} -65.8 \\ 33.6 \end{bmatrix}$$

Solution: $(-65.8, 33.6)$

89. $A = \begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}: \begin{vmatrix} 2 & 5 \\ -4 & 3 \end{vmatrix} = (2)(3) - (-4)(5) = 26$

91. $A = \begin{bmatrix} 10 & -2 \\ 18 & 8 \end{bmatrix}: \begin{vmatrix} 10 & -2 \\ 18 & 8 \end{vmatrix} = (10)(8) - (18)(-2)$

$= 116$

93. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

(a) $M_{11} = 4$
$M_{12} = 7$
$M_{21} = -1$
$M_{22} = 2$

(b) $C_{11} = M_{11} = 4$
$C_{12} = -M_{12} = -7$
$C_{21} = -M_{21} = 1$
$C_{22} = M_{22} = 2$

95. $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$

(a)

$$M_{11} = \begin{vmatrix} 5 & 0 \\ 8 & 6 \end{vmatrix} = 30$$

$$M_{12} = \begin{vmatrix} -2 & 0 \\ 1 & 6 \end{vmatrix} = -12$$

$$M_{13} = \begin{vmatrix} -2 & 5 \\ 1 & 8 \end{vmatrix} = -21$$

$$M_{21} = \begin{vmatrix} 2 & -1 \\ 8 & 6 \end{vmatrix} = 20$$

$$M_{22} = \begin{vmatrix} 3 & -1 \\ 1 & 6 \end{vmatrix} = 19$$

$$M_{23} = \begin{vmatrix} 3 & 2 \\ 1 & 8 \end{vmatrix} = 22$$

$$M_{31} = \begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = 5$$

$$M_{32} = \begin{vmatrix} 3 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$M_{33} = \begin{vmatrix} 3 & 2 \\ -2 & 5 \end{vmatrix} = 19$$

(b)

$C_{11} = M_{11} = 30$
$C_{12} = -M_{12} = 12$
$C_{13} = M_{13} = -21$
$C_{21} = -M_{21} = -20$
$C_{22} = M_{22} = 19$
$C_{23} = -M_{23} = -22$
$C_{31} = M_{31} = 5$
$C_{32} = -M_{32} = 2$
$C_{33} = M_{33} = 19$

97. Expand using Row 1.

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & -3 \end{vmatrix} = -2\begin{vmatrix} -1 & 0 \\ 1 & -3 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -2(3) - 0(6) + 0(1)$$

$$= -6$$

99. Expand using Row 3.

$$\begin{vmatrix} 4 & 1 & -1 \\ 2 & 3 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(5) + 1(10) + 0(10)$$

$$= 15$$

101. Expand using Column 2.

$$\begin{vmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{vmatrix} = -4\begin{vmatrix} -6 & 2 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ -6 & 2 \end{vmatrix}$$

$$= -4(-34) - 3(2) = 130$$

103. $\begin{cases} 5x - 2y = 6 \\ -11x + 3y = -23 \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & -2 \\ -23 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -11 & 3 \end{vmatrix}} = \frac{-28}{-7} = 4,$$

$$y = \frac{\begin{vmatrix} 5 & 6 \\ -11 & -23 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -11 & 3 \end{vmatrix}} = \frac{-49}{-7} = 7$$

Solution: $(4, 7)$

105. $$\begin{cases} -2x + 3y - 5z = -11 \\ 4x - y + z = -3 \\ -x - 4y + 6z = 15 \end{cases}$$

$$D = \begin{vmatrix} -2 & 3 & -5 \\ 4 & -1 & 1 \\ -1 & -4 & 6 \end{vmatrix} = -2(-1)^2\begin{vmatrix} -1 & 1 \\ -4 & 6 \end{vmatrix} + 4(-1)^3\begin{vmatrix} 3 & -5 \\ -4 & 6 \end{vmatrix} - 1(-1)^4\begin{vmatrix} 3 & -5 \\ -1 & 1 \end{vmatrix} = -2(-2) - 4(-2) - (-2) = 14$$

$$x = \frac{\begin{vmatrix} -11 & 3 & -5 \\ -3 & -1 & 1 \\ 15 & -4 & 6 \end{vmatrix}}{14} = \frac{-11(-1)^2\begin{vmatrix} -1 & 1 \\ -4 & 6 \end{vmatrix} - 3(-1)^3\begin{vmatrix} 3 & -5 \\ -4 & 6 \end{vmatrix} + 15(-1)^4\begin{vmatrix} 3 & -5 \\ -1 & 1 \end{vmatrix}}{14} = \frac{-11(-2) + 3(-2) + 15(-2)}{14} = \frac{-14}{14} = -1$$

$$y = \frac{\begin{vmatrix} -2 & -11 & -5 \\ 4 & -3 & 1 \\ -1 & 15 & 6 \end{vmatrix}}{14} = \frac{-2(-1)^2\begin{vmatrix} -3 & 1 \\ 15 & 6 \end{vmatrix} + 4(-1)^3\begin{vmatrix} -11 & -5 \\ 15 & 6 \end{vmatrix} - 1(-1)^4\begin{vmatrix} -11 & -5 \\ -3 & 1 \end{vmatrix}}{14} = \frac{-2(-33) - 4(9) - 1(-26)}{14} = \frac{56}{14} = 4$$

$$z = \frac{\begin{vmatrix} -2 & 3 & -11 \\ 4 & -1 & -3 \\ -1 & -4 & 15 \end{vmatrix}}{14} = \frac{-2(-1)^2\begin{vmatrix} -1 & -3 \\ -4 & 15 \end{vmatrix} + 4(-1)^3\begin{vmatrix} 3 & -11 \\ -4 & 15 \end{vmatrix} - 1(-1)^4\begin{vmatrix} 3 & -11 \\ -1 & -3 \end{vmatrix}}{14} = \frac{-2(-27) - 4(1) - 1(-20)}{14} = \frac{70}{14} = 5$$

Solution: $(-1, 4, 5)$

107. $(1, 0), (5, 0), (5, 8)$

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 8 & 1 \end{vmatrix} = \tfrac{1}{2}\left(1\begin{vmatrix} 0 & 1 \\ 8 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 0 \\ 5 & 8 \end{vmatrix}\right) = \tfrac{1}{2}(-8 + 40) = \tfrac{1}{2}(32) = 16 \text{ square units}$$

109. $(-1, 7), (3, -9), (-3, 15)$

$$\begin{vmatrix} -1 & 7 & 1 \\ 3 & -9 & 1 \\ -3 & 15 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -9 \\ -3 & 15 \end{vmatrix} - \begin{vmatrix} -1 & 7 \\ -3 & 15 \end{vmatrix} + \begin{vmatrix} -1 & 7 \\ 3 & -9 \end{vmatrix}$$

$$= 18 - 6 - 12 = 0$$

The points are collinear.

111. $(-4, 0), (4, 4)$

$$\begin{vmatrix} x & y & 1 \\ -4 & 0 & 1 \\ 4 & 4 & 1 \end{vmatrix} = 0$$

$$1\begin{vmatrix} -4 & 0 \\ 4 & 4 \end{vmatrix} - 1\begin{vmatrix} x & y \\ 4 & 4 \end{vmatrix} + 1\begin{vmatrix} x & y \\ -4 & 0 \end{vmatrix} = 0$$

$$-16 - (4x - 4y) + 4y = 0$$

$$-4x + 8y - 16 = 0$$

$$x - 2y + 4 = 0$$

113. $\left(-\frac{5}{2}, 3\right), \left(\frac{7}{2}, 1\right)$

$$\begin{vmatrix} x & y & 1 \\ -\frac{5}{2} & 3 & 1 \\ \frac{7}{2} & 1 & 1 \end{vmatrix} = 0$$

$$1\begin{vmatrix} -\frac{5}{2} & 3 \\ \frac{7}{2} & 1 \end{vmatrix} - 1\begin{vmatrix} x & y \\ \frac{7}{2} & 1 \end{vmatrix} + 1\begin{vmatrix} x & y \\ -\frac{5}{2} & 3 \end{vmatrix} = 0$$

$$-13 - \left(x - \tfrac{7}{2}y\right) + \left(3x + \tfrac{5}{2}y\right) = 0$$

$$2x + 6y - 13 = 0$$

115. The area of the parallelogram with vertices: $(0, 0), (2, 0), (1, 4)$ and $(3, 4) \Rightarrow a = 2, b = 0, c = 1$ and $d = 4$.

$$A = \left\|\begin{matrix} 2 & 0 \\ 1 & 4 \end{matrix}\right\| = |8 - 0| = 8 \text{ square units.}$$

117. $A^{-1} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

$$\begin{bmatrix} -5 & 11 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 19 & 5 & 5 \end{bmatrix} \quad \text{S E E}$$

$$\begin{bmatrix} 370 & -265 & 225 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 25 & 15 \end{bmatrix} \quad \text{\_ Y O}$$

$$\begin{bmatrix} -57 & 48 & -33 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 6 \end{bmatrix} \quad \text{U \_ F}$$

$$\begin{bmatrix} 32 & -15 & 20 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 9 & 4 \end{bmatrix} \quad \text{R I D}$$

$$\begin{bmatrix} 245 & -171 & 147 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix} \quad \text{A Y \_}$$

Message: SEE YOU FRIDAY

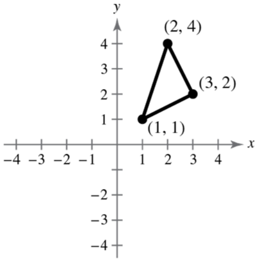
119. False. The matrix must be square.

121. If A is a square matrix, the cofactor C_{ij} of the entry a_{ij} is $(-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting the ith row and jth column of A. The determinant of A is the sum of the entries of any row or column of A multiplied by their respective cofactors.

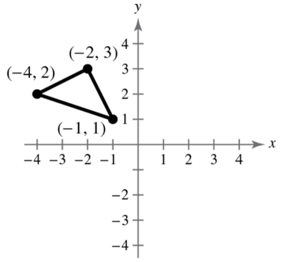
Problem Solving for Chapter 8

1. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}$

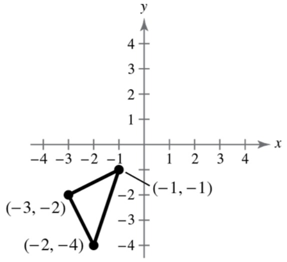
(a) $AT = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 3 \end{bmatrix} \quad AAT = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -4 & -2 \end{bmatrix}$



Original Triangle



AT Triangle



AAT Triangle

The transformation A interchanges the x and y coordinates and then takes the negative of the x coordinate. A represents a counterclockwise rotation by 90°.

(b) AAT is rotated clockwise 90° to obtain AT. AT is then rotated clockwise 90° to obtain T.

3. (a) $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$

A *is* idempotent.

(b) $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A$

A is *not* idempotent.

(c) $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A$

A is *not* idempotent.

(d) $A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} \neq A$

A is *not* idempotent.

5. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(a)
$$\begin{aligned} A^2 - 2A + 5I &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

(b)
$$A^{-1} = \frac{1}{(1) - (-4)}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\frac{1}{5}(2I - A) = \frac{1}{5}\left[\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right] = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

So, $A^{-1} = \frac{1}{5}(2I - A)$.

(c)
$$\begin{aligned} A^2 - 2A + 5I &= 0 \\ A^2 - 2A &= -5I \\ (A - 2I)A &= -5I \\ -\frac{1}{5}(A - 2I)A &= I \\ \frac{1}{5}(2I - A)A &= I \end{aligned}$$

So, $A^{-1} = \frac{1}{5}(2I - A)$.

7. $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

$A^T = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$, $\quad B^T = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 4 \\ -5 & -1 \end{bmatrix}$, $\quad (AB)^T = \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$

So, $(AB)^T = B^T A^T$.

9. If $A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$ is singular then

$ad - bc = -12 + 2x = 0$.

So, $x = 6$.

11. $(a - b)(b - c)(c - a)(a + b + c) = -a^3b + a^3c + ab^3 - ac^3 - b^3c + bc^3$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} b & c \\ b^3 & c^3 \end{vmatrix} - \begin{vmatrix} a & c \\ a^3 & c^3 \end{vmatrix} + \begin{vmatrix} a & b \\ a^3 & b^3 \end{vmatrix} = bc^3 - b^3c - ac^3 + a^3c + ab^3 - a^3b$$

So, $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$.

13. $$\begin{vmatrix} x & 0 & 0 & d \\ -1 & x & 0 & c \\ 0 & -1 & x & b \\ 0 & 0 & -1 & a \end{vmatrix} = x\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} - d\begin{vmatrix} -1 & x & 0 \\ 0 & -1 & x \\ 0 & 0 & -1 \end{vmatrix} = \underbrace{x\left(ax^2 + bx + c\right)}_{\text{From Exercise 12}} - d\left(-\begin{vmatrix} -1 & x \\ 0 & -1 \end{vmatrix}\right) = ax^3 + bx^2 + cx + d$$

15. $$\begin{array}{rcrcrcr} 4S & + & 4N & & & = & 184 \\ S & & & + & 6F & = & 146 \\ & & 2N & + & 4F & = & 104 \end{array}$$

$$D = \begin{vmatrix} 4 & 4 & 0 \\ 1 & 0 & 6 \\ 0 & 2 & 4 \end{vmatrix} = -64$$

$$S = \frac{\begin{vmatrix} 184 & 4 & 0 \\ 146 & 0 & 6 \\ 104 & 2 & 4 \end{vmatrix}}{-64} = \frac{-2048}{-64} = 32 \qquad N = \frac{\begin{vmatrix} 4 & 184 & 0 \\ 1 & 146 & 6 \\ 0 & 104 & 4 \end{vmatrix}}{-64} = \frac{-896}{-64} = 14 \qquad F = \frac{\begin{vmatrix} 4 & 4 & 184 \\ 1 & 0 & 146 \\ 0 & 2 & 104 \end{vmatrix}}{-64} = \frac{-1216}{-64} = 19$$

Element	*Atomic mass*
Sulfur	32
Nitrogen	14
Fluoride	19

17. $$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 1 & -3 \\ 1 & -1 & 4 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{11} & \frac{6}{11} & \frac{4}{11} \\ -\frac{7}{11} & \frac{2}{11} & \frac{5}{11} \\ -\frac{2}{11} & -\frac{1}{11} & \frac{3}{11} \end{bmatrix}$$

$$\begin{bmatrix} 23 & 13 & -34 \\ 31 & -34 & 63 \\ 25 & -17 & 61 \\ 24 & 14 & -37 \\ 41 & -17 & -8 \\ 20 & -29 & 40 \\ 38 & -56 & 116 \\ 13 & -11 & 1 \\ 22 & -3 & -6 \\ 41 & -53 & 85 \\ 28 & -32 & 16 \end{bmatrix} \begin{bmatrix} \frac{1}{11} & \frac{6}{11} & \frac{4}{11} \\ -\frac{7}{11} & \frac{2}{11} & \frac{5}{11} \\ -\frac{2}{11} & -\frac{1}{11} & \frac{3}{11} \end{bmatrix} \begin{bmatrix} 0 & 18 & 5 \\ 13 & 5 & 13 \\ 2 & 5 & 18 \\ 0 & 19 & 5 \\ 16 & 20 & 5 \\ 13 & 2 & 5 \\ 18 & 0 & 20 \\ 8 & 5 & 0 \\ 5 & 12 & 5 \\ 22 & 5 & 14 \\ 20 & 8 & 0 \end{bmatrix}$$

0 18 5 13 5 13 2 5 18 0
_ R E M E M B E R _

19 5 16 20 5 13 2 5 18 0
S E P T E M B E R _

20 8 5 0 5 12 5 22 5 14 20 8 0
T H E _ E L E V E N T H _

Message: REMEMBER SEPTEMBER THE ELEVENTH

19. $$A = \begin{bmatrix} 6 & 4 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{16} & -\frac{7}{16} & \frac{5}{8} \\ \frac{3}{16} & \frac{11}{16} & -\frac{9}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{3}{4} \end{bmatrix}$$

$|A| = 16$ and $\left|A^{-1}\right| = \dfrac{1}{16}$

Conjecture: $\left|A^{-1}\right| = \dfrac{1}{|A|}$

Practice Test for Chapter 8

1. Put the matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 9 \end{bmatrix}$$

For Exercises 2–4, use matrices to solve the system of equations.

2. $\begin{cases} 3x + 5y = 3 \\ 2x - y = -11 \end{cases}$

3. $\begin{cases} 2x + 3y = -3 \\ 3x + 2y = 8 \\ x + y = 1 \end{cases}$

4. $\begin{cases} x + 3z = -5 \\ 2x + y = 0 \\ 3x + y - z = 3 \end{cases}$

5. Multiply $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & -7 \\ -1 & 2 \end{bmatrix}$.

6. Given $A = \begin{bmatrix} 9 & 1 \\ -4 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 \\ 3 & 5 \end{bmatrix}$, find $3A - 5B$.

7. Find $f(A)$.

$f(x) = x^2 - 7x + 8$, $A = \begin{bmatrix} 3 & 0 \\ 7 & 1 \end{bmatrix}$

8. True or false:

$(A + B)(A + 3B) = A^2 + 4AB + 3B^2$ where A and B are matrices.

(Assume that A^2, AB, and B^2 exist.)

For Exercises 9–10, find the inverse of the matrix, if it exists.

9. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 6 & 5 \\ 6 & 10 & 8 \end{bmatrix}$

11. Use an inverse matrix to solve the systems.

(a) $\begin{cases} x + 2y = 4 \\ 3x + 5y = 1 \end{cases}$

(b) $\begin{cases} x + 2y = 3 \\ 3x + 5y = -2 \end{cases}$

For Exercises 12–14, find the determinant of the matrix.

12. $\begin{bmatrix} 6 & -1 \\ 3 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & -1 \\ 5 & 9 & 0 \\ 6 & 2 & -5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 3 & 5 & -1 & 1 \\ 2 & 0 & 6 & 1 \end{bmatrix}$

15. Evaluate $\begin{vmatrix} 6 & 4 & 3 & 0 & 6 \\ 0 & 5 & 1 & 4 & 8 \\ 0 & 0 & 2 & 7 & 3 \\ 0 & 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$.

16. Use a determinant to find the area of the triangle with vertices $(0, 7), (5, 0),$ and $(3, 9)$.

17. Use a determinant to find the equation of the line passing through $(2, 7)$ and $(-1, 4)$.

For Exercises 18–20, use Cramer's Rule to find the indicated value.

18. Find x.

$$\begin{cases} 6x - 7y = 4 \\ 2x + 5y = 11 \end{cases}$$

19. Find z.

$$\begin{cases} 3x + z = 1 \\ y + 4z = 3 \\ x - y = 2 \end{cases}$$

20. Find y.

$$\begin{cases} 721.4x - 29.1y = 33.77 \\ 45.9x + 105.6y = 19.85 \end{cases}$$

CHAPTER 9

Sequences, Series, and Probability

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C H A P T E R 9
Sequences, Series, and Probability

Section 9.1 Sequences and Series

1. recursively

3. series

5. Since $n! = 1 \cdot 2 \cdot 3 \cdots (n-1)(n)$, then
$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6!$.

7.
$$\begin{aligned} a_n &= 4n - 7 \\ a_1 &= 4(1) - 7 = -3 \\ a_2 &= 4(2) - 7 = 1 \\ a_3 &= 4(3) - 7 = 5 \\ a_4 &= 4(4) - 7 = 9 \\ a_5 &= 4(5) - 7 = 13 \end{aligned}$$

9.
$$\begin{aligned} a_n &= (-1)^{n+1} + 4 \\ a_1 &= (-1)^{1+1} + 4 = 5 \\ a_2 &= (-1)^{2+1} + 4 = 3 \\ a_3 &= (-1)^{3+1} + 4 = 5 \\ a_4 &= (-1)^{4+1} + 4 = 3 \\ a_5 &= (-1)^{5+1} + 4 = 5 \end{aligned}$$

11.
$$\begin{aligned} a_n &= \tfrac{2}{3} \\ a_1 &= \tfrac{2}{3} \\ a_2 &= \tfrac{2}{3} \\ a_3 &= \tfrac{2}{3} \\ a_4 &= \tfrac{2}{3} \\ a_5 &= \tfrac{2}{3} \end{aligned}$$

13.
$$\begin{aligned} a_n &= (-2)^n \\ a_1 &= (-2)^1 = -2 \\ a_2 &= (-2)^2 = 4 \\ a_3 &= (-2)^3 = -8 \\ a_4 &= (-2)^4 = 16 \\ a_5 &= (-2)^5 = -32 \end{aligned}$$

15.
$$\begin{aligned} a_n &= \frac{1}{3}n^3 \\ a_1 &= \frac{1}{3}(1)^3 = \frac{1}{3} \\ a_2 &= \frac{1}{3}(2)^3 = \frac{8}{3} \\ a_3 &= \frac{1}{3}(3)^3 = 9 \\ a_4 &= \frac{1}{3}(4)^3 = \frac{64}{3} \\ a_5 &= \frac{1}{3}(5)^3 = \frac{125}{3} \end{aligned}$$

17.
$$\begin{aligned} a_n &= \frac{n}{n+2} \\ a_1 &= \frac{1}{1+2} = \frac{1}{3} \\ a_2 &= \frac{2}{2+2} = \frac{1}{2} \\ a_3 &= \frac{3}{3+2} = \frac{3}{5} \\ a_4 &= \frac{4}{4+2} = \frac{2}{3} \\ a_5 &= \frac{5}{5+2} = \frac{5}{7} \end{aligned}$$

19.
$$\begin{aligned} a_n &= n(n-1)(n-2) \\ a_1 &= (1)(0)(-1) = 0 \\ a_2 &= (2)(1)(0) = 0 \\ a_3 &= (3)(2)(1) = 6 \\ a_4 &= (4)(3)(2) = 24 \\ a_5 &= (5)(4)(3) = 60 \end{aligned}$$

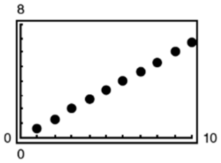
21. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

$$a_1 = (-1)^1\frac{1}{1+1} = -\frac{1}{2}$$
$$a_2 = (-1)^2\frac{2}{2+1} = \frac{2}{3}$$
$$a_3 = (-1)^3\frac{3}{3+1} = -\frac{3}{4}$$
$$a_4 = (-1)^4\frac{4}{4+1} = \frac{4}{5}$$
$$a_5 = (-1)^5\frac{5}{5+1} = -\frac{5}{6}$$

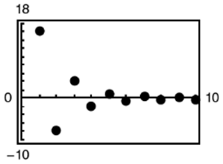
23. $a_{25} = (-1)^{25}(3(25) - 2) = -73$

25. $a_{11} = \dfrac{4(11)}{2(11)^2 - 3} = \dfrac{44}{239}$

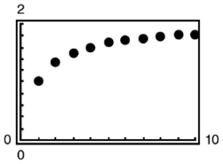
27. $a_n = \frac{2}{3}n$



29. $a_n = 16(-0.5)^{n-1}$



31. $a_n = \dfrac{2n}{n+1}$



33. 3, 7, 11, 15, 19, …

n:	1	2	3	4	5	…	n
Terms:	3	7	11	15	19	…	a_n

Apparent pattern:

Each term is one less than four times n, which implies that $a_n = 4n - 1$.

35. 3, 10, 29, 66, 127, …

n:	1	2	3	4	5	…	n
Terms:	3	10	29	66	127	…	a_n

Apparent pattern:

Each term is more than n cubed, which implies that $a_n = n^3 + 2$.

37. 1, −1, 1, −1, 1, …

n:	1	2	3	4	5	…	n
Terms:	1	−1	1	−1	1	…	a_n

Apparent pattern:

Each term is either 1 or −1 which implies that $a_n = (-1)^{n+1}$.

39. $-\dfrac{2}{3}, \dfrac{3}{4}, -\dfrac{4}{5}, \dfrac{5}{6}, -\dfrac{6}{7}, \ldots$

$$a_n = (-1)^n\left(\frac{n+1}{n+2}\right)$$

41. $\dfrac{2}{1}, \dfrac{3}{3}, \dfrac{4}{5}, \dfrac{5}{7}, \dfrac{6}{9}, \ldots$

$$a_n = \frac{n+1}{2n-1}$$

43. $1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

n:	1	2	3	4	5	…	n
Terms:	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	…	a_n

Apparent pattern:

Each term is the reciprocal of $n!$, which implies that $a_n = \dfrac{1}{n!}$.

45. $\dfrac{1}{1}, \dfrac{3}{1}, \dfrac{9}{2}, \dfrac{27}{6}, \dfrac{81}{24}, \ldots$

n:	1	2	3	4	…	n
Terms:	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{2}$	$\frac{27}{6}$	…	a_n

Apparent pattern:

Each term is a power of three, 3^{n-1} divided by a factorial, $(n-1)!$. After trial and error,

$$a_n = \frac{3^{n-1}}{(n-1)!}.$$

47. $a_1 = 28$ and $a_{k+1} = a_k - 4$

$a_1 = 28$

$a_2 = a_1 - 4 = 28 - 4 = 24$

$a_3 = a_2 - 4 = 24 - 4 = 20$

$a_4 = a_3 - 4 = 20 - 4 = 16$

$a_5 = a_4 - 4 = 16 - 4 = 12$

49. $a_1 = 81$ and $a_{k+1} = \frac{1}{3}a_k$

$a_1 = 81$

$a_2 = \frac{1}{3}a_1 = \frac{1}{3}(81) = 27$

$a_3 = \frac{1}{3}a_2 = \frac{1}{3}(27) = 9$

$a_4 = \frac{1}{3}a_3 = \frac{1}{3}(9) = 3$

$a_5 = \frac{1}{3}a_4 = \frac{1}{3}(3) = 1$

51. $a_0 = 1, a_1 = 2, a_k = a_{k-2} + \frac{1}{2}a_{k-1}$

$a_0 = 1$

$a_1 = 2$

$a_2 = a_0 + \frac{1}{2}a_1 = 1 + \frac{1}{2}(1) = 2$

$a_3 = a_1 + \frac{1}{2}a_2 = 2 + \frac{1}{2}(2) = 3$

$a_4 = a_2 + \frac{1}{2}a_3 = 2 + \frac{1}{2}(3) = \frac{7}{2}$

53. $a_1 = 1, a_2 = 1, a_k = a_{k-1} + a_{k-2}, k \geq 1$

$a_1 = 1$

$a_2 = 1$

$a_3 = 1 + 1 = 2$

$a_4 = 2 + 1 = 3$

$a_5 = 3 + 2 = 5$

$a_6 = 5 + 3 = 8$

$a_7 = 8 + 5 = 13$

$a_8 = 13 + 8 = 21$

$a_9 = 21 + 13 = 34$

$a_{10} = 34 + 21 = 55$

$a_{11} = 55 + 34 = 89$

$a_{12} = 89 + 55 = 144$

$b_1 = \frac{1}{1} = 1$

$b_2 = \frac{2}{1} = 2$

$b_3 = \frac{3}{2}$

$b_4 = \frac{5}{3}$

$b_5 = \frac{8}{5}$

$b_6 = \frac{13}{8}$

$b_7 = \frac{21}{13}$

$b_8 = \frac{34}{21}$

$b_9 = \frac{55}{34}$

$b_{10} = \frac{89}{55}$

55. $a_n = \frac{5}{n!}$

$a_0 = \frac{5}{0!} = \frac{5}{1} = 5$

$a_1 = \frac{5}{1!} = \frac{5}{1} = 5$

$a_2 = \frac{5}{2!} = \frac{5}{2}$

$a_3 = \frac{5}{3!} = \frac{5}{6}$

$a_4 = \frac{5}{4!} = \frac{5}{24}$

57. $a_n = \frac{(-1)^n(n+3)!}{n!}$

$a_0 = \frac{(-1)^0(0+3)!}{0!} = \frac{3!}{0!} = 6$

$a_1 = \frac{(-1)^1(1+3)!}{1!} = \frac{4!}{1!} = -24$

$a_2 = \frac{(-1)^2(2+3)!}{2!} = \frac{5!}{2!} = 60$

$a_3 = \frac{(-1)^3(3+3)!}{3!} = -\frac{6!}{3!} = -120$

$a_4 = \frac{(-1)^4(4+3)!}{4!} = \frac{7!}{4!} = 210$

59. $\frac{4!}{6!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4}}{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5 \cdot 6} = \frac{1}{5 \cdot 6} = \frac{1}{30}$

61. $\frac{(n+1)!}{n!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdots n} \cdot (n+1)}{\cancel{1 \cdot 2 \cdot 3 \cdots n}} = \frac{n+1}{1}$

$= n + 1$

63. $\sum_{i=1}^{5} (2i - 1) = (2 - 1) + (4 - 1) + (6 - 1) + (8 - 1) + (10 - 1) = 25$

65. $\sum_{j=3}^{5} \frac{1}{j^2 - 3} = \frac{1}{3^2 - 3} + \frac{1}{4^2 - 3} + \frac{1}{5^2 - 3} = \frac{124}{429}$

67. $\sum_{k=2}^{5} (k + 1)^2(k - 3) = (3)^2(-1) + (4)^2(0) + (5)^2(1) + (6)^2(2) = 88$

69. $\sum_{i=1}^{4} \frac{i!}{2^i} = \frac{1!}{2^1} + \frac{2!}{2^2} + \frac{3!}{2^3} + \frac{4!}{2^4} = \frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} + \frac{3}{2} = \frac{13}{4}$

71. $\sum_{k=0}^{4} \frac{(-1)^k}{k!} = \frac{3}{8}$

73. $\sum_{n=0}^{25} \frac{1}{4^n} \approx 1.33$

75. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(9)} = \sum_{i=1}^{9} \frac{1}{3i}$

77. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \left[2\left(\frac{3}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right] = \sum_{i=1}^{8}\left[2\left(\frac{i}{8}\right) + 3\right]$

79. $3 - 9 + 27 - 81 + 243 - 729 = \sum_{i=1}^{6} (-1)^{i+1}3i$

81. $\frac{1^2}{2} + \frac{2^2}{6} + \frac{3^2}{24} + \frac{4^2}{120} + \frac{5^2}{720} + \frac{6^2}{5040} + \frac{7^2}{40{,}320} = \sum_{i=1}^{7} \frac{i^2}{(i + 1)!}$

83. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64} = \sum_{i=1}^{5} \frac{2^i - 1}{2^{i+1}}$

85. (a) $\sum_{i=1}^{3} \left(\frac{1}{2}\right)^i = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{7}{8}$

(b) $\sum_{i=1}^{4} \left(\frac{1}{2}\right)^i = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{15}{16}$

(c) $\sum_{i=1}^{5} \left(\frac{1}{2}\right)^i = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 = \frac{31}{32}$

87. (a) $\sum_{n=1}^{3} 4\left(-\frac{1}{2}\right)^n = 4\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right)^3 = -\frac{3}{2}$

(b) $\sum_{n=1}^{4} 4\left(-\frac{1}{2}\right)^n = 4\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^4 = -\frac{5}{4}$

(c) $\sum_{n=1}^{5} 4\left(-\frac{1}{2}\right)^n = 4\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^4 + 4\left(-\frac{1}{2}\right)^5 = -\frac{11}{8}$

89. $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i = 0.6 + 0.06 + 0.006 + 0.0006 + \cdots = \frac{2}{3}$

91. By using a calculator,

$$\sum_{k=1}^{10} 7\left(\frac{1}{10}\right)^k \approx 0.7777777777$$

$$\sum_{k=1}^{50} 7\left(\frac{1}{10}\right)^k \approx 0.7777777778$$

$$\sum_{k=1}^{100} 7\left(\frac{1}{10}\right)^k \approx \frac{7}{9}.$$

The terms approach zero as $n \to \infty$.

So, $\sum_{k=1}^{\infty} 7\left(\frac{1}{10}\right)^k = \frac{7}{9}.$

93. (a) $A_1 = \$10{,}087.50$

$A_2 \approx \$10{,}175.77$

$A_3 \approx \$10{,}264.80$

$A_4 \approx \$10{,}354.62$

$A_5 \approx \$10{,}445.22$

$A_6 \approx \$10{,}536.62$

$A_7 \approx \$10{,}628.81$

$A_8 \approx \$10{,}721.82$

(b) $A_{40} \approx \$14{,}169.09$

(c) No; The balance after 20 years, $A_{80} = \$20{,}076.31$ is not twice the balance after 10 years, $A_{40} \approx \$14{,}169.09$.

95. True, $\sum_{i=1}^{4} \left(i^2 + 2i\right) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$ by the Properties of Sums.

97. $\dfrac{327.15 + 785.69 + 433.04 + 265.38 + 604.12 + 590.30}{6} \approx \$500.95$

99.
$$\begin{aligned}\sum_{i=1}^{n} \left(x_i - \overline{x}\right)^2 &= \sum_{i=1}^{n} \left(x_i^2 - 2x_i\overline{x} + \overline{x}^2\right) = \sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + n\overline{x}^2\\ &= \sum_{i=1}^{n} x_i^2 - 2 \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i + n \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\\ &= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i\left(-\frac{2}{n} + \frac{1}{n}\right) = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\end{aligned}$$

101. The error was not summing the constant, $\sum_{k=1}^{4} 3 = 12.$

$$\begin{aligned}\sum_{k=1}^{4} \left(3 + 2k^2\right) &= \sum_{k=1}^{4} 3 + \sum_{k=1}^{4} 2k^2 = (3 + 3 + 3 + 3) + \left(2(1)^2 + 2(2)^2 + 2(2)^3 + 2(4)^2\right)\\ &= (12) + (2 + 8 + 18 + 32) = 75\end{aligned}$$

103. $b_n = \dfrac{(-1)^{n+1}}{2n + 1}$

First four terms: $b_1 = \frac{1}{3}$

$b_2 = -\frac{1}{5}$

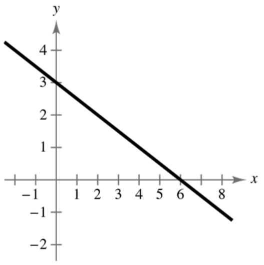
$b_3 = \frac{1}{7}$

$b_4 = -\frac{1}{9}$

The terms b_n are the opposite of the terms in Example 2: $b_n = -a_n$.

105. $m = -\frac{1}{2}, (0, 3)$

$y = -\frac{1}{2}x + 3$

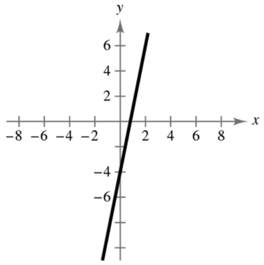


107. $m = 5, (1, 1)$

$y = 5x + b$

$1 = 5(1) + b \Rightarrow b = -4$

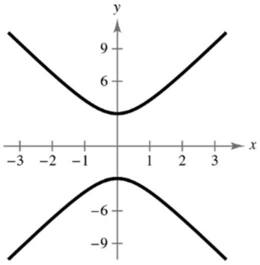
$y = 5x - 4$



109. $\frac{y^2}{9} - x^2 = 1$

Vertices: $(0, \pm 3)$

Asymptotes: $y = \pm 3x$

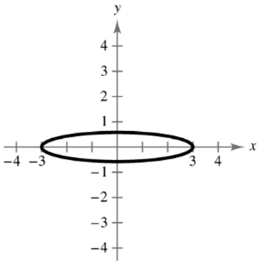


111. $x^2 + 27y^2 = 9$

$\frac{x^2}{9} + \frac{y^2}{(1/3)} = 1$

Vertices: $(\pm 3, 0)$

Intercepts: $\left(0, \pm\frac{1}{3}\right), (\pm 3, 0)$



113. Initial point: $(4, 1)$

Terminal point: $(6, -3)$

$\mathbf{v} = (6 - 4)\mathbf{i} + (-3 - 1)\mathbf{j} = 2\mathbf{i} - 4\mathbf{j}$

115. Initial point: $(5, -5)$

Terminal point: $(-4, 0)$

$\mathbf{v} = (-4 - 5)\mathbf{i} + [0 - (-5)]\mathbf{j} = -9\mathbf{i} + 5\mathbf{j}$

Section 9.2 Arithmetic Sequences and Partial Sums

1. $a_n = a_1 + (n - 1)d$

3. A sequence is arithmetic when the differences between consecutive terms are the same or constant, which is known as the common difference.

5. $1, 2, 4, 8, 16, \ldots$

Not an arithmetic sequence

7. $10, 8, 6, 4, 2, \ldots$

Arithmetic sequence, $d = -2$

9. $\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \ldots$

Arithmetic sequence, $d = \frac{1}{4}$

11. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Not an arithmetic sequence

13. $a_n = 5 + 3n$

$8, 11, 14, 17, 20$

Arithmetic sequence, $d = 3$

15. $a_n = 3 - 4(n - 2)$

$7, 3, -1, -5, -9$

Arithmetic sequence, $d = -4$

17. $a_n = (-1)^n$

$-1, 1, -1, 1, -1$

Not an arithmetic sequence

19. $a_n = (2^n)n$

$2, 8, 24, 64, 160$

Not an arithmetic sequence

21. $a_1 = 1, d = 3$

$a_n = a_1 + (n - 1)d = 1 + (n - 1)(3) = 3n - 2$

23. $a_1 = 100, d = -8$

$a_n = a_1 + (n-1)d = 100 + (n-1)(-8)$

$= -8n + 108$

25. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$

$d = -\frac{5}{2}$

$a_n = a_1 + (n-1)d = 4 + (n-1)\left(-\frac{5}{2}\right) = -\frac{5}{2}n + \frac{13}{2}$

27. $a_1 = 5, a_4 = 15$

$a_4 = a_1 + 3d \Rightarrow 15 = 5 + 3d \Rightarrow d = \frac{10}{3}$

$a_n = a_1 + (n-1)d = 5 + (n-1)\left(\frac{10}{3}\right) = \frac{10}{3}n + \frac{5}{3}$

29. $a_3 = 94, a_6 = 103$

$a_6 = a_3 + 3d \Rightarrow 103 = 94 + 3d \Rightarrow d = 3$

$a_1 = a_3 - 2d \Rightarrow a_1 = 94 - 2(3) = 88$

$a_n = a_1 + (n-1)d = 88 + (n-1)(3)$

$= 3n + 85$

31. $a_1 = 5, d = 6$

$a_1 = 5$

$a_2 = 5 + 6 = 11$

$a_3 = 11 + 6 = 17$

$a_4 = 17 + 6 = 23$

$a_5 = 23 + 6 = 29$

33. $a_1 = 2, a_{12} = -64$

$-64 = 2 + (12 + 1)d$

$-66 = 11d$

$-6 = d$

$a_1 = 2$

$a_2 = 2 - 6 = -4$

$a_3 = -4 - 6 = -10$

$a_4 = -10 - 6 = -16$

$a_5 = -16 - 6 = -22$

35. $a_8 = 26, a_{12} = 42$

$a_{12} = a_8 + 4d$

$42 = 26 + 4d \Rightarrow d = 4$

$a_8 = a_1 + 7d$

$26 = a_1 + 28 \Rightarrow a_1 = -2$

$a_1 = -2$

$a_2 = -2 + 4 = 2$

$a_3 = 2 + 4 = 6$

$a_4 = 6 + 4 = 10$

$a_5 = 10 + 4 = 14$

37. $a_1 = 15, a_{n+1} = a_n + 4$

$a_2 = 15 + 4 = 19$

$a_3 = 19 + 4 = 23$

$a_4 = 23 + 4 = 27$

$a_5 = 27 + 4 = 31$

39. $a_{n+1} = a_n - 2 \Rightarrow a_n = a_{n+1} + 2$

$a_5 = 7$

$a_4 = 7 + 2 = 9$

$a_3 = 9 + 2 = 11$

$a_2 = 11 + 2 = 13$

$a_1 = 13 + 2 = 15$

41. $a_1 = 5, a_2 = -1 \Rightarrow d = -1 - 5 = -6$

$a_n = a_1 + (n-1)d \Rightarrow a_{10} = 5 + 9(-6) = -49$

43. $a_1 = \dfrac{1}{8}, a_2 = \dfrac{3}{4} \Rightarrow d = \dfrac{3}{4} - \dfrac{1}{8} = \dfrac{5}{8}$

$a_n = a_1 + (n-1)d \Rightarrow a_7 = \dfrac{1}{8} + 6\left(\dfrac{5}{8}\right) = \dfrac{31}{8}$

45. $S_{10} = \frac{10}{2}(2 + 20) = 110$

47. $S_5 = \frac{5}{2}(-1 + (-9)) = -25$

49. $a_n = 2n - 1$

$a_1 = 1, a_{100} = 199$

$S_{100} = \frac{100}{2}(1 + 199) = 10{,}000$

51. $8, 20, 32, 44, \ldots$

$a_1 = 8, d = 12, n = 50$

$a_{50} = 8 + 49(12) = 596$

$S_{10} = \frac{50}{2}(8 + 596) = 15{,}100$

53. $0, -9, -18, -27, \ldots, n = 40$

$a_1 = 0, d = -9$

$a_{40} = 0 + (39)(-9) = -351$

$S_{100} = \frac{40}{2}(0 + (-351)) = -7020$

55. $a_1 = 1, a_{50} = 50, n = 50$

$\sum_{n=1}^{50} n = \frac{50}{2}(1 + 50) = 1275$

57. $a_1 = 9, a_{500} = 508, n = 500$

$\sum_{n=1}^{500} (n + 8) = \frac{500}{2}(9 + 508) = 129{,}250$

59. $a_1 = 14, a_{100} = -580, n = 100$

$$\sum_{n=1}^{100} (-6n + 20) = \frac{100}{2}(14 + (-580)) = -28{,}300$$

61. $a_n = -\frac{3}{4}n + 8$

$d = -\frac{3}{4}$ so the sequence is decreasing and $a_1 = 7\frac{1}{4}$.

Matches (b).

62. $a_n = 3n - 5$

$d = 3$ so the sequence is increasing and $a_1 = -2$.

Matches (d).

63. $a_n = 2 + \frac{3}{4}n$

$d = \frac{3}{4}$ so the sequence is increasing and $a_1 = 2\frac{3}{4}$.

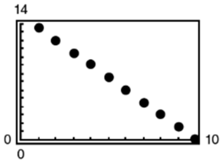
Matches (c).

64. $a_n = 25 - 3n$

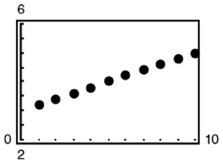
$d = -3$ so the sequence is decreasing and $a_1 = 22$.

Matches (a).

65. $a_n = 15 - \frac{3}{2}n$



67. $a_n = 0.2n + 3$



69. (a) $a_1 = 32{,}500, d = 1500$

$a_6 = a_1 + 5d = 32{,}500 + 5(1500) = \$40{,}000$

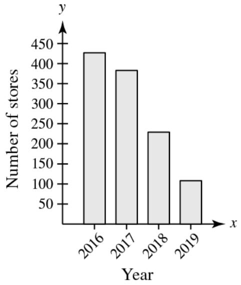
(b) $S_6 = \frac{6}{2}[32{,}500 + 40{,}000] = \$217{,}500$

71. $a_1 = 15, d = 3, n = 36$

$a_{36} = 15 + 35(3) = 120$

$S_{36} = \frac{36}{2}(15 + 120) = 2430$ seats

73. (a)



(b) Change from 2016 to 2017: –39

Change from 2017 to 2018: –159

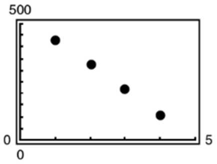
Change from 2018 to 2019: –121

Average change: $\frac{(-39 - 159 - 121)}{3} = 106.33$

$a_n = (427 + 106.33) - 106.33n$

$= 533.33 - 106.33n, n = 1, 2, 3, 4$

(c)



(d) $\sum_{n=1}^{4}(533.33 - 106.33n) = 1070.02$

The total number of new stores is approximately 1070.

75. $a_1 = 16, a_2 = 48, a_3 = 80, a_4 = 112$

$d = 32$

$a_n = dn + c = 32n + c$

$c = a_n - dn$

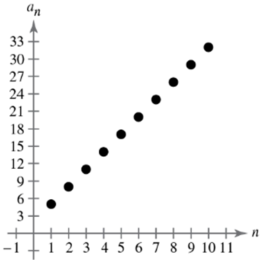
$c = a_1 - d = 16 - 32 = -16$

$a_n = 32n - 16$

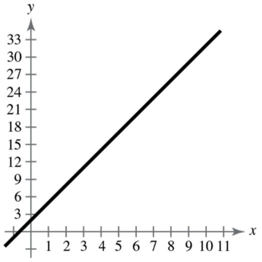
$\text{Distance} = \sum_{n=1}^{7} (32n - 16) = 784$ ft

77. True; given a_1 and a_2 then $d = a_2 - a_1$ and $a_n = a_1 + (n - 1)d$.

79. (a) $a_n = 2 + 3n$



(b) $y = 3x + 2$



(c) The graph of $a_n = 2 + 3n$ contains only points at the positive integers. The graph of $y = 3x + 2$ is a solid line which contains these points.

(d) The slope $m = 3$ is equal to the common difference $d = 3$. In general, these should be equal.

81. $a_1 = x, d = 2x$

$a_n = x + (n - 1)2x$

$a_n = 2xn - x$

$a_1 = 2x(1) - x = x$ $\quad a_6 = 2x(6) - x = 11x$

$a_2 = 2x(2) - x = 3x$ $\quad a_7 = 2x(7) - x = 13x$

$a_3 = 2x(3) - x = 5x$ $\quad a_8 = 2x(8) - x = 15x$

$a_4 = 2x(4) - x = 7x$ $\quad a_9 = 2x(9) - x = 17x$

$a_5 = 2x(5) - x = 9x$ $\quad a_{10} = 2x(10) - x = 19x$

83. The error was the value of $n = 50, a_{50} = 1 + (50 - 1)(2) = 99$. So $S_{50} = \frac{50}{2}(1 + 99) = 2500$.

85.
$$\begin{aligned}(5/7)^x &= 15/14\\ \log(5/7)^x &= \log(15/14)\\ x\log(5/7) &= \log(15/14)\\ x &= \frac{\log(15/14)}{\log(5/7)}\\ x &\approx -0.205\end{aligned}$$

87.
$$\begin{aligned}[3/5](1 - 4^x) &= -3\\ 1 - 4^x &= -5\\ 4^x &= 6\\ x\log 4 &= \log 6\\ x &= \frac{\log 6}{\log 4}\\ x &\approx 1.292\end{aligned}$$

89.
$$\begin{aligned}3\log_5 x &= 6\\ \log_5 x &= 2\\ 5^2 &= x\\ x &= 25\end{aligned}$$

91.
$$\begin{aligned}\log_2 5 &= x - 7\\ x &= 7 + \log_2 5 \approx 9.322\end{aligned}$$

93.
$$\begin{aligned}\frac{14(1 - x^3)}{1 - x^2} &= \frac{14(1 - x)(x^2 + x + 1)}{(1 - x)(1 + x)}\\ &= \frac{14(x^2 + x + 1)}{1 + x}\\ &= 14\left(\frac{1}{1 + x} + x\right)\\ &= 14x + \frac{14}{(1 + x)}, x \neq 1\end{aligned}$$

95.
$$\frac{1 - x}{4 - x^2} = \frac{x - 1}{x^2 - 4}$$

$$\frac{x - 1}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2}$$

$$x - 1 = A(x + 2) + B(x - 2)$$

$x = 2$: $1 = 4A \Rightarrow A = \frac{1}{4}$

$x = -2$: $-3 = -4B \Rightarrow B = \frac{3}{4}$

$$\frac{1 - x}{4 - x^2} = \frac{1}{4(x - 2)} + \frac{3}{4(x + 2)}$$

Section 9.3 Geometric Sequences and Series

1. ratio

3. You can find the $(n+1)$th term by multiplying the nth term by the common ratio.

5. $3, 6, 12, 24, \ldots$

Geometric sequence, $r = 2$

7. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, \ldots$

Geometric sequence, $r = 3$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Not a geometric sequence

11. $1, -\sqrt{7}, 7, -7\sqrt{7}, \ldots$

Geometric sequence, $r = -\sqrt{7}$

13. $a_1 = 4, r = 3$

$$\begin{aligned} a_1 &= 4 \\ a_2 &= 4(3) = 12 \\ a_3 &= 12(3) = 36 \\ a_4 &= 36(3) = 108 \\ a_5 &= 108(3) = 324 \end{aligned}$$

15. $a_1 = 1, r = \frac{1}{2}$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1\left(\tfrac{1}{2}\right) = \tfrac{1}{2} \\ a_3 &= \tfrac{1}{2}\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \\ a_4 &= \tfrac{1}{4}\left(\tfrac{1}{2}\right) = \tfrac{1}{8} \\ a_5 &= \tfrac{1}{8}\left(\tfrac{1}{2}\right) = \tfrac{1}{16} \end{aligned}$$

17. $a_1 = 1, r = e$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1(e) = e \\ a_3 &= (e)(e) = e^2 \\ a_4 &= (e^2)(e) = e^3 \\ a_5 &= (e^3)(e) = e^4 \end{aligned}$$

19. $a_1 = 3, r = \sqrt{5}$

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 3\left(\sqrt{5}\right)^1 = 3\sqrt{5} \\ a_3 &= 3\left(\sqrt{5}\right)^2 = 15 \\ a_4 &= 3\left(\sqrt{5}\right)^3 = 15\sqrt{5} \\ a_5 &= 3\left(\sqrt{5}\right)^4 = 75 \end{aligned}$$

21. $a_1 = 2, r = 3x$

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2(3x) = 6x \\ a_3 &= 6x(3x) = 18x^2 \\ a_4 &= 18x^2(3x) = 54x^3 \\ a_5 &= 54x^3(3x) = 162x^4 \end{aligned}$$

23. $a_1 = 4, r = \frac{1}{2}, n = 10$

$$\begin{aligned} a_n &= a_1 r^{n-1} = 4\left(\tfrac{1}{2}\right)^{n-1} \\ a_{10} &= 4\left(\tfrac{1}{2}\right)^9 = \left(\tfrac{1}{2}\right)^7 = \tfrac{1}{128} \end{aligned}$$

25. $a_1 = 6, r = -\dfrac{1}{3}, n = 12$

$$\begin{aligned} a_n &= a_1 r^{n-1} = 6\left(-\frac{1}{3}\right)^{n-1} \\ a_{12} &= 6\left(-\frac{1}{3}\right)^{11} = -\frac{2}{3^{10}} = -\frac{2}{59{,}049} \end{aligned}$$

27. $a_1 = 100, r = e^x, n = 9$

$$\begin{aligned} a_n &= a_1 r^{n-1} = 100\left(e^x\right)^{n-1} = 100e^{x(n-1)} \\ a_9 &= 100\left(e^x\right)^8 = 100e^{8x} \end{aligned}$$

29. $a_1 = 1, r = \sqrt{2}, n = 12$

$$\begin{aligned} a_n &= 1\left(\sqrt{2}\right)^{n-1} = \left(\sqrt{2}\right)^{n-1} \\ a_{12} &= \left(\sqrt{2}\right)^{12-1} = 32\sqrt{2} \end{aligned}$$

31. $64, 32, 16, \ldots$

$$\begin{aligned} r &= \frac{32}{64} = \frac{1}{2} \\ a_n &= 64\left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

33. $9, 18, 36, \ldots$

$$r = \frac{18}{9} = 2$$

$$a_n = 9(2)^{n-1}$$

35. $6, -9, \frac{27}{2}, \ldots$

$$r = \frac{-9}{6} = -\frac{3}{2}$$

$$a_n = 6\left(-\tfrac{3}{2}\right)^{n-1}$$

37. $6, 18, 54, \ldots$

$$r = \frac{18}{6} = 3$$

$$a_n = 6(3)^{n-1}$$

$$a_8 = 6(3)^{8-1}$$

$$= 6(3)^7$$

$$= 13{,}122$$

39. $\frac{1}{3}, \frac{-1}{6}, \frac{1}{12}, \ldots$

$$r = \frac{-\frac{1}{6}}{\frac{1}{3}} = -\frac{1}{2}$$

$$a_n = \frac{1}{3}\left(-\tfrac{1}{2}\right)^{n-1}$$

$$a_9 = \frac{1}{3}\left(-\tfrac{1}{2}\right)^{9-1}$$

$$= \frac{1}{3}\left(-\tfrac{1}{2}\right)^{8}$$

$$= \frac{1}{768}$$

41. $a_1 = 16, a_4 = \frac{27}{4}$

$$a_4 = a_1r^3$$

$$\tfrac{27}{4} = 16r^3$$

$$\tfrac{27}{64} = r^3$$

$$\tfrac{3}{4} = r$$

$$a_n = 16\left(\tfrac{3}{4}\right)^{n-1}$$

$$a_3 = 16\left(\tfrac{3}{4}\right)^{2} = 9$$

43. $a_4 = -18, a_7 = \dfrac{2}{3}$

$$a_7 = a_4r^3$$

$$\frac{2}{3} = -18r^3$$

$$-\frac{1}{27} = r^3$$

$$-\frac{1}{3} = r$$

$$a_6 = \frac{a_7}{r} = \frac{2/3}{-1/3} = -2$$

45. $a_n = 18\left(\tfrac{2}{3}\right)^{n-1}$

$a_1 = 18$ and $r = \frac{2}{3}$

Because $0 < r < 1$, the sequence is decreasing.

Matches (a).

46. $a_n = 18\left(-\tfrac{2}{3}\right)^{n-1}$

Because $r = \left(-\tfrac{2}{3}\right) > -1$, the sequence alternates as it approaches 0.

Matches (c).

47. $a_n = 18\left(\tfrac{3}{2}\right)^{n-1}$

Because $a_1 = 18$ and $r = \frac{3}{2} > 1$, the sequence is increasing.

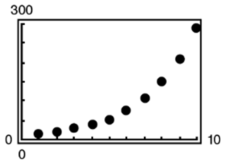
Matches (b).

48. $a_n = 18\left(-\tfrac{3}{2}\right)^{n-1}$

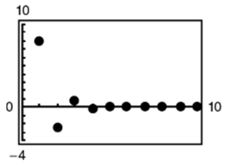
Because $r = \left(-\tfrac{3}{2}\right) < -1$, the sequence alternates as it approaches ∞.

Matches (d).

49. $a_n = 14(1.4)^{n-1}$



51. $a_n = 8(-0.3)^{n-1}$



53. $\sum_{n=1}^{7} 4^{n-1} = 1 + 4^1 + 4^2 + 4^3 + 4^4 + 4^5 + 4^6 \Rightarrow a_1 = 1, r = 4$

$$S_7 = \frac{1\left(1 - 4^7\right)}{1 - 4} = 5461$$

55. $\sum_{n=1}^{6} (-7)^{n-1} = 1 + (-7) + (-7)^2 + \cdots + (-7)^5 \Rightarrow a_1 = 1, r = -7$

$$S_6 = \frac{1\left(1 - (-7)^6\right)}{1 - (-7)} = -14{,}706$$

57. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n = \sum_{n=1}^{21} 3\left(\frac{3}{2}\right)^{n-1} = 3 + 3\left(\frac{3}{2}\right)^1 + 3\left(\frac{3}{2}\right)^2 + \cdots + 3\left(\frac{3}{2}\right)^{20} \Rightarrow a_1 = 3, r = \frac{3}{2}$

$$S_{21} = 3\left[\frac{1 - \left(\frac{3}{2}\right)^{21}}{1 - \frac{3}{2}}\right] = -6\left[1 - \left(\frac{3}{2}\right)^{21}\right] \approx 29{,}921.311$$

59. $\sum_{n=0}^{5} 200(1.05)^n = 200 + \sum_{n=1}^{5} 200(1.05)^5 = 200 + \left[200(1.05)^1 + 200(1.05)^2 + \cdots + 200(1.05)^5\right]$

$a_1 = 210, r = 1.05$

$$S_5 = 200 + 210\left[\frac{1 - (1.05)^5}{1 + (1.05)}\right] \approx 1360.383$$

61. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^n = 2 + 2\left(-\frac{1}{4}\right) + 2\left(-\frac{1}{4}\right)^2 + \cdots + 2\left(-\frac{1}{4}\right)^{40} \Rightarrow a_1 = 2, r = -\frac{1}{4}, n = 41$

$$S_{41} = 2\left[\frac{1 - \left(-\frac{1}{4}\right)^{41}}{1 - \left(-\frac{1}{4}\right)}\right] = \frac{8}{5}\left[1 - \left(-\frac{1}{4}\right)^{41}\right] \approx 1.6 = \frac{8}{5}$$

63. $10 + 30 + 90 + \cdots + 7290$

$r = 3$ and $7290 = 10(3)^{n-1}$

$729 = 3^{n-1}$

$6 = n - 1 \Rightarrow n = 7$

So, the sum can be written as $\sum_{n=1}^{7} 10(3)^{n-1}$.

65. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

$r = 4$ and $102.4 = 0.1(4)^{n-1}$

$1024 = 4^{n-1} \Rightarrow 5 = n - 1 \Rightarrow n = 6$

So, the sum can be written as $\sum_{n=1}^{6} 0.1(4)^{n-1}$.

67. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots$

$a_1 = 1, r = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{a_1}{1 - r} = \frac{1}{1 - \left(\frac{1}{2}\right)} = 2$$

69. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = 2 + 2\left(-\frac{2}{3}\right)^1 + 2\left(-\frac{2}{3}\right)^2 + \cdots$

$a_1 = 2, r = -\frac{2}{3}$

$$\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{a_1}{1 - r} = \frac{2}{1 - \left(-\frac{2}{3}\right)} = \frac{6}{5}$$

71. $\sum_{n=0}^{\infty} (0.8)^n = 1 + (0.8)^1 + (0.8)^2 + \cdots$

$a_1 = 1, r = 0.8$

$$\sum_{n=0}^{\infty} (0.8)^n = \frac{1}{1 - 0.8} = 5$$

73. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots = \sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1 - \frac{3}{4}} = 32$

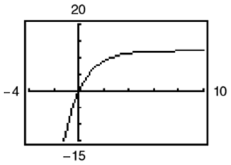
75. $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{9}(-3)^n$

The sum is undefined because $|r| = |-3| = 3 > 1$.

77. $0.\overline{36} = \sum_{n=0}^{\infty} 0.36(0.01)^n = \frac{0.36}{1 - 0.01} = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$

79.
$$\begin{aligned} 0.3\overline{18} &= 0.3 + \sum_{n=0}^{\infty} 0.018(0.01)^n = \frac{3}{10} + \frac{0.018}{1 - 0.01} \\ &= \frac{3}{10} + \frac{0.018}{0.99} = \frac{3}{10} + \frac{18}{990} = \frac{3}{10} + \frac{2}{110} \\ &= \frac{35}{110} = \frac{7}{22} \end{aligned}$$

81. $f(x) = 6\left[\frac{1 - (0.5)^x}{1 - (0.5)}\right], \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n = \frac{6}{1 - \frac{1}{2}} = 12$



The horizontal asymptote of $f(x)$ is $y = 12$.

This corresponds to the sum of the series.

83. $V_5 = 175{,}000(0.70)^5 = \$29{,}412.25$

85. Let $N = 12t$ be the total number of deposits.

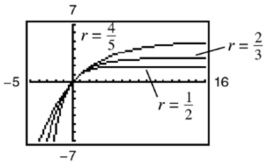
$$\begin{aligned} A &= P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^N \\ &= \left(1 + \frac{r}{12}\right)\left[P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{N-1}\right] \\ &= P\left(1 + \frac{r}{12}\right)\sum_{n=1}^{N}\left(1 + \frac{r}{12}\right)^{n-1} \\ &= P\left(1 + \frac{r}{12}\right)\left[\frac{1 - \left(1 + \frac{r}{12}\right)^N}{1 - \left(1 + \frac{r}{12}\right)}\right] \\ &= P\left(1 + \frac{r}{12}\right)\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^N\right] \\ &= P\left(\frac{12}{r} + 1\right)\left[-1 + \left(1 + \frac{r}{12}\right)^N\right] \\ &= P\left[\left(1 + \frac{r}{12}\right)^N - 1\right]\left(1 + \frac{12}{r}\right) \\ &= P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right) \end{aligned}$$

87. $27^2\left(\frac{1}{9}\right) + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right) + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^2 + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^3 = \sum_{n=0}^{3} 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^n = \frac{2465}{9} = 273\frac{8}{9}$ square inches

89. False. A sequence is geometric if the ratios of consecutive terms are the same.

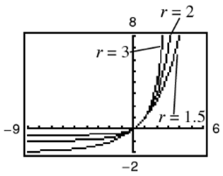
91. $y = \left(\frac{1 - r^x}{1 - r}\right)$

(a)



As $x \to \infty, y \to \frac{1}{1 - r}$.

(b)



As $x \to \infty, y \to \infty$.

93. The term were divided in the wrong order. The common ratio is $\frac{a_2}{a_1} = \frac{42}{21} = 2$, not $\frac{a_1}{a_2} = \frac{1}{2}$.

95. 97 is prime. The only positive factors are itself and 1.

97. 65,537 is prime. The only positive factors are itself and 1.

99. $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

101. $k^2 + (2k + 2 - 1) = k^2 + 2k + 1 = (k + 1)^2$

Section 9.4 Mathematical Induction

1. mathematical induction

3. To find the first differences of a sequence, subtract consecutive terms of the sequence.

5. $P_k = \frac{5}{k(k+1)}$

$P_{k+1} = \frac{5}{(k+1)[(k+1)+1]} = \frac{5}{(k+1)(k+2)}$

7. $P_k = k^2(k+3)^2$

$P_{k+1} = (k+1)^2[(k+1)+3]^2 = (k+1)^2(k+4)^2$

9. 1. When $n = 1, S_1 = 2 = 1(1+1)$.

2. Assume that

$S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k+1)$.

Then,

$$\begin{aligned} S_{k+1} &= 2 + 4 + 6 + 8 + \cdots + 2k + 2(k+1) \\ &= S_k + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2). \end{aligned}$$

So, we conclude that the formula is valid for all positive integer values of *n*.

11. 1. When $n = 1, S_1 = 1 = 2^1 - 1$.

2. Assume that

$S_k = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1$.

Then,

$S_{k+1} = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^k = S_k + 2^k = 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1$.

So, we conclude that this formula is valid for all positive integer values of *n*.

13. 1. When $n = 1, S_1 = 1 = \frac{1(1+1)}{2}$.

2. Assume that

$S_k = 1 + 2 + 3 + 4 + \cdots + k = \frac{k(k+1)}{2}$.

Then,

$S_{k+1} = 1 + 2 + 3 + 4 + \cdots + k + (k+1) = S_k + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

So, we conclude that this formula is valid for all positive integer values of *n*.

15. 1. When $n = 1, S_1 = 1^2 = \dfrac{1(2(1) - 1)(2(1) + 1)}{3}$

2. Assume that

$$S_k = 1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$$

Then,

$$\begin{aligned} S_{k+1} &= 1^2 + 3^2 + \cdots + (2k - 1)^2 + (2k + 1)^2 \\ &= S_k + (2k + 1)^2 = \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2 \\ &= (2k + 1)\left[\frac{k(2k - 1)}{3} + (2k + 1)\right] = \frac{2k + 1}{3}\left[2k^2 - k + 6k + 3\right] \\ &= \frac{2k + 1}{3}(2k + 3)(k + 1) = \frac{(k + 1)(2(k + 1) - 1)(2(k + 1) + 1)}{3} \end{aligned}$$

So, we conclude that this formula is valid for all positive integer values of *n*.

17. 1. When $n = 1, S_1 = 1 = \dfrac{(1)^2(1 + 1)^2\left(2(1)^2 + 2(1) - 1\right)}{12}.$

2. Assume that

$$S_k = \sum_{i=1}^{k} i^5 = \frac{k^2(k + 1)^2\left(2k^2 + 2k - 1\right)}{12}.$$

Then,

$$\begin{aligned} S_{k+1} &= \sum_{i=1}^{k+1} i^5 = \left(\sum_{i=1}^{k} i^5\right) + (k + 1)^5 \\ &= \frac{k^2(k + 1)^2\left(2k^2 + 2k - 1\right)}{12} + \frac{12(k + 1)^5}{12} \\ &= \frac{(k + 1)^2\left[k^2\left(2k^2 + 2k - 1\right) + 12(k + 1)^3\right]}{12} \\ &= \frac{(k + 1)^2\left[2k^4 + 2k^3 - k^2 + 12\left(k^3 + 3k^2 + 3k + 1\right)\right]}{12} \\ &= \frac{(k + 1)^2\left[2k^4 + 14k^3 + 35k^2 + 36k + 12\right]}{12} \\ &= \frac{(k + 1)^2\left(k^2 + 4k + 4\right)\left(2k^2 + 6k + 3\right)}{12} \\ &= \frac{(k + 1)^2(k + 2)^2\left[2(k + 1)^2 + 2(k + 1) - 1\right]}{12}. \end{aligned}$$

So, we conclude that this formula is valid for all positive integer values of *n*.

Note: The easiest way to complete the last two steps is to "work backwards." Start with the desired expression for S_{k+1} and multiply out to show that it is equal to the expression you found for $S_k + (k + 1)^5$.

19. 1. When $n = 4, 4! = 24$ and $2^4 = 16$, thus $4! > 2^4$.

2. Assume $k! > 2^k, k > 4$.

Then, $(k + 1)! = k!(k + 1) > 2^k(2)$ since $k! > 2^k$ and $k + 1 > 2$.

Thus, $(k + 1)! > 2^{k+1}$.

So, by extended mathematical induction, the inequality is valid for all integers *n* such that $n \geq 4$.

21. 1. When $n = 2, \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} \approx 1.707$ and $\sqrt{2} \approx 1.414$, thus $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} > \sqrt{2}$.

2. Assume that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}, k > 2.$$

Then,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Now it is sufficient to show that

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}, k > 2,$$

or equivalently (multiplying by $\sqrt{k+1}$),

$$\sqrt{k}\sqrt{k+1} + 1 > k + 1.$$

This is true because

$$\sqrt{k}\sqrt{k+1} + 1 > \sqrt{k}\sqrt{k} + 1 = k + 1.$$

Therefore,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

So, by extended mathematical induction, the inequality is valid for all integers *n* such that $n \geq 2$.

23. 1. When $n = 1, \left[1^3 + 3(1)^2 + 2(1)\right] = 6$ and 3 is a factor.

2. Assume that 3 is a factor of $k^3 + 3k^2 + 2k$.

Then,
$$\begin{aligned}(k + 1)^3 + 3(k + 1)^2 + 2(k + 1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2\\ &= \left(k^3 + 3k^2 + 2k\right) + \left(3k^2 + 9k + 6\right)\\ &= \left(k^3 + 3k^2 + 2k\right) + 3\left(k^2 + 3k + 2\right).\end{aligned}$$

Because 3 is a factor of each term, 3 is a factor of the sum.

So, 3 is a factor of $\left(n^3 + 3n^2 + 2n\right)$ for every positive integer *n*.

25. Prove 3 is a factor of $2^{2n+1} + 1$ for all positive integers *n*.

1. When $n = 1, 2^{2\cdot1+1} + 1 = 2^3 + 1 = 8 + 1 = 9$ and 3 is a factor.

2. Assume 3 is a factor of $2^{2k+1} + 1$.

Then, $2^{2(k+1)+1} + 1 = 2^{2k+2+1} + 1 = 2^{(2k+1)+2} + 1 = 2^{2k+1} \cdot 2^2 + 1 = 4 \cdot 2^{2k+1} + 1 = 4\left(2^{2k+1} + 1\right) - 3$.

Because 3 is a factor of each term, 3 is a factor of the sum.

So, 3 is a factor of $2^{2n+1} + 1$ for all positive integers *n*.

27. $S_n = 1 + 5 + 9 + 13 + \cdots + (4n - 3)$

$S_1 = 1 = 1 \cdot 1$

$S_2 = 1 + 5 = 6 = 2 \cdot 3$

$S_3 = 1 + 5 + 9 = 15 = 3 \cdot 5$

$S_4 = 1 + 5 + 9 + 13 = 28 = 4 \cdot 7$

From this sequence, it appears that $S_n = n(2n - 1)$.

This can be verified by mathematical induction.

1. The formula has already been verified for $n = 1$.
2. Assume that the formula is valid for $n = k$: $1 + 5 + 9 + 13 + \cdots + (4n - 3) = n(2n - 1)$.

$$\begin{aligned} \text{Then, } S_{k+1} &= \left[1 + 5 + 9 + 13 + \cdots + (4k - 3)\right] + \left[4(k + 1) - 3\right] \\ &= k(2k - 1) + (4k + 1) \\ &= 2k^2 + 3k + 1 \\ &= (k + 1)(2k + 1) \\ &= (k + 1)\left[2(k + 1) - 1\right] \end{aligned}$$

So, the formula is valid.

29. $S_n = \dfrac{1}{4} + \dfrac{1}{12} + \dfrac{1}{24} + \dfrac{1}{40} + \cdots + \dfrac{1}{2n(n + 1)}$

$S_1 = \dfrac{1}{4} = \dfrac{1}{2(2)}$

$S_2 = \dfrac{1}{4} + \dfrac{1}{12} = \dfrac{4}{12} = \dfrac{2}{6} = \dfrac{2}{2(3)}$

$S_3 = \dfrac{1}{4} + \dfrac{1}{12} + \dfrac{1}{24} = \dfrac{9}{24} = \dfrac{3}{8} = \dfrac{3}{2(4)}$

$S_4 = \dfrac{1}{4} + \dfrac{1}{12} + \dfrac{1}{24} + \dfrac{1}{40} = \dfrac{16}{40} = \dfrac{4}{10} = \dfrac{4}{2(5)}$

From the sequence, it appears that $S_n = \dfrac{n}{2(n + 1)}$.

This can be verified by mathematical induction.

1. The formula has already by verified for $n = 1$.
2. Assume that the formula is valid for $n = k$: $\dfrac{1}{4} + \dfrac{1}{12} + \dfrac{1}{24} + \dfrac{1}{40} + \cdots + \dfrac{1}{2n(n + 1)} = \dfrac{n}{2(n + 1)}$

$$\text{Then, } S_{k+1} = \left[\frac{1}{4} + \frac{1}{12} + \frac{1}{40} + \cdots + \frac{1}{2k(k + 1)}\right] + \frac{1}{2(k + 1)(k + 2)} = \frac{k}{2(k + 1)} + \frac{1}{2(k + 1)(k + 2)}$$

$$= \frac{k(k + 2) + 1}{2(k + 1)(k + 2)} = \frac{k^2 + 2k + 1}{2(k + 1)(k + 2)} = \frac{(k + 1)^2}{2(k + 1)(k + 2)} = \frac{k + 1}{2(k + 2)}.$$

So, the formula is valid.

31. $\displaystyle\sum_{n=1}^{15} n = \frac{15(15 + 1)}{2} = 120$

33. $\displaystyle\sum_{n=1}^{5} n^4 = \frac{5(5 + 1)\left[2(5) + 1\right]\left[3(5)^2 + 3(5) - 1\right]}{30} = 979$

35. $$\begin{aligned} \sum_{n=1}^{6}\left(n^2 - n\right) &= \sum_{n=1}^{6} n^2 - \sum_{n=1}^{6} n \\ &= \frac{6(6 + 1)\left[2(6) + 1\right]}{6} - \frac{6(6 + 1)}{2} \\ &= 91 - 21 = 70 \end{aligned}$$

37. 5, 14, 23, 32, 41, 50, …

Linear

Note: This is an arithmetic sequence.

$a_1 = 5, d = 9$

$$a_n = 5 + (n - 1)(9)$$
$$a_n = 9n - 4$$

39. 4, 10, 20, 34, 52, 74, …

Quadratic

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 10 \\ 9a + 3b + c = 20 \end{cases}$$

Solving this system yields $a = 2, b = 0$, and $c = 2$.

So, $a_n = 2n^2 + 2$.

41. $a_1 = 0, a_n = a_{n-1} + 3$

$a_1 = a_1 = 0$

$a_2 = a_1 + 3 = 0 + 3 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 3 = 6 + 3 = 9$

$a_5 = a_4 + 3 = 9 + 3 = 12$

$a_6 = a_5 + 3 = 12 + 3 = 15$

a_n: 0 3 6 9 12 15
First differences: 3 3 3 3 3
Second differences: 0 0 0 0

Because the first differences are equal, the sequence has a linear model.

43. $a_1 = 4, a_n = a_{n-1} + 3n$

$a_1 = a_1 = 4$

$a_2 = a_1 + 3n = 4 + 3(2) = 10$

$a_3 = a_2 + 3n = 10 + 3(3) = 19$

$a_4 = a_3 + 3n = 19 + 3(4) = 31$

$a_5 = a_4 + 3n = 31 + 3(5) = 46$

$a_6 = a_5 + 3n = 46 + 3(6) = 64$

a_n: 4 10 19 31 46 64
First differences: 6 9 12 15 18
Second differences: 3 3 3 3

Because the second differences are all the same, the sequence has a quadratic model.

45. $a_0 = 3, a_1 = 3, a_4 = 15$

Let $a_n = an^2 + bn + c$.

Then:

$$\begin{array}{rclcl} a_0 = a(0)^2 + b(0) + c = 3 & \Rightarrow & & & c = 3 \\ a_1 = a(1)^2 + b(1) + c = 3 & \Rightarrow & a + b + c & = & 3 \\ & & a + b & = & 0 \\ a_4 = a(4)^2 + b(4) + c = 15 & \Rightarrow & 16a + 4b + c & = & 15 \\ & & 16a + 4b & = & 12 \\ & & 4a + b & = & 3 \end{array}$$

By elimination:

$$\begin{array}{rl} -a - b &= 0 \\ \underline{4a + b} &\underline{= 3} \\ 3a &= 3 \\ a = 1 &\Rightarrow b = -1 \end{array}$$

So, $a_n = n^2 - n + 3$.

47. First differences:

$19.65 - 19.65 = 0$

$19.63 - 19.65 = -0.02$

$19.59 - 19.63 = -0.04$

$19.53 - 19.59 = -0.06$

$19.45 - 19.53 = -0.08$

A linear model is not appropriate because the first differences are not equal.

49. False. It has $n - 2$ second differences.

51. The formula $\dfrac{n(n+1)(2n+1)}{6}$ should be used.

So, $1^2 + 2^2 + \cdots + 9^2 = \dfrac{9(9+1)(18+1)}{6} = 285.$

53. At some point, the pattern may fail. Answers will vary.

55. $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

57.
$$\begin{aligned}(y-2)^4 &= (y-2)^2(y-2)^2\\ &= \left(y^2 - 4y + 4\right)\left(y^2 - 4y + 4\right)\\ &= y^4 - 8y^3 + 24y^2 - 32y + 16\end{aligned}$$

59.
$$\begin{aligned}\left(x^2+4\right)^3 &= \left(x^2\right)^3 + 3\left(x^2\right)^2(4) + 3x^2(4)^2 + 4^3\\ &= x^6 + 12x^4 + 48x^2 + 64\end{aligned}$$

61. $(x+y)^0 = 1$

63.
$$\begin{aligned}\frac{10!}{7!\,3!} &= \frac{10\cdot 9\cdot 8\cdot 7!}{7!\,3!}\\ &= \frac{10\cdot 9\cdot 8}{6} = 120\end{aligned}$$

65. $\dfrac{8!}{0!\,8!} = \dfrac{1}{0!} = 1$

67.
$$\begin{aligned}\frac{5x}{4} + \frac{1}{2} &= x - \frac{1}{2}\\ \frac{x}{4} &= -1\\ x &= -4\end{aligned}$$

Check:
$$\begin{aligned}\frac{5(-4)}{4} + \frac{1}{2} &\overset{?}{=} -4 - \frac{1}{2}\\ -5 + \frac{1}{2} &\overset{?}{=} -4 - \frac{1}{2}\\ -\frac{9}{2} &= -\frac{9}{2} \quad \checkmark\end{aligned}$$

Section 9.5 The Binomial Theorem

1. expanding

3. Both the Binomial Theorem and Pascal's Triangle can be used to find binominal coefficients.

5. ${}_5C_3 = \dfrac{5!}{3!\cdot 2!} = \dfrac{5\cdot 4}{2\cdot 1} = 10$

7. ${}_{12}C_0 = \dfrac{12!}{0!\cdot 12!} = 1$

9. $\dbinom{10}{4} = \dfrac{10!}{6!\cdot 4!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!(24)} = 210$

11. $\dbinom{100}{98} = \dfrac{100!}{2!\cdot 98!} = \dfrac{100\cdot 99}{2\cdot 1} = 4950$

13.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 ⑳ 15 6 1

$\dbinom{6}{3} = 20$, the 4th entry in the 6th row.

15.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 ⑤ 10 10 5 1

$\dbinom{5}{1} = 5$, the 2nd entry in the 5th row.

17.
$$\begin{aligned}(x+1)^6 &= {}_6C_0x^6 + {}_6C_1x^5(1) + {}_6C_2x^4(1)^2 + {}_6C_3x^3(1)^3 + {}_6C_4x^2(1)^4 + {}_6C_5x(1)^5 + {}_6C_6(1)^6\\ &= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1\end{aligned}$$

19.
$$\begin{aligned}(y-3)^3 &= {}_3C_0y^3 - {}_3C_1y^2(3) + {}_3C_2y(3)^2 - {}_3C_3(3)^3\\ &= 1y^3 - 3y^2(3) + 3y(3)^2 - 1(3)^3\\ &= y^3 - 9y^2 + 27y - 27\end{aligned}$$

21.
$$\begin{aligned}(r+3s)^3 &= {}_3C_0r^3 + {}_3C_1r^2(3s) + {}_3C_2r(3s)^2 + {}_3C_3(3s)^3\\ &= 1r^3 + 3r^2(3s) + 3r(3s)^2 + 1(3s)^3\\ &= r^3 + 9r^2s + 27rs^2 + 27s^3\end{aligned}$$

23.
$$\begin{aligned}(3a-4b)^5 &= {}_5C_0(3a)^5 - {}_5C_1(3a)^4(4b) + {}_5C_2(3a)^3(4b)^2 - {}_5C_3(3a)^2(4b)^3 + {}_5C_4(3a)(4b)^4 - {}_5C_5(4b)^5\\ &= (1)(243a^5) - 5(81a^4)(4b) + 10(27a^3)(16b^2) - 10(9a^2)(64b^3) + 5(3a)(256b^4) - (1)(1024b^5)\\ &= 243a^5 - 1620a^4b + 4320a^3b^2 - 5760a^2b^3 + 3840ab^4 - 1024b^5\end{aligned}$$

25.
$$\begin{aligned}(a+6)^4 &= {}_4C_0a^4 + {}_4C_1a^3(6) + {}_4C_2a^2(6)^2 + {}_4C_3a(6)^3 + {}_4C_4(6)^4\\ &= 1a^4 + 4a^3(6) + 6a^2(6)^2 + 4a(6)^3 + 1(6)^4\\ &= a^4 + 24a^3 + 216a^2 + 864a + 1296\end{aligned}$$

27.
$$\begin{aligned}(y-1)^6 &= {}_6C_0y^6 - {}_6C_1y^5(1) + {}_6C_2y^4(1)^2 - {}_6C_3y^3(1)^3 + {}_6C_4y^2(1)^4 - {}_6C_5y(1)^5 + {}_6C_6(1)^6\\ &= 1y^6 - 6y^5(1) + 15y^4(1)^2 - 20y^3(1)^3 + 15y^2(1)^4 - 6y(1)^5 + 1(1)^6\\ &= y^6 - 6y^5 + 15y^4 - 20y^3 + 15y^2 - 6y + 1\end{aligned}$$

29. 4th Row of Pascal's Triangle: 1 4 6 4 1
$$\begin{aligned}(3-2z)^4 &= 3^4 - 4(3)^3(2z) + 6(3)^2(2z)^2 - 4(3)(2z)^3 + (2z)^4\\ &= 81 - 216z + 216z^2 - 96z^3 + 16z^4\end{aligned}$$

31. 5th Row of Pascal's Triangle: 1 5 10 10 5 1
$$\begin{aligned}(x+2y)^5 &= 1x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + 1(2y)^5\\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5\end{aligned}$$

33.
$$\begin{aligned}(x^2+y^2)^4 &= {}_4C_0(x^2)^4 + {}_4C_1(x^2)^3(y^2) + {}_4C_2(x^2)^2(y^2)^2 + {}_4C_3(x^2)(y^2)^3 + {}_4C_4(y^2)^4\\ &= (1)(x^8) + (4)(x^6y^2) + (6)(x^4y^4) + (4)(x^2y^6) + (1)(y^8)\\ &= x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8\end{aligned}$$

35.
$$\begin{aligned}\left(\frac{1}{x}+y\right)^5 &= {}_5C_0\left(\frac{1}{x}\right)^5 + {}_5C_1\left(\frac{1}{x}\right)^4y + {}_5C_2\left(\frac{1}{x}\right)^3y^2 + {}_5C_3\left(\frac{1}{x}\right)^2y^3 + {}_5C_4\left(\frac{1}{x}\right)y^4 + {}_5C_5y^5\\ &= 1\left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4y + 10\left(\frac{1}{x}\right)^3y^2 + 10\left(\frac{1}{x}\right)^2y^3 + 5\left(\frac{1}{x}\right)y^4 + 1y^5\\ &= \frac{1}{x^5} + \frac{5y}{x^4} + \frac{10y^2}{x^3} + \frac{10y^3}{x^2} + \frac{5y^4}{x} + y^5\end{aligned}$$

37.
$$\begin{aligned}2(x-3)^4 + 5(x-3)^2 &= 2\left[{}_4C_0x^4 - {}_4C_1x^3(3) + {}_4C_2x^2(3)^2 - {}_4C_3x(3)^3 + {}_4C_4(3)^4\right] + 5\left[{}_2C_0x^2 + {}_2C_1x(3) + {}_2C_2(3)^2\right]\\ &= 2\left[x^4 - 4(x^3)(3) + 6(x^2)(3)^2 - 4(x)(3^3) + 3^4\right] + 5\left[x^2 - 2(x)(3) + 3^2\right]\\ &= 2(x^4 - 12x^3 + 54x^2 - 108x + 81) + 5(x^2 - 6x + 9)\\ &= 2x^4 - 24x^3 + 113x^2 - 246x + 207\end{aligned}$$

39. The 4th term in the expansion of $(x + y)^{10}$ is $_{10}C_3x^{10-3}y^3 = 120x^7y^3$.

41. The 3rd term in the expansion of $(x - 6y)^5$ is $_5C_2x^{5-2}(-6y)^2 = 10x^3(36y^2) = 360x^3y^2$.

43. The 8th term in the expansion of $(4x + 3y)^9$ is

$$_9C_7(4x)^{9-7}(3y)^7 = 36(16x^2)(2187y^7) = 1{,}259{,}712x^2y^7.$$

45. The 10th term in the expansion of $(10x - 3y)^{12}$ is

$$_{12}C_9(10x)^{12-9}(-3y)^9 = 220(1000x^3)(-19{,}683y^9) = -4{,}330{,}260{,}000x^3y^9.$$

47. The new term involving ax^3 in the expansion of $(x + 2)^6$ is $_6C_3x^3(2)^3 = \dfrac{6!}{3!\cdot 3!}\cdot 2^3x^3 = 160x^3$.

The coefficient is 160.

49. The term involving x^2y^8 in the expansion of $(4x - y)^{10}$ is $_{10}C_8(4x)^2(-y)^8 = \dfrac{10!}{(10-8)!8!}\cdot 16x^2y^8 = 720x^2y^8$.

The coefficient is 720.

51. The term involving x^4y^5 in the expansion of $(2x - 5x)^9$ is

$$_9C_5(2x)^4(-5y)^5 = 126(16x^4)(-3125y^5) = -6{,}300{,}000x^4y^5.$$

The coefficient is $-6{,}300{,}000$.

53. The term involving $x^8y^6 = (x^2)^4y^6$ in the expansion of $(x^2 + y)^{10}$ is $_{10}C_6(x^2)^4y^6 = \dfrac{10!}{4!6!}(x^2)^4y^6 = 210x^8y^6$.

The coefficient is 210.

55. $$(\sqrt{x} + 5)^3 = (\sqrt{x})^3 + 3(\sqrt{x})^2(5) + 3(\sqrt{x})(5^2) + 5^3 = x^{3/2} + 15x + 75x^{1/2} + 125$$

57. $$(x^{2/3} - y^{1/3})^3 = (x^{2/3})^3 - 3(x^{2/3})^2(y^{1/3}) + 3(x^{2/3})(y^{1/3})^2 - (y^{1/3})^3 = x^2 - 3x^{4/3}y^{1/3} + 3x^{2/3}y^{2/3} - y$$

59. $$(3\sqrt{t} + \sqrt[4]{t})^4 = (3\sqrt{t})^4 + 4(3\sqrt{t})^3(\sqrt[4]{t}) + 6(3\sqrt{t})^2(\sqrt[4]{t})^2 + 4(3\sqrt{t})(\sqrt[4]{t})^3 + (\sqrt[4]{t})^4$$
$$= 81t^2 + 108t^{7/4} + 54t^{3/2} + 12t^{5/4} + t$$

61. $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2,\ h \neq 0$$

63. $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^6 - x^6}{h} = \frac{x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 - x^6}{h}$$
$$= \frac{h(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)}{h} = 6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5,\ h \neq 0$$

65. $$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}},\ h \neq 0\end{aligned}$$

67. $$\begin{aligned}(1+i)^4 &= {}_4C_0(1)^4 + {}_4C_1(1)^3 i + {}_4C_2(1)^2 i^2 + {}_4C_3(1)i^3 + {}_4C_4 i^4\\ &= 1 + 4i - 6 - 4i + 1\\ &= -4\end{aligned}$$

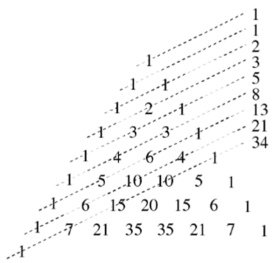
69. $$\begin{aligned}\left(2-\sqrt{-4}\right)^6 &= (2-2i)^6\\ &= {}_6C_0 2^6 - {}_6C_1 2^5(2i) + {}_6C_2 2^4(2i)^2 - {}_6C_3 2^3(2i)^3 + {}_6C_4 2^2(2i)^4 - {}_6C_5 2(2i)^5 + {}_6C_6(2i)^6\\ &= (1)(64) - (6)(32)(2i) + 15(16)(-4) - 20(8)(-8i) + 15(4)(16) - 6(2)(32i) - 64\\ &= 64 - 384i - 960 + 1280i + 960 - 384i - 64\\ &= 512i\end{aligned}$$

71. $$\begin{aligned}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 &= \frac{1}{8}\left[(-1)^3 + 3(-1)^2\left(\sqrt{3}i\right) + 3(-1)\left(\sqrt{3}i\right)^2 + \left(\sqrt{3}i\right)^3\right]\\ &= \frac{1}{8}\left[-1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i\right]\\ &= 1\end{aligned}$$

73. $$\begin{aligned}(1.02)^8 &= (1+0.02)^8\\ &= 1 + 8(0.02) + 28(0.02)^2 + 56(0.02)^3 + 70(0.02)^4 + 56(0.02)^5 + 28(0.02)^6 + 8(0.02)^7 + (0.02)^8\\ &= 1 + 0.16 + 0.0112 + 0.000448 + \cdots\\ &\approx 1.172\end{aligned}$$

75. $$\begin{aligned}(2.99)^{12} &= (3-0.01)^{12}\\ &= 3^{12} - 12(3)^{11}(0.01) + 66(3)^{10}(0.01)^2 - 220(3)^9(0.01)^3 + 495(3)^8(0.01)^4\\ &\quad -792(3)^7(0.01)^5 + 924(3)^6(0.01)^6 - 792(3)^5(0.01)^7 + 495(3)^4(0.01)^8\\ &\quad -220(3)^3(0.01)^9 + 66(3)^2(0.01)^{10} - 12(3)(0.01)^{11} + (0.01)^{12}\\ &\approx 531{,}441 - 21{,}257.64 + 389.7234 - 4.3303 + 0.0325 - 0.0002 + \cdots \approx 510{,}568.785\end{aligned}$$

77.



The first nine terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, …

After the first two terms, the next terms are formed by adding the previous two terms.

$a_1 = 1, a_2 = 1$

$a_3 = a_1 + a_2 = 1 + 1 = 2$

$a_4 = a_2 + a_3 = 1 + 2 = 3$

$a_5 = a_3 + a_4 = 2 + 3 = 5$

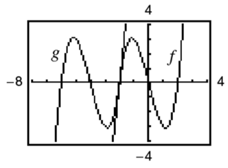
$a_6 = a_4 + a_5 = 3 + 5 = 8$

$a_7 = a_5 + a_6 = 5 + 8 = 13$

This is called the Fibonacci sequence.

79. $f(x) = x^3 - 4x$

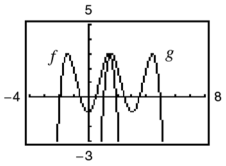
$$\begin{aligned} g(x) &= f(x + 4) \\ &= (x + 4)^3 - 4(x + 4) \\ &= x^3 + 3x^2(4) + 3x(4)^2 + (4)^3 - 4x - 16 \\ &= x^3 + 12x^2 + 48x + 64 - 4x - 16 \\ &= x^3 + 12x^2 + 44x + 48 \end{aligned}$$



The graph of *g* is the same as the graph of *f* shifted four units to the left.

81. $f(x) = -x^4 + 4x^2 - 1, g(x) = f(x - 3)$

$$\begin{aligned} g(x) &= f(x - 3) \\ &= -(x - 3)^4 + 4(x - 3)^2 - 1 \\ &= -\left(x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4\right) + 4\left(x^2 - 6x + 9\right) - 1 \\ &= -x^4 + 12x^3 - 54x^2 + 108x - 81 + 4x^2 - 24x + 36 - 1 \\ &= -x^4 + 12x^3 - 50x^2 + 84x - 46 \end{aligned}$$



The graph of *g* is the same as the graph of *f* shifted three units to the right.

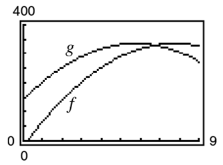
83. $f(t) = -5.893t^2 + 91.35t - 18.0, 2 \le t \le 9$

$(t = 2 \Leftrightarrow 2012)$

(a) $$\begin{aligned} g(t) &= f(t + 2) \\ &= -5.893(t + 2)^2 + 91.35(t + 2) - 18.0 \\ &= -5.893(t^2 + 4t + 4) + 91.35(t + 2) - 18.0 \\ &= -5.893t^2 + 67.78t + 141.1, 0 \le t \le 7 \end{aligned}$$

$(t = 2 \Leftrightarrow 2014)$

(b)



(c) Twitter exceeded 300 million monthly active users when $t \approx 5$, or in 2015.

85. True. The coefficients from the Binomial Theorem can be used to find the numbers in Pascal's Triangle.

87. False. The coefficient of the x-term of $(n - 4)^9$ is

${}_9C_1(4)^8 = -589{,}824$. The coefficient of the x^2-term is

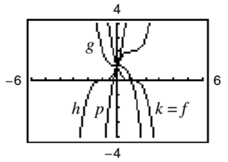
${}_9C_2(4)^7 = 589{,}824$

89. The first and last numbers in each row are 1. Every other number in each row is formed by adding the two numbers immediately above the number.

91. The functions $f(x) = (1 - x)^3$ and

$k(x) = 1 - 3x + 3x^2 + x^3$

f and k have identical graphs, because $k(x)$ is the expansion of $f(x)$.



93. ${}_7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 = \frac{7!}{3!4!}\left(\frac{1}{16}\right)\left(\frac{1}{8}\right) = 35\left(\frac{1}{16}\right)\left(\frac{1}{8}\right) \approx 0.273$

95.
$$\begin{aligned}
{}_nC_{n-r} &= \frac{n!}{(n-(n-r))!(n-r)!}\\
&= \frac{n!}{r!(n-r)!}\\
&= \frac{n!}{(n-r)!r!}\\
&= {}_nC_r
\end{aligned}$$

97.
$$\begin{aligned}
{}_nC_r + {}_nC_{r-1} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}\\
&= \frac{n!(n-r+1)!(r-1)! + n!(n-r)!r!}{(n-r)!r!(n-r+1)!(r-1)!}\\
&= \frac{n!\left[(n-r+1)!(r-1)! + r!(n-r)!\right]}{(n-r)!r!(n-r+1)!(r-1)!}\\
&= \frac{n!(\cancel{r-1})!\left[(n-r+1)! + r(n-r)!\right]}{(n-r)!r!(n-r+1)!(\cancel{r-1})!}\\
&= \frac{n!(\cancel{n-r})!\left[(n-r+1) + r\right]}{(\cancel{n-r})!r!(n-r+1)!}\\
&= \frac{n![n+1]}{r!(n-r+1)!}\\
&= \frac{(n+1)!}{\left[(n+1)-r\right]!r!}\\
&= {}_{n+1}C_r
\end{aligned}$$

99.
$$\begin{aligned}
f(x) &= \frac{1}{3}x - 4\\
\text{Average rate of change} &= \frac{\left[f(2) - f(-2)\right]}{\left[2 - (-2)\right]}\\
&= \frac{\left[\left(\frac{2}{3} - 4\right) - \left(-\frac{2}{3} - 4\right)\right]}{4}\\
&= \frac{\left(\frac{4}{3}\right)}{4} = \frac{1}{3}
\end{aligned}$$

101.
$$\begin{aligned}
f(x) &= 6 - 3x^2\\
\text{Average rate of change} &= \frac{\left[f(3) - f(0)\right]}{(3 - 0)}\\
&= \frac{\left[(6 - 27) - 6\right]}{3}\\
&= \frac{-27}{3}\\
&= -9
\end{aligned}$$

103.

$$\begin{aligned} f(x) &= 2x^3 - x^2 - x \\ \text{Average rate of change} &= \frac{[f(4) - f(1)]}{(4 - 1)} \\ &= \frac{[(128 - 16 - 4) - (2 - 1 - 1)]}{3} \\ &= \frac{108}{3} \\ &= 36 \end{aligned}$$

105. $\dfrac{6!}{7!} = \dfrac{6!}{7 \cdot 6!} = \dfrac{1}{7}$

107.

$$\begin{aligned} \frac{(n + 9)!}{(n + 7)!} &= \frac{(n + 9)(n + 8)(n + 7)!}{(n + 7)!} \\ &= (n + 9)(n + 8) \end{aligned}$$

109.

$$\sin(330°) = -\frac{1}{2}$$

$$\cos(330°) = \frac{\sqrt{3}}{2}$$

$$\tan(330°) = \frac{\left(-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

111.

$$\sin\left(-\frac{3\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{3\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{3\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) = 1$$

113.

$$\sin\left(\frac{11\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{11\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(\frac{11\pi}{3}\right) = \left(\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = -\sqrt{3}$$

Section 9.6 Counting Principles

1. distinguishable permutations

3. A given set of *n* elements forms exactly one combination, whereas different arrangements of the elements may form numerous permutations.

5. Odd integers: 1, 3, 5, 7, 9, 11
6 ways

7. Prime integers: 2, 3, 5, 7, 11
5 ways

9. Divisible by 4: 4, 8, 12
3 ways

11. Sum is 9: $1 + 8, 2 + 7, 3 + 6, 4 + 5, 5 + 4,$
$6 + 3, 7 + 2, 8 + 1$
8 ways

13. Amplifiers: 3 choices
Compact disc players: 2 choices
Speakers: 5 choices
Total: $3 \cdot 2 \cdot 5 = 30$ ways

15. Math courses: 2
Science courses: 3
Social sciences and humanities courses: 5
Total: $2 \cdot 3 \cdot 5 = 30$ schedules

17. $2^6 = 64$

19. $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$
distinct license plate numbers

21. (a) $9 \cdot 9 \cdot 8 = 648$
(b) $9 \cdot 10 \cdot 2 = 180$
(c) $6 \cdot 10 \cdot 10 = 600$

23. (a) $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$
(b) $8 \cdot 1 \cdot 6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 384$

25. $5! = 120$ ways

27. ${}_5P_2 = \dfrac{5!}{(5 - 2)!} = \dfrac{5!}{3!} = 5 \cdot 4 = 20$

29. $_{12}P_2 = \dfrac{12!}{(12-2)!} = \dfrac{12!}{10!} = 12 \cdot 11 = 132$

31. $_{15}P_3 = 2730$

33. $_{50}P_4 = 5{,}527{,}200$

35. The number of permutations of 9 possible donors taken 3 at a time is

$$\begin{aligned} {}_9P_3 &= \frac{9!}{(9-3)!} \\ &= \frac{9!}{6!} = 9 \cdot 8 \cdot 7 \\ &= 504 \end{aligned}$$

possible orders.

37. $_{15}P_9 = \dfrac{15!}{6!} = 1{,}816{,}214{,}400$

different batting orders

39. $\dfrac{7!}{2!1!3!1!} = \dfrac{7!}{2!3!} = 420$

41. $\dfrac{7!}{2!1!1!1!1!1!} = \dfrac{7!}{2!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

43. ABCD BACD CABD DABC
ABDC BADC CADB DACB
ACBD BCAD CBAD DBAC
ACDB BCDA CBDA DBCA
ADBC BDAC CDAB DCAB
ADCB BDCA CDBA DCBA

45. $_6C_4 = \dfrac{6!}{4!(6-4)!} = \dfrac{6!}{4!2!} = 15$

47. $_9C_9 = \dfrac{9!}{9!(9-9)!} = \dfrac{9!}{9!0!} = 1$

49. $_{16}C_2 = 120$

51. $_{20}C_6 = 38{,}760$

53. There are $_6C_2 = 15$ different combinations: AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF

55. $_{40}C_{12} = \dfrac{40!}{28!12!} = 5{,}586{,}853{,}480$ ways

57. There are 22 good units and 3 defective units.

(a) $_{22}C_4 = \dfrac{22!}{4!8!} = 7315$ ways

(b) $_{22}C_2 \cdot {}_3C_2 = \dfrac{22!}{2!20!} \cdot \dfrac{3!}{2!1!} = 231 \cdot 3 = 693$ ways

(c)
$$\begin{aligned} {}_{22}C_4 + {}_{22}C_3 \cdot {}_3C_1 + {}_{22}C_2 \cdot {}_3C_2 &= \frac{22!}{4!18!} + \frac{22!}{3!19!} \cdot \frac{3!}{1!2!} + \frac{22!}{2!20!} \cdot \frac{3!}{2!1!} \\ &= 7315 + 1540 \cdot 3 + 231 \cdot 3 \\ &= 12{,}628 \text{ ways} \end{aligned}$$

59. (a) Select type of card for three of a kind: $_{13}C_1$

Select three of four cards for three of a kind: $_4C_3$

Select type of card for pair: $_{12}C_1$

Select two of four cards for pair: $_4C_2$

$$_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_1 \cdot {}_4C_2 = \frac{13!}{(13-1)!1!} \cdot \frac{4!}{(4-3)!3!} \cdot \frac{12!}{(12-1)!1!} \cdot \frac{4!}{(4-2)!2!} = 3744$$

(b) Select two jacks: $_4C_2$

Select three aces: $_4C_3$

$$_4C_2 \cdot {}_4C_3 = \frac{4!}{(4-2)!2!} \cdot \frac{4!}{(4-3)!3!} = 24$$

61. $_7C_1 \cdot {}_{12}C_3 \cdot {}_{20}C_2 = \dfrac{7!}{(7-1)!1!} \cdot \dfrac{12!}{(12-3)!3!} \cdot \dfrac{20!}{(20-2)!2!} = 292{,}600$

63. ${}_5C_2 - 5 = 10 - 5 = 5$ diagonals

65. ${}_8C_2 - 8 = 28 - 8 = 20$ diagonals

67. $4 \cdot {}_{n+1}P_2 = {}_{n+2}P_3$ **Note:** $n \geq 1$ for this to be defined.

$$\begin{aligned}
4 \cdot \frac{(n+1)!}{(n-1)!} &= \frac{(n+2)!}{(n-1)!} \\
4(n+1)(n) &= (n+2)(n+1)n \qquad \text{(We can divide by } (n+1)n \text{ because } n \neq -1 \text{ and } n \neq 0.) \\
4 &= n + 2 \\
2 &= n
\end{aligned}$$

69. ${}_{n+1}P_3 = 4 \cdot {}_nP_2$ **Note:** $n \geq 2$ for this to be defined.

$$\begin{aligned}
\frac{(n+1)!}{(n-2)!} &= 4 \cdot \frac{n!}{(n-2)!} \\
(n+1)(n)(n-1) &= 4(n)(n-1) \qquad \text{(We can divide by } n(n-1) \text{ because } n \neq 0, \text{ and } n \neq 1.) \\
n + 1 &= 4 \\
n &= 3
\end{aligned}$$

71. $14 \cdot {}_nP_3 = {}_{n+2}P_4$ **Note:** $n \geq 3$ for this to be defined.

$$\begin{aligned}
14\left(\frac{n!}{(n-3)!}\right) &= \frac{(n+2)!}{(n-2)!} \\
14n(n-1)(n-2) &= (n+2)(n+1)n(n-1) \qquad \text{(We can divide here by } n(n-1) \text{ because } n \neq 0, n \neq 1.) \\
14(n-2) &= (n+2)(n+1) \\
14n - 28 &= n^2 + 3n + 2 \\
0 &= n^2 - 11n + 30 \\
0 &= (n-5)(n-6) \\
n &= 5 \quad \text{or} \quad n = 6
\end{aligned}$$

73. ${}_nP_4 = 10 \cdot {}_{n-1}P_3$ **Note:** $n \geq 4$ for this to be defined.

$$\begin{aligned}
\frac{n!}{(n-4)!} &= 10 \cdot \frac{(n-1)!}{(n-4)!} \\
n(n-1)(n-2)(n-3) &= 10(n-1)(n-2)(n-3) \qquad \left(\begin{array}{l}\text{We can divide by } (n-1)(n-2)(n-3) \text{ because} \\ n \neq 1, n \neq 2, \text{ and } n \neq 3.\end{array}\right) \\
n &= 10
\end{aligned}$$

75. ${}_9C_2 = \dfrac{9!}{2!7!} = 36$ lines

77. False.

It is an example of a combination.

79. ${}_{10}P_6 > {}_{10}C_6$

Changing the order of any of the six elements selected results in a different permutation but the same combination.

81. ${}_nP_{n-1} = \dfrac{n!}{(n-(n-1))!} = \dfrac{n!}{1!} = \dfrac{n!}{0!} = {}_nP_n$

83.
$$\begin{aligned}
{}_nC_{n-1} &= \frac{n!}{(n-(n-1))!(n-1)!} = \frac{n!}{(1)!(n-1)!} \\
&= \frac{n!}{(n-1)!1!} = {}_nC_1
\end{aligned}$$

85.
$$\begin{aligned}
\frac{6!(16!)}{22!} &= \frac{6!(16!)}{22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot (16!)} \\
&= \frac{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot 2}{22 \cdot 21 \cdot \cancel{20} \cdot 19 \cdot \cancel{18} \cdot 17} \\
&= \frac{1}{74{,}613}
\end{aligned}$$

87. $$3\left(\frac{20!(9!)}{29!}\right) = 3\left(\frac{20!(9!)}{29\cdot 28\cdot 27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21\cdot (20!)}\right)$$
$$= \cancel{3}\left(\frac{\cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot 2}{29\cdot \cancel{28}\cdot \cancel{27}\cdot 26\cdot 25\cdot \cancel{24}\cdot 23\cdot 22\cdot 21}\right)$$
$$= \frac{1}{3{,}338{,}335}$$

89. The 3rd term in the expansion of $(2x + 3y)^5$ is

$${}_5C_2(2x)^3(3y)^2 = 10(2^3)(3^2)x^3y^2 = 720x^3y^2.$$

91. The 7th term in the expansion of $(8x - 5y)^8$ is

$$\begin{aligned}{}_8C_6(8x)^2(-5y)^6 &= 28(8^2)(-5)^6x^2y^6\\ &= 28{,}000{,}000\,x^2y^6.\end{aligned}$$

93. The 6th term in the expansion of $(-3x - 9y)^{13}$ is

$$\begin{aligned}{}_{13}C_5(-3x)^8(-9y)^5 &= 1287(-3)^8(-9)^5x^8y^5\\ &= -498{,}610{,}169{,}343\,x^8y^5.\end{aligned}$$

95. $y = ax^2 + bx + c$

$$\begin{cases}-8 = a(-2)^2 + b(-2) + c = 4a - 2b + c & \text{Equation 1}\\ 10 = a(4)^2 + b(4) + c = 16a + 4b + c & \text{Equation 2}\\ 12 = a(8)^2 + b(8) + c = 64a + 8b + c & \text{Equation 3}\end{cases}$$

$$\begin{cases}4a - 2b + c = -8 & \\ 12b - 3c = 42 & -4\text{Eq. } 1 + \text{Eq. } 2\\ 40b - 15c = 140 & -16\text{Eq. } 1 + \text{Eq. } 3\end{cases}$$

$$\begin{cases}4a - 2b + c = -8 & \\ 12b - 3c = 42 & \\ -\dfrac{15}{2}c = 0 & -5\text{Eq. } 2 + \dfrac{3}{2}\text{Eq. } 3\end{cases}$$

$c = 0$

$12b - 3(0) = 42 \Rightarrow b = \dfrac{7}{2}$

$4a - 2\left(\dfrac{7}{2}\right) + 0 = -8 \Rightarrow a = -\dfrac{1}{4}$

Solution: $a = -\dfrac{1}{4}, b = \dfrac{7}{2}, c = 0$

The quadratic equation is $y = -\dfrac{1}{4}x^2 + \dfrac{7}{2}x.$

Section 9.7 Probability

1. experiment; outcomes

3. event

5. $n(E)$; $n(S)$

7. The probability of a certain event is 1.

9. $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

11. {ABC, ACB, BAC, BCA, CAB, CBA}

13. {AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}

15. $E = \{HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

17. $E = \{HHH, HHT, HTH, HTT\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

19. $E = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}$$

21. $E = \{K\clubsuit, K\blacklozenge, K\heartsuit, K\spadesuit, Q\clubsuit, Q\blacklozenge, Q\heartsuit, Q\spadesuit, J\clubsuit, J\blacklozenge, J\heartsuit, J\spadesuit\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

23. $E = \{K\blacklozenge, K\heartsuit, Q\blacklozenge, Q\heartsuit, J\blacklozenge, J\heartsuit\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{52} = \frac{3}{26}$$

25. $E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$$

27. Use the complement.

$$E' = \{(5, 6), (6, 5), (6, 6)\}$$

$$P(E') = \frac{n(E')}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$P(E) = 1 - P(E') = 1 - \frac{1}{12} = \frac{11}{12}$$

29.

$$\begin{aligned}
E_3 &= \{(1, 2), (2, 1)\}, n(E_3) = 2\\
E_5 &= \{(1, 4), (2, 3), (3, 2), (4, 1)\}, n(E_5) = 4\\
E_7 &= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}, n(E_7) = 6\\
E &= E_3 \cup E_5 \cup E_7\\
n(E) &= 2 + 4 + 6 = 12\\
P(E) &= \frac{n(E)}{n(S)} = \frac{12}{36} = \frac{1}{3}
\end{aligned}$$

31. $P(E) = \dfrac{{}_3C_2}{{}_6C_2} = \dfrac{3}{15} = \dfrac{1}{5}$

33. $P(E) = \dfrac{{}_4C_2}{{}_6C_2} = \dfrac{6}{15} = \dfrac{2}{5}$

35. (a) $0.124(6{,}000{,}00) = 744{,}000$ unemployed in the 16–19 age group

(b) 17.1% or $\frac{171}{1000}$

(c) 53.4% or $\frac{534}{1000} = \frac{267}{500}$

(d) 11.8% + 5.3% = 17.1% or $\frac{171}{1000}$

37. (a) $\dfrac{104}{128} = \dfrac{13}{16}$

(b) $\dfrac{24}{128} = \dfrac{3}{16}$

(c) $\dfrac{52 - 48}{128} = \dfrac{1}{32}$

39. $1 - 0.37 - 0.44 = 0.19 = 19\%$

41. (a) $\dfrac{{}_{15}C_{10}}{{}_{20}C_{10}} = \dfrac{3003}{184{,}756} = \dfrac{21}{1292} \approx 0.016$

(b) $\dfrac{{}_{15}C_8 \cdot {}_5C_2}{{}_{20}C_{10}} = \dfrac{64{,}350}{184{,}756} = \dfrac{225}{646} \approx 0.348$

(c)
$$\begin{aligned}
\frac{{}_{15}C_9 \cdot {}_5C_1}{{}_{20}C_{10}} + \frac{{}_{15}C_{10}}{{}_{20}C_{10}} + \frac{25{,}025 + 3003}{184{,}756} &= \frac{28{,}028}{184{,}756}\\
&= \frac{49}{323}\\
&\approx 0.152
\end{aligned}$$

43. (a) $\dfrac{1}{{}_5P_5} = \dfrac{1}{120}$

(b) $\dfrac{1}{{}_4P_4} = \dfrac{1}{24}$

45. (a) $\frac{20}{52} = \frac{5}{13}$

(b) $\frac{26}{52} = \frac{1}{2}$

(c) $\frac{16}{52} = \frac{4}{13}$

47. (a) $\dfrac{{}_9C_4}{{}_{12}C_4} = \dfrac{126}{495} = \dfrac{14}{55}$ (4 good units)

(b) $\dfrac{{}_9C_2 \cdot {}_3C_2}{{}_{12}C_4} = \dfrac{108}{495} = \dfrac{12}{55}$ (2 good units)

(c) $\dfrac{{}_9C_3 \cdot {}_3C_1}{{}_{12}C_4} = \dfrac{252}{495} = \dfrac{28}{55}$ (3 good units)

At least 2 good units: $\dfrac{12}{55} + \dfrac{28}{55} + \dfrac{14}{55} = \dfrac{54}{55}$

49. (a) $P(EE) = \frac{20}{40} \cdot \frac{20}{40} = \frac{1}{4}$

(b) $P(EO \text{ or } OE) = 2\left(\frac{20}{40}\right)\left(\frac{20}{40}\right) = \frac{1}{2}$

(c) $P(N_1 < 30, N_2 < 30) = \frac{29}{40} \cdot \frac{29}{40} = \frac{841}{1600}$

(d) $P(N_1N_1) = \frac{30}{40} \cdot \frac{1}{40} = \frac{1}{40}$

51. $P(E') = 1 - P(E) = 1 - 0.73 = 0.27$

53. $P(E') = 1 - P(E) = 1 - \frac{1}{5} = \frac{4}{5}$

55. $P(E) = 1 - P(E') = 1 - 0.29 = 0.71$

57. $P(E) = 1 - P(E') = 1 - \frac{14}{25} = \frac{11}{25}$

59. (a) $P(SS) = (0.985)^2 \approx 0.9702$

(b) $P(FF) = (0.015)^2 \approx 0.0002$

(c) $P(S) = 1 - P(FF) = 1 - (0.015)^2 \approx 0.9998$

61. (a) $\dfrac{1}{38}$

(b) $\dfrac{18}{38} = \dfrac{9}{19}$

(c) $\dfrac{2}{38} + \dfrac{18}{38} = \dfrac{20}{38} = \dfrac{10}{19}$

(d) $\dfrac{1}{38} \cdot \dfrac{1}{38} = \dfrac{1}{1444}$

(e) $\dfrac{18}{38} \cdot \dfrac{18}{38} \cdot \dfrac{18}{38} = \dfrac{5832}{54{,}872} = \dfrac{729}{6859}$

63. True. Two events are independent if the occurrence of one has no effect on the occurrence of the other.

65. The only numbers from 1 to 15 greater than 12 are 13, 14, and 15. So, the probability of selecting one number greater than 12 is $\frac{3}{15}$ and the probability that both numbers are greater than 12 is $\left(\frac{3}{15}\right)\left(\frac{3}{15}\right) = \frac{9}{225} = \frac{1}{25}$.

67. $P(A) = 0.76$ and $P(B) = 0.58$

(a) A and B cannot be mutually exclusive, because $P(A) + P(B) = 0.76 + 0.58 = 1.34 > 1$.

(b) $0.76 \le P(A \cup B) \le 1$, where $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

69. If a weather forecast indicates that the probability of rain is 40%, this means the meteorological records indicates that over an extended period of time with similar weather conditions it will rain 40% of the time.

71. Let $x = 2\sin\theta$.

$$\begin{aligned}\sqrt{4 - x^2} &= \sqrt{2}\\ \sqrt{4 - (2\sin\theta)^2} &= \sqrt{2}\\ \sqrt{4 - 4\sin^2\theta} &= \sqrt{2}\\ \sqrt{4(1 - \sin^2\theta)} &= \sqrt{2}\\ \sqrt{4\cos^2\theta} &= \sqrt{2}\\ 2\cos\theta &= \sqrt{2}\\ \cos\theta &= \frac{\sqrt{2}}{2}\\ \sin\theta = \sqrt{1 - \cos^2\theta} &= \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \pm\frac{\sqrt{2}}{2}\end{aligned}$$

73. As $x \to \frac{\pi}{2}^-$, $\tan x \to \infty$ and $\cot x \to 0$.

75. $$\begin{aligned}-6i^3 + i^2 &= -6i^2 i + i^2\\ &= -6(-1)i + (-1)\\ &= 6i - 1\\ &= -1 + 6i\end{aligned}$$

Review Exercises for Chapter 9

1. $a_n = 3 + \frac{12}{n}$

$a_1 = 3 + \frac{12}{1} = 15$

$a_2 = 3 + \frac{12}{2} = 9$

$a_3 = 3 + \frac{12}{3} = 7$

$a_4 = 3 + \frac{12}{4} = 6$

$a_5 = 3 + \frac{12}{5} = \frac{27}{5}$

3. $a_n = \frac{120}{n!}$

$a_1 = \frac{120}{1!} = 120$

$a_2 = \frac{120}{2!} = 60$

$a_3 = \frac{120}{3!} = 20$

$a_4 = \frac{120}{4!} = 5$

$a_5 = \frac{120}{5!} = 1$

5. $-2, 2, -2, 2, -2, \ldots$

$a_n = 2(-1)^n$

7. $3, \frac{9}{2}, 9, \frac{81}{4}, \frac{243}{5}, \ldots$

Rewrite as $\frac{3}{1}, \frac{3^2}{2}, \frac{3^3}{3}, \frac{3^4}{4}, \frac{3^5}{5}, \ldots$

$a_n = \frac{3^n}{n}$

9. $\frac{3!}{5!} = \frac{3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{20}$

11. $$\begin{aligned}\frac{(n-1)!}{(n+1)!} &= \frac{(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1}{(n+1)(n)(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1}\\ &= \frac{1}{n(n+1)}\end{aligned}$$

13. $$\begin{aligned}\sum_{j=1}^{4} \frac{6}{j^2} &= \frac{6}{1^2} + \frac{6}{2^2} + \frac{6}{3^2} + \frac{6}{4^2}\\ &= 6 + \frac{3}{2} + \frac{2}{3} + \frac{3}{8}\\ &= \frac{205}{24}\end{aligned}$$

15. $\frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2(20)} = \sum_{k=1}^{20} \frac{1}{2k}$

17. $\sum_{i=1}^{\infty} \frac{4}{10^i} = \sum_{i=1}^{\infty} 4\left(\frac{1}{10^i}\right) = \frac{\frac{4}{10}}{1 - \frac{1}{10}} = \frac{4}{9}$

19. (a)

$$\begin{aligned} A_1 &= \$10{,}018.75 \\ A_2 &\approx \$10{,}037.54 \\ A_3 &\approx \$10{,}056.36 \\ A_4 &\approx \$10{,}075.21 \\ A_5 &\approx \$10{,}094.10 \\ A_6 &\approx \$10{,}113.03 \\ A_7 &\approx \$10{,}131.99 \\ A_8 &\approx \$10{,}150.99 \\ A_9 &\approx \$10{,}170.02 \\ A_{10} &\approx \$10{,}189.09 \end{aligned}$$

(b) The balance in the account after 10 years is $A_{120}\ 10{,}000\left(1 + \dfrac{0.0225}{12}\right)^{120} \approx \$12{,}520.59$

21. $5, -1, -7, -13, -19, \ldots$

Arithmetic sequence, $d = -6$

27. $a_3 = 96, a_7 = 24$

$$\begin{aligned} a_7 &= a_3 + 4d \Rightarrow 24 = 96 + 4d \Rightarrow -72 = 4d \Rightarrow d = -18 \\ a_1 &= a_3 - 2d \Rightarrow a_1 = 96 - 2(-18) = 132 \\ a_n &= 132 + (n-1)(-18) \\ &= -18n + 150 \end{aligned}$$

29. $a_1 = 4, d = 17$

$$\begin{aligned} a_1 &= 4 \\ a_2 &= 4 + 17 = 21 \\ a_3 &= 21 + 17 = 38 \\ a_4 &= 38 + 17 = 55 \\ a_5 &= 45 + 17 = 72 \end{aligned}$$

31. $\sum\limits_{k=1}^{100} 9k$ is arithmetic. Therefore, $a_1 = 9, a_{100} = 900,$

$S_{700} = \frac{100}{2}(9 + 900) = 45{,}450.$

33. $\sum\limits_{j=1}^{40} 2j = 2\sum\limits_{j=1}^{40} j = 2\left[\dfrac{40(41)}{2}\right] = 40(41) = 1640$

23. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, \ldots$

Not an arithmetic sequence.

25. $a_1 = 7, d = 12$

$$\begin{aligned} a_n &= 7 + (n-1)12 \\ &= 7 + 12n - 12 \\ &= 12n - 5 \end{aligned}$$

35.

$$\begin{aligned} \sum_{k=1}^{100}\left(\frac{2}{3}k + 4\right) &= \frac{2}{3}\sum_{k=1}^{100} k + \sum_{k=1}^{100} 4 \\ &= \frac{2}{3}\left[\frac{100(101)}{2}\right] + 4(100) \\ &= \frac{10{,}100}{3} + 400 \\ &= \frac{11{,}300}{3} \end{aligned}$$

37. $a_n = 43{,}800 + (n-1)(1950)$

(a) $a_5 = 43{,}800 + 4(1950) = \$51{,}600$

(b) $S_5 = \frac{5}{2}(43{,}800 + 51{,}600) = \$238{,}500$

39. $2, 6, 18, 54, 162, \ldots$

$r = \dfrac{6}{2} = 3$

Geometric sequence, $r = 3$

41. $\frac{1}{5}, -\frac{3}{5}, \frac{9}{5}, -\frac{27}{5}, \dots$

Geometric sequence, $r = -3$

43. $a_1 = 2, r = 15$

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2(15) = 30 \\ a_3 &= 30(15) = 450 \\ a_4 &= 450(15) = 6750 \\ a_5 &= 6750(15) = 101{,}250 \end{aligned}$$

45. $a_1 = 9, a_3 = 4$

$$\begin{aligned} a_3 &= a_1 r^2 \\ 4 &= 9r^2 \\ \tfrac{4}{9} &= r^2 \Rightarrow r = \pm\tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} a_1 &= 9 & a_1 &= 9 \\ a_2 &= 9\left(\tfrac{2}{3}\right) = 6 & a_2 &= 9\left(-\tfrac{2}{3}\right) = -6 \\ a_3 &= 6\left(\tfrac{2}{3}\right) = 4 \quad \text{or} & a_3 &= -6\left(-\tfrac{2}{3}\right) = 4 \\ a_4 &= 4\left(\tfrac{2}{3}\right) = \tfrac{8}{3} & a_4 &= 4\left(-\tfrac{2}{3}\right) = -\tfrac{8}{3} \\ a_5 &= \tfrac{8}{3}\left(\tfrac{2}{3}\right) = \tfrac{16}{9} & a_5 &= -\tfrac{8}{3}\left(-\tfrac{2}{3}\right) = \tfrac{16}{9} \end{aligned}$$

47. $a_1 = 100, r = 1.05$

$$\begin{aligned} a_n &= 100(1.05)^{n-1} \\ a_{10} &= 100(1.05)^9 \approx 155.133 \end{aligned}$$

49. $a_1 = 18, a_2 = -9$

$$\begin{aligned} a_2 &= a_1 r \\ -9 &= 18r \\ -\tfrac{1}{2} &= r \\ a_n &= 18\left(-\tfrac{1}{2}\right)^{n-1} \\ a_{10} &= 18\left(-\tfrac{1}{2}\right)^9 = \tfrac{-9}{256} \end{aligned}$$

51. $\displaystyle\sum_{i=1}^{7} 2^{i-1} = \frac{1-2^7}{1-2} = 127$

53. $\displaystyle\sum_{i=1}^{4} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

55. $\displaystyle\sum_{i=1}^{5} (2)^{i-1} = 1 + 2 + 4 + 8 + 16 = 31$

57. $\displaystyle\sum_{i=1}^{5} 10(0.6)^{i-1} = 23.056$

59. $\displaystyle\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^{i-1} = \frac{1}{1-\frac{7}{8}} = 8$

61. $$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots &= \sum_{i=0}^{\infty}\left(\frac{-1}{2}\right)^i \\ &= \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3} \end{aligned}$$

63. (a) $a_n = 120{,}000(0.7)^n$

(b) $a_5 = 120{,}000(0.7)^5 = \$20{,}168.40$

65. 1. When $n = 1, 3 = 1(1 + 2)$.

2. Assume that $S_k = 3 + 5 + 7 + \cdots + (2k + 1) = k(k + 2)$.

Then, $$\begin{aligned} S_{k+1} &= 3 + 5 + 7 + \cdots + (2k + 1) + \left[2(k + 1) + 1\right] = S_k + (2k + 3) \\ &= k(k + 2) + 2k + 3 \\ &= k^2 + 4k + 3 \\ &= (k + 1)(k + 3) \\ &= (k + 1)\left[(k + 1) + 2\right]. \end{aligned}$$

So, by mathematical induction, the formula is valid for all positive integer values of n.

67. 1. When $n = 1, a = a\left(\dfrac{1 - r}{1 - r}\right).$

2. Assume that $S_k = \displaystyle\sum_{i=0}^{k-1} ar^i = \frac{a(1 - r^k)}{1 - r}.$

Then

$$S_{k+1} = \sum_{i=0}^{k} ar^i = \left(\sum_{i=0}^{k-1} ar^i\right) + ar^k = \frac{a(1 - r^k)}{1 - r} + ar^k$$

$$= \frac{a(1 - r^k + r^k - r^{k+1})}{1 - r} = \frac{a(1 - r^{k+1})}{1 - r}.$$

So, by mathematical induction, the formula is valid for all positive integer values of *n*.

69. $S_1 = 9 = 1(9) = 1[2(1) + 7]$

$S_2 = 9 + 13 = 22 = 2(11) = 2[2(2) + 7]$

$S_3 = 9 + 13 + 17 = 39 = 3(13) = 3[2(3) + 7]$

$S_4 = 9 + 13 + 17 + 21 = 60 = 4(15) = 4[2(4) + 7]$

$S_n = n(2n + 7)$

1. When $n = 1, S_1 = 9 = 1(2 + 7)$

2. Assume $S_k = 9 + 13 + \ldots + [4k + 5] = k(2k + 7)$

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= k(2k + 7) + [4(k + 1) + 5] \\ &= 2k^2 + 7k + 4k + 9 \\ &= 2k^2 + 11k + 9 \\ &= (k + 1)(2k + 9) \\ &= (k + 1)[2(k + 1) + 7] \end{aligned}$$

So, the formula holds for all positive integers *n*.

71. $S_1 = 1$

$S_2 = 1 + \dfrac{3}{5} = \dfrac{8}{5}$

$S_3 = 1 + \dfrac{3}{5} + \dfrac{9}{25} = \dfrac{49}{25}$

$S_4 = 1 + \dfrac{3}{5} + \dfrac{9}{25} + \dfrac{27}{125} = \dfrac{272}{125}$

From these sums, there is no apparent pattern. Because the series is geometric, the formula for the sum is

$$S_n = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^n\right].$$

1. When $n = 1, S_1 = 1 = \dfrac{5}{2}\left[1 - \left(\dfrac{3}{5}\right)^1\right]$

2. Assume

$$S_k = 1 + \frac{3}{5} + \frac{9}{25} + \cdots + \left(\frac{3}{5}\right)^{k-1} = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^k\right]$$

$$= \frac{1 - \left(\frac{3}{5}\right)^k}{1 - \frac{3}{5}}$$

$$\begin{aligned} S_{k+1} = S_k + a_{k+1} &= \frac{1 - \left(\frac{3}{5}\right)^k}{1 - \frac{3}{5}} + \left(\frac{3}{5}\right)^{(k+1)-1} \\ &= \frac{1 - \left(\frac{3}{5}\right)^k}{1 - \frac{3}{5}} + \left(\frac{3}{5}\right)^k \\ &= \frac{1 - \left(\frac{3}{5}\right)^k + \left(1 - \frac{3}{5}\right)\left(\frac{3}{5}\right)^k}{1 - \frac{3}{5}} \\ &= \frac{1 - \left(\frac{3}{5}\right)^k + \left(\frac{3}{5}\right)^k - \left(\frac{3}{5}\right)^{k+1}}{1 - \frac{3}{5}} \\ &= \frac{1 - \left(\frac{3}{5}\right)^{k+1}}{1 - \frac{3}{5}} \\ &= \frac{5}{2}\left(1 - \left(\frac{3}{5}\right)^{k+1}\right) \end{aligned}$$

So, the formula holds for all positive integers *n*.

73. $\displaystyle\sum_{n=1}^{75} n = \frac{75(76)}{2} = 2850$

75. $a_1 = f(1) = 5, a_n = a_{n-1} + 5$

$a_1 = 5$

$a_2 = 5 + 5 = 10$

$a_3 = 10 + 5 = 15$

$a_4 = 15 + 5 = 20$

$a_5 = 20 + 5 = 25$

$$\begin{array}{rccccccccc} n: & 1 & & 2 & & 3 & & 4 & & 5 \\ a_n: & 5 & & 10 & & 15 & & 20 & & 25 \\ \text{First differences:} & & 5 & & 5 & & 5 & & 5 & \\ \text{Second differences:} & & & 0 & & 0 & & 0 & & \end{array}$$

Because the first differences are all the same, the sequence has a linear model.

77. ${}_6C_4 = \dfrac{6!}{2!4!} = 15$

79.

$$\begin{array}{ccccccccccccccc} &&&&&&&1&&&&&&& \\ &&&&&&1&&1&&&&&& \\ &&&&&1&&2&&1&&&&& \\ &&&&1&&3&&3&&1&&&& \\ &&&1&&4&&6&&4&&1&&& \\ &&1&&5&&10&&10&&5&&1&& \\ &1&&6&&15&&20&&15&&6&&1& \\ 1&&7&&\textcircled{21}&&35&&35&&21&&7&&1 \end{array}$$

$\dbinom{7}{2} = 21$, the 3rd entry in the 7th row.

81. $(x + 4)^4 = x^4 + 4x^3(4) + 6x^2(4)^2 + 4x(4)^3 + 4^4 = x^4 + 16x^3 + 96x^2 + 256x + 256$

83.
$$\begin{aligned} (4 - 5x)^3 &= {}_3C_0(4^3) + {}_3C_1(4^2)(-5x) + {}_3C_2(4)(-5x)^2 + {}_3C_3(-5x)^3 \\ &= 4^3 - 3(4)^2(5x) + 3(4)(5x)^2 - (5x)^3 \\ &= 64 - 240x + 300x^2 + 125x^3 \end{aligned}$$

85. The composite numbers between 1 and 14 are 4, 6, 8, 9, 10, 12, and 14.

There are 7 ways to select a composite number from the integers 1 to 14.

87. $(10)(10)(10)(10) = 10{,}000$ different telephone numbers

89. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

91. ${}_{32}C_{12} = \dfrac{32!}{20!12!} = 225{,}792{,}840$

93. (a) $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{2}{10} = \dfrac{1}{5} = 0.2$

(b) $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{6}{10} = \dfrac{3}{5} = 0.6$

95. (a) $25\% + 18\% = 43\%$

(b) $100\% - 18\% = 82\%$

97. $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{1296}$

99. $1 - \frac{13}{52} = 1 - \frac{1}{4} = \frac{3}{4}$

101. False.
$$\begin{aligned} \frac{(n + 2)!}{n!} &= \frac{(n + 2)(n + 1)\cancel{n!}}{\cancel{n!}} \\ &= (n + 2)(n + 1) \\ &\neq \frac{n + 2}{n} \end{aligned}$$

103. True. $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$ by the Properties of Sums.

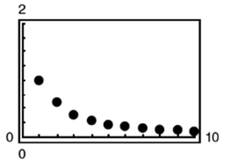
105. The domain of an infinite sequence is the set of natural numbers.

107. Each term of the sequence is defined in terms of preceding terms.

Problem Solving for Chapter 9

1. $a_n = \dfrac{n + 1}{n^2 + 1}$

(a)



(b) $a_n \to 0$ as $n \to \infty$

(c)

n	1	10	100	1000	10,000
a_n	1	0.1089	0.0101	0.0010	0.0001

(d) $a_n \to 0$ as $n \to \infty$

3. Distance: $\sum_{n=1}^{\infty} 20\left(\frac{1}{2}\right)^{n-1} = \frac{20}{1-\frac{1}{2}} = 40$

Time: $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$

In two seconds, both Achilles and the tortoise will be 40 feet away from Achilles' starting point.

5. Let $a_n = dn + c,$ an arithmetic sequence with a common difference of *d*.

(a) If *C* is added to each term, then the resulting sequence, $b_n = a_n + C = dn + c + C,$ is still arithmetic with a common difference of *d*.

(b) If each term is multiplied by a nonzero constant *C*, then the resulting sequence, $b_n = C(dn + c) = Cdn + Cc,$ is still arithmetic. The common difference is *Cd*.

(c) If each term is squared, the resulting sequence, $b_n = a_n^2 = (dn + c)^2,$ is not arithmetic.

7. $a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1 & \text{if } a_{n-1} \text{ is odd} \end{cases}$

(a) $a_1 = 7$

$a_2 = 3(7) + 1 = 22$

$a_3 = \frac{22}{3} = 11$

$a_4 = 3(11) + 1 = 34$

$a_5 = \frac{34}{2} = 17$

$a_6 = 3(17) + 1 = 52$

$a_7 = \frac{52}{2} = 26$

$a_8 = \frac{26}{2} = 13$

$a_9 = 3(13) + 1 = 40$

$a_{10} = \frac{40}{2} = 20$

$a_{11} = \frac{20}{2} = 10$

$a_{12} = \frac{10}{2} = 5$

$a_{13} = 3(5) + 1 = 16$

$a_{14} = \frac{16}{2} = 8$

$a_{15} = \frac{8}{2} = 4$

$a_{16} = \frac{4}{2} = 2$

$a_{17} = \frac{2}{2} = 1$

$a_{18} = 3(1) + 1 = 4$

$a_{19} = \frac{4}{2} = 2$

$a_{20} = \frac{2}{2} = 1$

(b) $a_1 = 4$

$a_2 = 2$

$a_3 = 1$

$a_4 = 4$

$a_5 = 2$

$a_6 = 1$

$a_7 = 4$

$a_8 = 2$

$a_9 = 1$

$a_{10} = 4$

$a_1 = 5$

$a_2 = 16$

$a_3 = 8$

$a_4 = 4$

$a_5 = 2$

$a_6 = 1$

$a_7 = 4$

$a_8 = 2$

$a_9 = 1$

$a_{10} = 4$

$a_1 = 12$

$a_2 = 6$

$a_3 = 3$

$a_4 = 10$

$a_5 = 5$

$a_6 = 16$

$a_7 = 8$

$a_8 = 4$

$a_9 = 2$

$a_{10} = 1$

Eventually the terms repeat: 4, 2, 1

9. The numbers $1, 5, 12, 22, 35, 51, \ldots$ can be written recursively as $P_n = P_{n-1} + (3n - 2).$ Show that $P_n = n(3n - 1)/2.$

1. For $n = 1$: $1 = \frac{1(3-1)}{2}$

2. Assume $P_k = \frac{k(3k-1)}{2}.$

$$\begin{aligned} \text{Then, } P_{k+1} &= P_k + [3(k+1) - 2] \\ &= \frac{k(3k-1)}{2} + (3k+1) = \frac{k(3k-1) + 2(3k+1)}{2} \\ &= \frac{3k^2 + 5k + 2}{2} = \frac{(k+1)(3k+2)}{2} \\ &= \frac{(k+1)[3(k+1) - 1]}{2}. \end{aligned}$$

So, by mathematical induction, the formula is valid for all integers $n \geq 1$.

11. Side lengths: $1, \frac{1}{2}, , \frac{1}{8}, \dots$

$$S_n = \left(\frac{1}{2}\right)^{n-1} \text{ for } n \geq 1$$

$$\text{Areas: } \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2, \frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^2, \frac{\sqrt{3}}{4}\left(\frac{1}{8}\right)^2, \dots$$

$$A_n = \frac{\sqrt{3}}{4}\left[\left(\frac{1}{2}\right)^{n-1}\right]^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^{2n-2} = \frac{\sqrt{3}}{4}S_n^2$$

13. $\frac{1}{3}$

15. (a) $$\text{Odds in favor of choosing a blue marble} = \frac{\text{number of blue marbles}}{\text{number of yellow marbles}} = \frac{3}{7}$$

$$\text{Odds against choosing a blue marble} = \frac{\text{number of yellow marbles}}{\text{number of blue marbles}} = \frac{7}{3}$$

(b) $$\text{Odds against choosing a red marble} = \frac{\text{number of non-red marbles}}{\text{number of red marbles}}$$

$$\frac{4}{1} = \frac{x}{6}$$

$$24 = x \quad (\text{number of non-red marbles})$$

$$\text{Total marbles} = 6 + 24 = 30$$

(c) $$P(E) = \frac{n(E)}{n(S)} = \frac{n(E)}{n(E) + n(E')} = \frac{n(E)/n(E')}{n(E)/n(E') + n(E')/n(E')}$$

$$P(E) = \frac{\text{odds in favor of } E}{\text{odds in favor of } E + 1}$$

(d) $$P(E) = \frac{n(E)}{n(S)} \qquad P(E') = \frac{n(E')}{n(S)}$$

$$n(S)P(E) = n(E) \qquad n(S)P(E') = n(E')$$

$$\text{Odds in favor of event } E = \frac{n(E)}{n(E')} = \frac{n(S)P(E)}{n(S)P(E')} = \frac{P(E)}{P(E')}$$

Practice Test for Chapter 9

1. Write out the first five terms of the sequence $a_n = \dfrac{2n}{(n+2)!}$.

2. Write an expression for the *n*th term of the sequence $\frac{4}{3}, \frac{5}{9}, \frac{6}{27}, \frac{7}{81}, \frac{8}{243}, \ldots$.

3. Find the sum $\displaystyle\sum_{i=1}^{6} (2i - 1)$.

4. Write out the first five terms of the arithmetic sequence where $a_1 = 23$ and $d = -2$.

5. Find a_n for the arithmetic sequence with $a_1 = 12, d = 3,$ and $n = 50$.

6. Find the sum of the first 200 positive integers.

7. Write out the first five terms of the geometric sequence with $a_1 = 7$ and $r = 2$.

8. Evaluate $\displaystyle\sum_{n=1}^{10} 6\left(\frac{2}{3}\right)^{n-1}$.

9. Evaluate $\displaystyle\sum_{n=0}^{\infty} (0.03)^n$.

10. Use mathematical induction to prove that $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$.

11. Use mathematical induction to prove that $n! > 2^n, n \geq 4$.

12. Evaluate ${}_{13}C_4$.

13. Expand $(x + 3)^5$.

14. Find the term involving x^7 in $(x - 2)^{12}$.

15. Evaluate ${}_{30}P_4$.

16. How many ways can six people sit at a table with six chairs?

17. Twelve cars run in a race. How many different ways can they come in first, second, and third place? (Assume that there are no ties.)

18. Two six-sided dice are tossed. Find the probability that the total of the two dice is less than 5.

19. Two cards are selected at random from a deck of 52 playing cards without replacement. Find the probability that the first card is a King and the second card is a black ten.

20. A manufacturer has determined that for every 1000 units it produces, 3 will be faulty. What is the probability that an order of 50 units will have one or more faulty units?

CHAPTER 10

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CHAPTER 10
Topics in Analytic Geometry

Section 10.1 Lines

1. inclination

3. The tangent of the angle between two nonperpendicular lines with slopes m_1 and m_2 is given by the formula

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

5. $m = \tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}$

7. $m = \tan\dfrac{3\pi}{4} = -1$

9. $m = \tan\dfrac{\pi}{3} = \sqrt{3}$

11. $m = \tan 0.39 \approx 0.4111$

13. $m = \tan 1.27 \approx 3.2236$

15. $m = \tan 1.81 \approx -4.1005$

17. $m = 1$

$$\begin{aligned}1 &= \tan\theta\\ \theta &= \frac{\pi}{4}\text{ radian} = 45^\circ\end{aligned}$$

19. $m = \dfrac{2}{3}$

$$\begin{aligned}\frac{2}{3} &= \tan\theta\\ \theta &= \arctan\left(\frac{2}{3}\right)\\ &\approx 0.5880\text{ radian} \approx 33.7^\circ\end{aligned}$$

21. $m = -1$

$$\begin{aligned}-1 &= \tan\theta\\ \theta &= 180^\circ + \arctan(-1)\\ &= \frac{3\pi}{4}\text{ radians} = 135^\circ\end{aligned}$$

23. $m = -\dfrac{3}{2}$

$$\begin{aligned}-\frac{3}{2} &= \tan\theta\\ \theta &= \tan^{-1}\left(-\frac{3}{2}\right) + \pi\\ &\approx 2.1588\text{ radians} \approx 123.7^\circ\end{aligned}$$

25. $\left(\sqrt{3}, 2\right), (0, 1)$

$$\begin{aligned}m &= \frac{1 - 2}{0 - \sqrt{3}} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}} &= \tan\theta\\ \theta &= \arctan\frac{1}{\sqrt{3}}\\ &= \frac{\pi}{6}\text{ radian} = 30^\circ\end{aligned}$$

27. $\left(-\sqrt{3}, -1\right), (0, -2)$

$$\begin{aligned}m &= \frac{-2 - (-1)}{0 - \left(-\sqrt{3}\right)} = \frac{-1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}} &= \tan\theta\\ \theta &= \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}\text{ radians} = 150^\circ\end{aligned}$$

29. $(6, 1), (10, 8)$

$$\begin{aligned}m &= \frac{8 - 1}{10 - 6} = \frac{7}{4}\\ \frac{7}{4} &= \tan\theta\\ \theta &= \arctan\frac{7}{4} \approx 1.0517\text{ radians} \approx 60.3^\circ\end{aligned}$$

31. $(-2, 20), (10, 0)$

$$\begin{aligned}m &= \frac{0 - 20}{10 - (-2)} = -\frac{20}{12} = -\frac{5}{3}\\ -\frac{5}{3} &= \tan\theta\\ \theta &= \pi + \arctan\left(-\frac{5}{3}\right) \approx 2.1112\text{ radians} \approx 121.0^\circ\end{aligned}$$

33. $\left(\frac{1}{4}, \frac{3}{2}\right), \left(\frac{1}{3}, \frac{1}{2}\right)$

$$m = \frac{\frac{1}{2} - \frac{3}{2}}{\frac{1}{3} - \frac{1}{4}} = -\frac{1}{\frac{1}{12}} = -12$$

$$-12 = \tan\theta$$

$$\begin{aligned}\theta &= \arctan(-12) + \pi \approx 1.6539 \text{ radians}\\ &\approx 94.8^\circ\end{aligned}$$

35. $2x + 2y - 5 = 0$

$$y = -x + \frac{5}{2} \Rightarrow m = -1$$

$$-1 = \tan\theta$$

$$\theta = \arctan(-1) = \frac{3\pi}{4} \text{ radians} = 135^\circ$$

37. $3x - 3y + 1 = 0$

$$y = x + \frac{1}{3} \Rightarrow m = 1$$

$$1 = \tan\theta$$

$$\theta = \arctan 1 = \frac{\pi}{4} \text{ radian} = 45^\circ$$

39. $x + \sqrt{3y} + 2 = 0$

$$y = -\frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}} \Rightarrow m = -\frac{1}{\sqrt{3}}$$

$$-\frac{1}{\sqrt{3}} = \tan\theta$$

$$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6} \text{ radians} = 150^\circ$$

41. $6x - 2y + 8 = 0$

$$y = 3x + 4 \Rightarrow m = 3$$

$$3 = \tan\theta$$

$$\theta = \arctan 3 \approx 1.2490 \text{ radians} \approx 71.6^\circ$$

43. $4x + 5y - 9 = 0$

$$y = -\tfrac{4}{5}x + \tfrac{9}{5} \Rightarrow m = -\tfrac{4}{5}$$

$$-\tfrac{4}{5} = \tan\theta$$

$$\begin{aligned}\theta &= \tan^{-1}\left(-\tfrac{4}{5}\right) + \pi\\ &\approx 2.4669 \text{ radians} \approx 141.3^\circ\end{aligned}$$

45. $3x + y = 3 \Rightarrow y = -3x + 3 \Rightarrow m_1 = -3$

$x - y = 2 \Rightarrow y = x - 2 \Rightarrow m_2 = 1$

$$\tan\theta = \left|\frac{1 - (-3)}{1 + (-3)(1)}\right| = 2$$

$$\theta = \arctan 2 \approx 1.1071 \text{ radians} \approx 63.4^\circ$$

47. $x - y = 0 \Rightarrow y = x \Rightarrow m_1 = 1$

$3x - 2y = -1 \Rightarrow y = \frac{3}{2}x + \frac{1}{2} \Rightarrow m_2 = \frac{3}{2}$

$$\tan\theta = \left|\frac{\frac{3}{2} - 1}{1 + \left(\frac{3}{2}\right)(1)}\right| = \frac{1}{5}$$

$$\theta = \arctan\frac{1}{5} \approx 0.1974 \text{ radian} \approx 11.3^\circ$$

49. $x - 2y = 7 \Rightarrow y = \frac{1}{2}x - \frac{7}{2} \Rightarrow m_1 = \frac{1}{2}$

$6x + 2y = 5 \Rightarrow y = -3x + \frac{5}{2} \Rightarrow m_2 = -3$

$$\tan\theta = \left|\frac{-3 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-3)}\right| = 7$$

$$\theta = \arctan 7 \approx 1.4289 \text{ radians} \approx 81.9^\circ$$

51. $x + 2y = 8 \Rightarrow y = -\frac{1}{2}x + 4 \Rightarrow m_1 = -\frac{1}{2}$

$x - 2y = 2 \Rightarrow y = \frac{1}{2}x - 1 \Rightarrow m_2 = \frac{1}{2}$

$$\tan\theta = \left|\frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{1 + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\right| = \frac{4}{3}$$

$$\theta = \arctan\left(\frac{4}{3}\right) \approx 0.9273 \text{ radian} \approx 53.1^\circ$$

53. $0.05x - 0.03y = 0.21 \Rightarrow y = \frac{5}{3}x - 7 \Rightarrow m_1 = \frac{5}{3}$

$0.07x + 0.02y = 0.16 \Rightarrow y = -\frac{7}{2}x + 8 \Rightarrow m_2 = -\frac{7}{2}$

$$\tan\theta = \left|\frac{\left(-\frac{7}{2}\right) - \left(\frac{5}{3}\right)}{1 + \left(\frac{5}{3}\right)\left(-\frac{7}{2}\right)}\right| = \frac{31}{29}$$

$$\theta = \arctan\left(\frac{31}{29}\right) \approx 0.8187 \text{ radian} \approx 46.9^\circ$$

55. Let $A = (1, 5), B = (3, 8)$, and $C = (4, 5)$.

Slope of AB: $m_1 = \frac{8 - 5}{3 - 1} = \frac{3}{2}$

Slope of BC: $m_2 = \frac{5 - 8}{4 - 3} = \frac{-3}{1} = -3$

Slope of AC: $m_3 = \frac{5 - 5}{4 - 1} = \frac{0}{3} = 0$

$$\tan A = \left|\frac{0 - \frac{3}{2}}{1 + \left(\frac{3}{2}\right)(0)}\right| = \frac{\frac{3}{2}}{1} = \frac{3}{2} \qquad \tan B = \left|\frac{\frac{3}{2} - (-3)}{1 + (-3)\left(\frac{3}{2}\right)}\right| = \frac{\frac{9}{2}}{\frac{7}{2}} = \frac{9}{7} \qquad \tan C = \left|\frac{-3 - 0}{1 + (0)(-3)}\right| = \frac{3}{1} = 3$$

$$A = \arctan\left(\frac{3}{2}\right) \approx 56.3^\circ \qquad B = \arctan\frac{9}{7} \approx 52.1^\circ \qquad C = \arctan 3 \approx 71.6^\circ$$

57. Let $A = (-4, -1), B = (3, 2)$, and $C = (1, 0)$.

Slope of AB: $m_1 = \frac{-1 - 2}{-4 - 3} = \frac{3}{7}$

Slope of BC: $m_2 = \frac{2 - 0}{3 - 1} = 1$

Slope of AC: $m_3 = \frac{-1 - 0}{-4 - 1} = \frac{1}{5}$

$$\tan A = \left|\frac{\frac{1}{5} - \frac{3}{7}}{1 + \left(\frac{3}{7}\right)\left(\frac{1}{5}\right)}\right| = \frac{\frac{8}{35}}{\frac{38}{35}} = \frac{4}{19}$$

$$A = \arctan\left(\frac{4}{19}\right) \approx 11.9^\circ$$

$$\tan B = \left|\frac{1 - \frac{3}{7}}{1 + \left(\frac{3}{7}\right)(1)}\right| = \frac{\frac{4}{7}}{\frac{10}{7}} = \frac{2}{5}$$

$$B = \arctan\left(\frac{2}{5}\right) \approx 21.8^\circ$$

$$\begin{aligned} C &= 180^\circ - A - B \\ &\approx 180^\circ - 11.9^\circ - 21.8^\circ = 146.3^\circ \end{aligned}$$

59. $(x_1, y_1) = (1, 2)$

$y = x + 2 \Rightarrow x - y + 2 = 0$

$$d = \frac{|(1)(1) + (-1)(2) + 2|}{\sqrt{1^2 + (-1)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

61. $(x_1, y_1) = (2, 3)$

$y = 2x - 3 \Rightarrow 2x - y - 3 = 0$

$$d = \frac{|2(2) + (-1)(3) + (-3)|}{\sqrt{2^2 + (-1)^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \approx 0.8944$$

63. $(x_1, y_1) = (-2, 4)$

$y = -x + 6 \Rightarrow x + y - 6 = 0$

$$d = \frac{|(1)(-2) + (1)(4) + (-6)|}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.8284$$

65. $(x_1, y_1) = (2, 3)$

$3x + y = 1 \Rightarrow 3x + y - 1 = 0$

$$d = \frac{|3(2) + (1)(3) + (-1)|}{\sqrt{3^2 + 1^2}} = \frac{8}{\sqrt{10}} = \frac{8\sqrt{10}}{10} = \frac{4\sqrt{10}}{5} \approx 2.5298$$

67. $(x_1, y_1) = (6, 2)$

$-3x + 4y = -5 \Rightarrow -3x + 4y + 5 = 0$

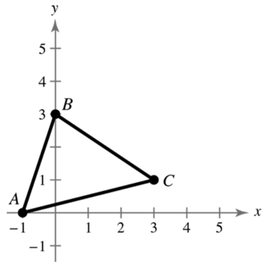
$$d = \frac{|(-3)(6) + (4)(2) + (5)|}{\sqrt{(-3)^2 + (4)^2}} = \frac{5}{\sqrt{25}} = 1$$

69. $(x_1, y_1) = (-2, 4)$

$4x + 3y = 5 \Rightarrow 4x + 3y - 5 = 0$

$$d = \frac{|(4)(-2) + (3)(4) + (-5)|}{\sqrt{4^2 + 3^2}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

71. (a)



(b) Slope of the line AC: $m = \frac{1 - 0}{3 - (-1)} = \frac{1}{4}$

Equation of the line AC: $y - 0 = \frac{1}{4}(x + 1)$

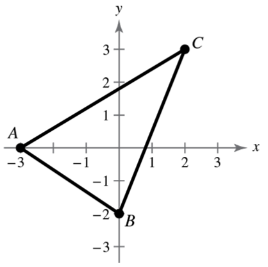
$x - 4y + 1 = 0$

Altitude from $B = (0, 3)$: $h = \frac{|(1)(0) + (-4)(3) + (1)|}{\sqrt{1^2 + (-4)^2}} = \frac{11}{\sqrt{17}} = \frac{11\sqrt{17}}{17}$

(c) Length of the base AC: $b = \sqrt{(3 + 1)^2 + (1 - 0)^2} = \sqrt{17}$

Area of the triangle: $A = \frac{1}{2}bh = \frac{1}{2}(\sqrt{17})\left(\frac{11}{\sqrt{17}}\right) = \frac{11}{2}$ units^2

73. (a)



(b) Slope of the line AC: $m = \frac{3 - 0}{2 + 3} = \frac{3}{5}$

Equation of the line AC: $y - 0 = \frac{3}{5}(x + 3)$

$3x - 5y + 9 = 0$

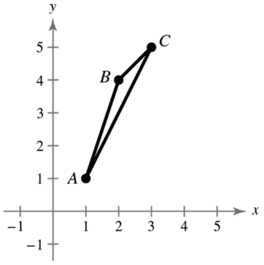
Altitude from $B = (0, -2)$: $h = \frac{|3(0) + (-5)(-2) + (9)|}{\sqrt{3^2 + (-5)^2}} = \frac{19}{\sqrt{34}} = \frac{19\sqrt{34}}{34}$

(c) Length of the base AC: $b = \sqrt{(2 + 3)^2 + (3 - 0)^2} = \sqrt{34}$

Area of the triangle: $A = \frac{1}{2}bh$

$= \frac{1}{2}(\sqrt{34})\left(\frac{19}{\sqrt{34}}\right) = \frac{19}{2}$ units^2

75. (a)



(b) Slope of the line AC: $m = \dfrac{5-1}{3-1} = \dfrac{4}{2} = 2$

Equation of the line AC: $y - 1 = 2(x - 1)$

$$2x - y - 1 = 0$$

Altitude from $B = (2, 4)$: $h = \dfrac{|(2)(2) + (-1)(4) + (-1)|}{\sqrt{2^2 + (-1)^2}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

(c) Length of the base AC: $b = \sqrt{(3-1)^2 + (5-1)^2} = \sqrt{20} = 2\sqrt{5}$

Area of the triangle: $A = \dfrac{1}{2}bh = \dfrac{1}{2}(2\sqrt{5})\left(\dfrac{\sqrt{5}}{5}\right) = 1 \text{ unit}^2$

77. $x + y = 1 \Rightarrow (0, 1)$ is a point on the line $\Rightarrow x_1 = 0$ and $y_1 = 1$

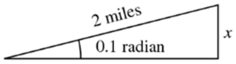
$x + y = 5 \Rightarrow A = 1, B = 1,$ and $C = -5$

$$d = \frac{|1(0) + 1(1) + (-5)|}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

79. Slope: $m = \tan 0.1 \approx 0.1003$

Change in elevation: $\sin 0.1 = \dfrac{x}{2(5280)}$

$$x \approx 1054 \text{ feet}$$



Not drawn to scale

81. (a)

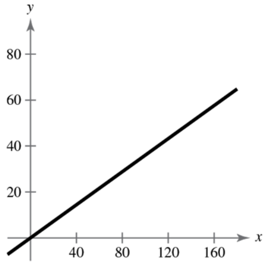
$$m = \tan\theta$$
$$0.36 = \tan\theta$$
$$\tan^{-1} 0.36 = \theta$$
$$\theta \approx 0.3456 \text{ radian, or } 19.8°$$

(b) Let z be the change in elevation.

$$\sin\theta = \frac{z}{170}$$
$$170\sin\theta = z$$
$$z = 170\sin(19.8°) \approx 57.6 \text{ ft}$$

(c) The slope is $m = 0.36$ and the y-intercept is $(0, 0)$. So, an equation is $y = 0.36x$.

(d)



83. True. The inclination of a line is related to its slope by $m = \tan\theta$. If the line has an inclination of 0 radians, then the slope is 0 radians.

85. False. Substitute $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ into the formula for the angle between two lines.

87. False. By definition, the inclination of a nonhorizontal line is the positive angle θ measured counterclockwise from the x-axis to the line. So, the angle θ can be acute, right or obtuse. The angle θ between two lines is less than $\pi/2$ because, if $\theta > \dfrac{\pi}{2}$, then $\tan\theta < 0$.

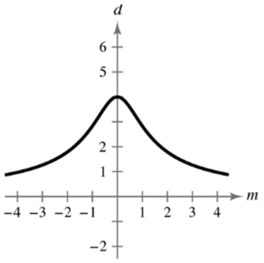
Because the formula for the angle between two lines involves absolute value, then $\tan\theta$ will always be positive. So, θ cannot be larger than $\pi/2$.

89. (a) $(0, 0) \Rightarrow x_1 = 0$ and $y_1 = 0$

$y = mx + 4 \Rightarrow 0 = mx - y + 4$

$$d = \frac{|m(0) + (-1)(0) + 4|}{\sqrt{m^2 + (-1)^2}} = \frac{4}{\sqrt{m^2 + 1}}$$

(b)



(c) The maximum distance of 4 occurs when the slope m is 0 and the line through $(0, 4)$ is horizontal.

(d) The graph has a horizontal asymptote at $d = 0$. As the slope becomes larger, the distance between the origin and the line, $y = mx + 4$, becomes smaller and approaches 0.

91. $x^2 + 12x + 35 = (x + 5)(x + 7)$

93. $x^2 - 20x + 96 = (x - 12)(x - 8)$

95. $11x^2 - 33x - 44 = 11(x^2 - 3x - 4) = 11(x - 4)(x + 1)$

97. $f(x) = x^2 - 13x + 40$

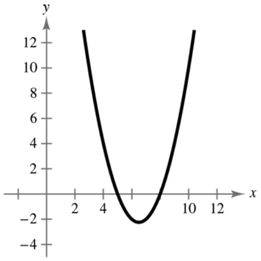
$$= \left(x^2 - 13x + \frac{169}{4}\right) + 40 - \frac{169}{4}$$

$$= \left(x - \frac{13}{2}\right)^2 - \frac{9}{4}$$

Vertex: $\left(\frac{13}{2}, \frac{-9}{4}\right)$

Axis of symmetry: $x = \frac{13}{2}$

$f(x) = (x - 5)(x - 8) \Rightarrow x$−intercepts: $(5, 0), (8, 0)$



99. $f(x) = x^2 + 29x + 208$

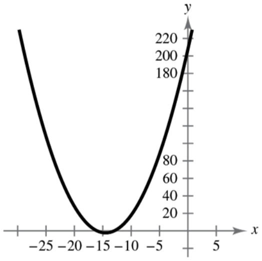
$$= \left(x^2 + 29x + \tfrac{841}{4}\right) + 208 - \tfrac{841}{4}$$

$$= \left(x + \tfrac{29}{2}\right)^2 - \tfrac{9}{4}$$

Vertex: $\left(\frac{-29}{2}, \frac{-9}{4}\right)$

Axis of symmetry: $x = \frac{-29}{2}$

$f(x) = (x + 13)(x + 16) \Rightarrow x$−intercepts: $(-13, 0), (-16, 0)$



101. $f(x) = x^2 + \frac{7}{3}x - \frac{98}{9}$

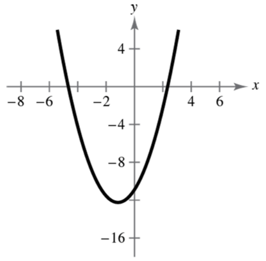
$$= \left(x^2 + \tfrac{7}{3}x + \tfrac{49}{36}\right) - \tfrac{98}{9} - \tfrac{49}{36}$$

$$= \left(x + \tfrac{7}{6}\right)^2 - \tfrac{49}{4}$$

Vertex: $\left(\frac{-7}{6}, \frac{-49}{4}\right)$

Axis of symmetry: $x = \frac{-7}{6}$

$f(x) = \frac{1}{9}(3x - 7)(3x + 14) \Rightarrow x$−intercepts: $\left(\frac{7}{3}, 0\right), \left(\frac{-14}{3}, 0\right)$



103. $(8, 19), (5, 14)$

(a) $\sqrt{(8 - 5)^2 + (19 - 14)^2} = \sqrt{3^2 + 5^2} = \sqrt{34}$

(b) $\left(\frac{8 + 5}{2}, \frac{19 + 14}{2}\right) = \left(\frac{13}{2}, \frac{33}{2}\right)$

105. $(-17, 11), (-19, -10)$

(a) $\sqrt{(-17 - (-19))^2 + [11 - (-10)]^2}$

$$= \sqrt{2^2 + 21^2}$$

$$= \sqrt{445}$$

(b) $\left(\frac{-17 - 19}{2}, \frac{11 - 10}{2}\right) = \left(-18, \frac{1}{2}\right)$

107. $\left(\frac{1}{2}, \frac{-5}{7}\right), \left(\frac{3}{2}, \frac{-10}{7}\right)$

(a) $\sqrt{\left(\frac{1}{2} - \frac{3}{2}\right)^2 + \left(\frac{-5}{7} + \frac{10}{7}\right)^2}$

$$= \sqrt{1 + \tfrac{25}{49}}$$

$$= \sqrt{\tfrac{74}{49}}$$

$$= \frac{\sqrt{74}}{7}$$

(b) $\left(\frac{\frac{1}{2} + \frac{3}{2}}{2}, \frac{\frac{-5}{7} - \frac{10}{7}}{2}\right)$

$$= \left(1, \tfrac{-15}{14}\right)$$

Section 10.2 Introduction to Conics: Parabolas

1. conic

3. parabola; directrix; focus

5. The focal chord passes through the focus and has endpoints on the parabola.

7. The vertex is the midpoint between the focus and directrix.

9. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

The graph opens to the right because p is positive. So, the equation matches graph (c).

10. $x^2 = 2y$

Vertex: $(0, 0)$

$p = \dfrac{1}{2} > 0$

The graph opens upward because p is positive. So, the equation matches graph (a).

11. $x^2 = -8y$

Vertex: $(0, 0)$

$p = -2 < 0$

The graph opens downward because p is negative. So, the equation matches graph (b).

12. $y^2 = -12x$

Vertex: $(0, 0)$

$p = -3 < 0$

The graph opens to the left because p is negative. So, the equation matches graph (d).

13. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Graph opens upward.

$x^2 = 4py$

Focus: $(0, 1)$

$p = 1$

$x^2 = 4(1)y$

$x^2 = 4y$

15. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Focus: $\left(0, \tfrac{1}{2}\right) \Rightarrow p = \tfrac{1}{2}$

$x^2 = 4py$

$x^2 = 4\left(\tfrac{1}{2}\right)y$

$x^2 = 2y$

17. Focus: $(-2, 0) \Rightarrow p = -2$

$y^2 = 4px$

$y^2 = 4(-2)x$

$y^2 = -8x$

19. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Directrix: $y = 2 \Rightarrow p = -2$

$x^2 = 4py$

$x^2 = 4(-2)y$

$x^2 = -8y$

21. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Directrix: $x = -1 \Rightarrow p = 1$

$y^2 = 4px$

$y^2 = 4(1)x$

$y^2 = 4x$

23. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Vertical axis
Passes through: $(4, 6)$

$x^2 = 4py$

$4^2 = 4p(6)$

$16 = 24p$

$p = \tfrac{2}{3}$

$x^2 = 4\left(\tfrac{2}{3}\right)y$

$x^2 = \tfrac{8}{3}y$

25. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Horizontal axis

Passes through: $(-2, 5)$

$$\begin{aligned} y^2 &= 4px \\ 5^2 &= 4p(-2) \\ 25 &= -8p \\ p &= -\tfrac{25}{8} \\ y^2 &= 4\left(-\tfrac{25}{8}\right)x \\ y^2 &= -\tfrac{25}{2}x \end{aligned}$$

27. Vertex: $(-4, 0) \Rightarrow h = -4, k = 0$

Focus: $(-3, 0) \Rightarrow p = 1$

$$\begin{aligned} (y - k)^2 &= 4p(x - h) \\ [y - (0)]^2 &= 4(1)[x - (-4)] \\ y^2 &= 4(x + 4) \end{aligned}$$

29. Vertex: $(6, 3) \Rightarrow h = 6, k = 3$

Focus: $(4, 3) \Rightarrow p = -2$

$$\begin{aligned} (y - k)^2 &= 4p(x - h) \\ (y - 3)^2 &= 4(-2)(x - 6) \\ (y - 3)^2 &= -8(x - 6) \end{aligned}$$

31. Vertex: $(0, 2)$

Directrix: $y = 4$

Vertical axis

$$\begin{aligned} p &= 2 - 4 = -2 \\ (x - 0)^2 &= 4(-2)(y - 2) \\ x^2 &= -8(y - 2) \end{aligned}$$

33. Focus: $(2, 2)$

Directrix: $x = -2$

Horizontal axis

Vertex: $(0, 2)$

$$\begin{aligned} p &= 2 - 0 = 2 \\ (y - 2)^2 &= 4(2)(x - 0) \\ (y - 2)^2 &= 8x \end{aligned}$$

35. Vertex: $(3, -3) \Rightarrow h = 3, k = -3$

Vertical Axis; Passes through $(0, 0)$

$$\begin{aligned} (x - h)^2 &= 4p(y - k) \\ (x - 3)^2 &= 4p(y + 3) \\ (0 - 3)^2 &= 4p(0 + 3) \\ 9 &= 12p \\ p &= \frac{3}{4} \\ (x - 3)^2 &= 3(y + 3) \end{aligned}$$

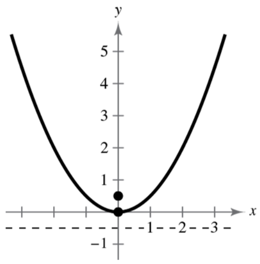
37. $x^2 = 2y$

$x^2 = 4\left(\tfrac{1}{2}\right)y \Rightarrow h = 0, k = 0, p = \tfrac{1}{2}$

Vertex: $(0, 0)$

Focus: $\left(0, \tfrac{1}{2}\right)$

Directrix: $y = -\tfrac{1}{2}$



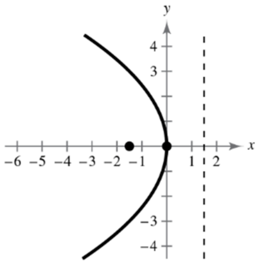
39. $y^2 = -6x$

$y^2 = 4\left(-\tfrac{3}{2}\right)x \Rightarrow h = 0, k = 0, p = -\tfrac{3}{2}$

Vertex: $(0, 0)$

Focus: $\left(-\tfrac{3}{2}, 0\right)$

Directrix: $x = \tfrac{3}{2}$



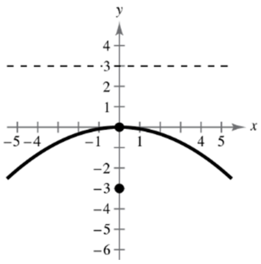
41. $x^2 + 12y = 0$

$x^2 = -12y = 4(-3)y \Rightarrow h = 0, k = 0, p = -3$

Vertex: $(0, 0)$

Focus: $(0, -3)$

Directrix: $y = 3$



43. $(x-1)^2 + 8(y+2) = 0$

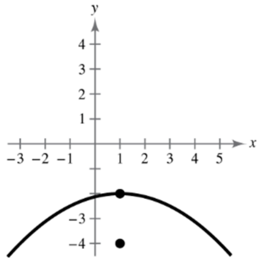
$$(x-1)^2 = 4(-2)(y+2)$$

$h = 1, k = -2, p = -2$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



45. $(y+7)^2 = 4\left(x - \frac{3}{2}\right)$

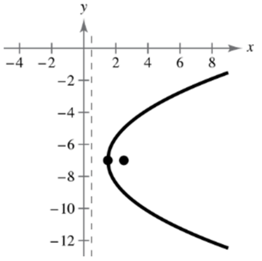
$(y+7)^2 = 4(1)\left(x - \frac{3}{2}\right)$

$h = \frac{3}{2}, k = -7, p = 1$

Vertex: $\left(\frac{3}{2}, -7\right)$

Focus: $\left(\frac{5}{2}, -7\right)$

Directrix: $x = \frac{1}{2}$



47.

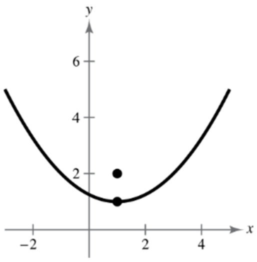
$$y = \tfrac{1}{4}(x^2 - 2x + 5)$$
$$4y = x^2 - 2x + 5$$
$$4y - 5 + 1 = x^2 - 2x + 1$$
$$4y - 4 = (x-1)^2$$
$$(x-1)^2 = 4(1)(y-1)$$

$h = 1, k = 1, p = 1$

Vertex: $(1, 1)$

Focus: $(1, 2)$

Directrix: $y = 0$



49. $y^2 + 6y + 8x + 25 = 0$

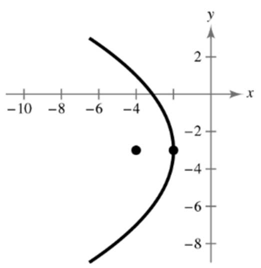
$$y^2 + 6y + 9 = -8x - 25 + 9$$
$$(y+3)^2 = 4(-2)(x+2)$$

$h = -2, k = -3, p = -2$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



51. $x^2 + 4x - 6y = -10$

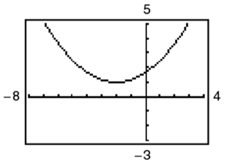
$$x^2 + 4x + 4 = 6y - 10 + 4$$
$$x^2 + 4x + 4 = 6y - 6$$
$$(x+2)^2 = 6(y-1)$$
$$(x+2)^2 = 4\left(\frac{3}{2}\right)(y-1)$$

$h = -2, k = 1, p = \frac{3}{2}$

Vertex: $(-2, 1)$

Focus: $\left(-2, \frac{5}{2}\right)$

Directrix: $y = -\frac{1}{2}$



53. $y^2 + x + y = 0$

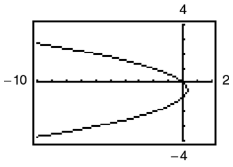
$$y^2 + y + \tfrac{1}{4} = -x + \tfrac{1}{4}$$
$$\left(y + \tfrac{1}{2}\right)^2 = 4\left(-\tfrac{1}{4}\right)\left(x - \tfrac{1}{4}\right)$$

$h = \frac{1}{4}, k = -\frac{1}{2}, p = -\frac{1}{4}$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$



55. $x^2 = 8y$

$x^2 = 4(2)y \Rightarrow p = 2$

Focus: $(0, 2)$

$$d_1 = 2 - b$$
$$d_2 = \sqrt{(6-0)^2 + \left(\frac{9}{2} - 2\right)^2} = \sqrt{36 + \frac{25}{4}}$$
$$2 - b = \frac{13}{2}$$
$$b = -\frac{9}{2}$$
$$m = \frac{-(9/2) - (9/2)}{0 - 6} = \frac{3}{2}$$

Tangent line: $y = \frac{3}{2}x - \frac{9}{2}$

57. $x^2 = 2y$

$x^2 = 4\left(\frac{1}{2}\right)y \Rightarrow p = \frac{1}{2}$

Focus: $\left(0, \frac{1}{2}\right)$

Point: $(-4, 8)$

$d_1 = \frac{1}{2} - b$

$d_2 = \sqrt{(-4 - 0)^2 + \left(8 - \frac{1}{2}\right)^2} = \frac{17}{2}$

$d_1 = d_2 \Rightarrow b = -8$

$m = \frac{8 - (-8)}{-4 - 0} = -4$

Tangent line: $y = -4x - 8$

59. $y = -2x^2$

$x^2 = -\frac{1}{2}y \Rightarrow p = -\frac{1}{8}$

Point: $(-1, -2)$

Focus: $\left(0, -\frac{1}{8}\right)$

$d_1 = b - \left(-\frac{1}{8}\right) = b + \frac{1}{8}$

$d_2 = \sqrt{(-1 - 0)^2 + \left(-2 - \left(-\frac{1}{8}\right)\right)^2}$

$= \frac{17}{8}$

$d_1 = d_2 \Rightarrow b = 2$

$m = \frac{-2 - 2}{-1 - 0} = 4$

$y = 4x + 2$

61. $y^2 = 4px, p = 1.5$

$y^2 = 4(1.5)x$

$y^2 = 6x$

63. Vertex: $(0, 0)$

$(y - 0)^2 = 4p(x - 0)$

$y^2 = 4px$

At $(1000, 800)$: $800^2 = 4p(1000) \Rightarrow p = 160$

$y^2 = 4(160)x$

$y^2 = 640x$

65. (a) $x^2 = 4py$

$32^2 = 4p\left(\frac{1}{12}\right)$

$1024 = \frac{1}{3}p$

$3072 = p$

$x^2 = 4(3072)y$

$x^2 = 12{,}288y$ (in feet)

(b) $\frac{1}{24} = \frac{x^2}{12{,}288}$

$\frac{12{,}288}{24} = x^2$

$512 = x^2$

$x \approx 22.6$ feet

67. Vertex: $(0, 48) \Rightarrow h = 0, k = 48$

Passes through $\left(10\sqrt{3}, 0\right)$

Vertical axis

$(x - 0)^2 = 4p(y - 48)$

$\left(10\sqrt{3} - 0\right)^2 = 4p(0 - 48)$

$300 = -192p$

$-\frac{25}{16} = p$

$x^2 = 4\left(-\frac{25}{16}\right)(y - 48)$

$x^2 = -\frac{25}{4}(y - 48)$

69. $x^2 = 4p(y - 12)$

$(4, 10)$ on curve:

$16 = 4p(10 - 12) = -8p \Rightarrow p = -2$

$x^2 = 4(-2)(y - 12) = -8y + 96$

$y = \frac{-x^2 + 96}{8}$

$y = 0$ if $x^2 = 96 \Rightarrow x = 4\sqrt{6}$

So, the width is about $2\left(4\sqrt{6}\right) \approx 19.6$ meters.

71. (a) $x^2 = 4py$

$60^2 = 4p(20) \Rightarrow p = 45$

Focus: $(0, 45)$

(b) $x^2 = 4(45)y$ or $y = \frac{1}{180}x^2$

73. (a) $V = 17{,}500\sqrt{2}$ mi/h

$\approx 24{,}750$ mi/h

(b) $p = -4100, (h, k) = (0, 4100)$

$$(x - 0)^2 = 4(-4100)(y - 4100)$$

$$x^2 = -16{,}400(y - 4100)$$

75. (a) $x^2 = -\dfrac{v^2}{16}(y - s)$

$$x^2 = -\frac{(28)^2}{16}(y - 100)$$

$$x^2 = -49(y - 100)$$

(b) The egg hits the ground when $y = 0$.

$$x^2 = -49(0 - 100)$$

$$x^2 = 4900$$

$$x = 70$$

The egg travels 70 feet.

77. False. It is not possible for a parabola to intersect its directrix. If the graph crossed the directrix there would exist points closer to the directrix than the focus.

79. $x^2 = 4py$, (x_1, y_1) on parabola

$$y - y_1 = \frac{x_1}{2p}(x - x_1)$$

Slope: $m = \dfrac{x_1}{2p}$

$$x = 0 \Rightarrow y - y_1 = \frac{x_1}{2p}(-x_1)$$

$$\Rightarrow y = y_1 - \frac{x_1^2}{(2p)} = \frac{2py_1 - x_1^2}{2p}$$

y-intercept: $\left(0, \dfrac{2py_1 - x_1^2}{2p}\right)$

$$y = 0 \Rightarrow -y_1 = \frac{x_1}{2p}(x - x_1) = \frac{x_1}{2p}x - \frac{x_1^2}{2p}$$

$$\Rightarrow \frac{x_1^2}{(2p)} - y_1 = \frac{x_1}{2p}x$$

$$\Rightarrow x = x_1 - \frac{2py_1}{x_1} = \frac{x_1^2 - 2py_1}{x_1}$$

x-intercept: $\left(\dfrac{x_1^2 - 2py_1}{x_1}, 0\right)$

81. Both (a) and (b) are parabolas with vertical axes, while (c) is a parabola with a horizontal axis.

So, equations (a) and (b) are equivalent when $p = \dfrac{1}{4a}$.

(a) $y = a(x - h)^2 + k$

(b)

$$(x - h)^2 = 4p(y - k)$$

$$(x - h)^2 = 4py - 4pk$$

$$(1/4p)\left((x - h)^2 + 4pk\right) = 4py(1/4p)$$

$$(1/4p)(x - h)^2 + k = y = a(x - h)^2 + k$$

$$a = \left(\frac{1}{4p}\right)$$

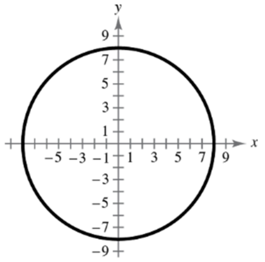
$$4a = (1/p)$$

$$p = \frac{1}{4a}$$

83. $x^2 + y^2 = 64 = 8^2$

center: $(0, 0)$

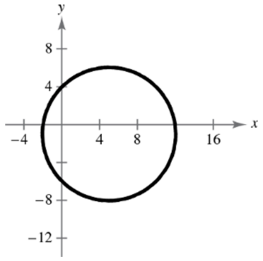
radius: $r = 8$



85. $(x - 5)^2 + (y + 1)^2 = 50 = \left(5\sqrt{2}\right)^2$

center: $(5, -1)$

radius: $r = 5\sqrt{2}$



87. $\dfrac{4}{2x^{-3}} = \dfrac{4x^3}{2} = 2x^3, x \neq 0$

89. $\left(\dfrac{a^3}{2b^2}\right)^{-4} = \left(\dfrac{2b^2}{a^3}\right)^4 = \dfrac{16b^8}{a^{12}}, b \neq 0$

91. $f(x) = x^2$

$$y = f(x - 4) + 2 = (x - 4)^2 + 2$$

93. $f(x) = |x|$

$$y = f(-x + 3) - 1 = |-x + 3| - 1$$

Section 10.3 Ellipses

1. ellipse; foci

3. The two axes are the major axis and the minor axis. The major axis passes through the vertices.

5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

Center: $(0, 0)$

$a = 3, b = 2$

Vertical major axis

Matches graph (b).

6. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Center: $(0, 0)$

$a = 3, b = 2$

Horizontal major axis

Matches graph (c).

7. $\dfrac{(x - 2)^2}{16} + (y + 1)^2 = 1$

Center: $(2, -1)$

$a = 4, b = 1$

Horizontal major axis

Matches graph (a).

8. $\dfrac{(x + 2)^2}{9} + \dfrac{(y + 2)^2}{4} = 1$

Center: $(-2, -2)$

$a = 3, b = 2$

Horizontal major axis

Matches graph (d).

9. Center: $(0, 0)$

$a = 4, b = 2$

Vertical major axis

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$
$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

11. Center: $(0, 0)$

Vertices: $(\pm 7, 0) \Rightarrow a = 7$

Foci: $(\pm 2, 0) \Rightarrow c = 2$

$b^2 = a^2 - c^2 = 49 - 4 = 45$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{49} + \frac{y^2}{45} = 1$$

13. Center: $(0, 0)$

Foci: $(\pm 4, 0) \Rightarrow c = 4$

Length of horizontal major axis: $10 \Rightarrow a = 5$

$b^2 = a^2 - c^2 = 25 - 16 = 9$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

15. Major axis vertical

Passes through: $(0, 6)$ and $(3, 0)$

$a = 6, b = 3$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$
$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

17. Vertices: $(\pm 6, 0) \Rightarrow a = 6$

Major axis horizontal

Passes through: $(4, 1)$

$$\frac{x^2}{36} + \frac{y^2}{b^2} = 1$$
$$\frac{4^2}{36} + \frac{1^2}{b^2} = 1$$
$$16b^2 + 36 = 36b^2$$
$$36 = 20b^2$$
$$\frac{9}{5} = b^2$$
$$\frac{x^2}{36} + \frac{y^2}{\frac{9}{5}} = 1 \text{ or } \frac{x^2}{36} + \frac{5y^2}{9} = 1$$

19. Center: $(2, 3)$

$a = 3, b = 1$

Vertical major axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{1} + \frac{(y-3)^2}{9} = 1$$

21. Vertices: $(2, 0), (10, 0) \Rightarrow a = 4$

Horizontal major axis

Length of minor axis: $4 \Rightarrow b = 2$

Center: $(6, 0) \Rightarrow h = 6, k = 0$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-6)^2}{16} + \frac{(y-0)^2}{4} = 1$$

$$\frac{(x-6)^2}{16} + \frac{y^2}{4} = 1$$

23. Foci: $(0, 0), (0, 8) \Rightarrow c = 4$

Major axis of length: $16 \Rightarrow a = 8$

$b^2 = a^2 - c^2 = 64 - 16 = 48$

Center: $(0, 4) = (h, k)$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{x^2}{48} + \frac{(y-4)^2}{64} = 1$$

25. Center: $(1, 3)$

Vertex: $(-2, 3) \Rightarrow a = 3$

Major axis horizontal

Length of minor axis: $4 \Rightarrow b = 2$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1$$

27. Center: $(1, 4)$

Vertices: $(1, 0)$ and $(1, 8) \Rightarrow a = 4$

Major axis vertical

$$\begin{aligned} a &= 2c \\ 4 &= 2c \\ c &= 2 \\ b^2 &= a^2 - c^2 \\ b^2 &= 16 - 4 = 12 \end{aligned}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-1)^2}{12} + \frac{(y-4)^2}{16} = 1$$

29. Vertices: $(0, 2), (4, 2) \Rightarrow a = 2$

Center: $(2, 2)$

Endpoints of the minor axis: $(2, 3), (2, 1) \Rightarrow b = 1$

Horizontal major axis:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y-2)^2}{1} = 1$$

31. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

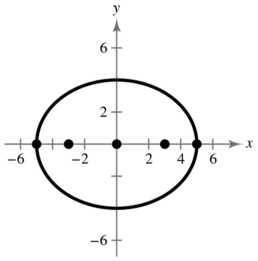
$a = 5, b = 4, c = 3$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $(\pm 3, 0)$

Eccentricity: $e = \dfrac{3}{5}$



33. $9x^2 + y^2 = 36$

$$\frac{x^2}{4} + \frac{y^2}{36} = 1$$

$a = 6, b = 2$

$c^2 = a^2 - b^2$

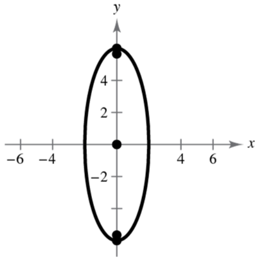
$= 36 - 4 = 32$

Center: $(0, 0)$

Vertices: $(0, \pm 6)$

Foci: $(0, \pm 4\sqrt{2})$

Eccentricity: $e = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$



35. $\frac{(x - 4)^2}{16} + \frac{(y + 1)^2}{25} = 1$

$a = 5, b = 4$

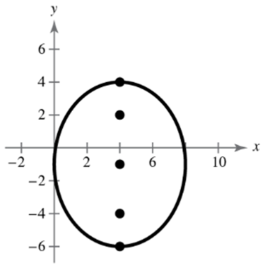
$c^2 = a^2 - b^2 = 25 - 16 = 9 \Rightarrow c = 3$

Center: $(4, -1)$

Vertices: $(4, 4), (4, -6)$

Foci: $(4, 2), (4, -4)$

Eccentricity: $e = \frac{3}{5}$



37. $\frac{(x + 5)^2}{9/4} + (y - 1)^2 = 1$

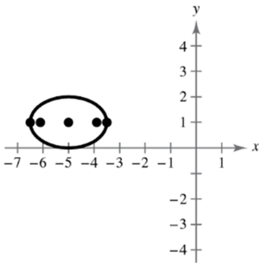
$a = \frac{3}{2}, b = 1, c = \frac{\sqrt{5}}{2}$

Center: $(-5, 1)$

Vertices: $\left(-\frac{7}{2}, 1\right), \left(-\frac{13}{2}, 1\right)$

Foci: $\left(-5 + \frac{\sqrt{5}}{2}, 1\right), \left(-5 - \frac{\sqrt{5}}{2}, 1\right)$

Eccentricity: $e = \frac{\sqrt{5}}{3}$



39. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$

$$9(x + 2)^2 + 4(y - 3)^2 = 36$$

$$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$$

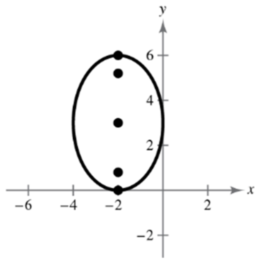
$a = 3, b = 2, c = \sqrt{5}$

Center: $(-2, 3)$

Vertices: $(-2, 6), (-2, 0)$

Foci: $(-2, 3 \pm \sqrt{5})$

Eccentricity: $e = \frac{\sqrt{5}}{3}$



41.
$$\begin{aligned} x^2 + 5y^2 - 8x - 30y - 39 &= 0 \\ \left(x^2 - 8x + 16\right) + 5\left(y^2 - 6y + 9\right) &= 39 + 16 + 45 \\ (x - 4)^2 + 5(y - 3)^2 &= 100 \\ \frac{(x - 4)^2}{100} + \frac{(y - 3)^2}{20} &= 1 \end{aligned}$$

$a = 10, b = \sqrt{20} = 2\sqrt{5},$

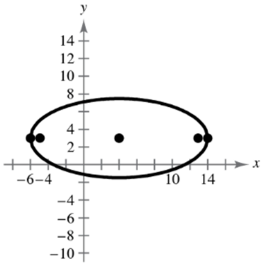
$c = \sqrt{80} = 4\sqrt{5}$

Center: $(4, 3)$

Foci: $\left(4 \pm 4\sqrt{5}, 3\right)$

Vertices: $(14, 3), (-6, 3)$

Eccentricity: $e = \dfrac{4\sqrt{5}}{10} = \dfrac{2\sqrt{5}}{5}$



43.
$$\begin{aligned} 6x^2 + 2y^2 + 18x - 10y + 2 &= 0 \\ 6\left(x^2 + 3x + \frac{9}{4}\right) + 2\left(y^2 - 5y + \frac{25}{4}\right) &= -2 + \frac{27}{2} + \frac{25}{2} \\ 6\left(x + \frac{3}{2}\right)^2 + 2\left(y - \frac{5}{2}\right)^2 &= 24 \\ \frac{\left(x + \frac{3}{2}\right)^2}{4} + \frac{\left(y - \frac{5}{2}\right)^2}{12} &= 1 \end{aligned}$$

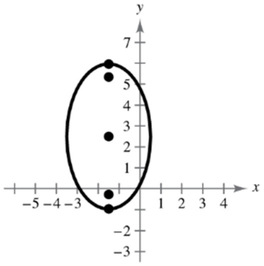
$a = \sqrt{12} = 2\sqrt{3}, b = 2, c = \sqrt{8} = 2\sqrt{2}$

Center: $\left(-\dfrac{3}{2}, \dfrac{5}{2}\right)$

Vertices: $\left(-\dfrac{3}{2}, \dfrac{5}{2} \pm 2\sqrt{3}\right)$

Foci: $\left(-\dfrac{3}{2}, \dfrac{5}{2} \pm 2\sqrt{2}\right)$

Eccentricity: $e = \dfrac{2\sqrt{2}}{2\sqrt{3}} = \dfrac{\sqrt{6}}{3}$



45.
$$\begin{aligned} 12x^2 + 20y^2 - 12x + 40y - 37 &= 0 \\ 12\left(x^2 - x + \frac{1}{4}\right) + 20\left(y^2 + 2y + 1\right) &= 37 + 3 + 20 \\ 12\left(x - \frac{1}{2}\right)^2 + 20(y + 1)^2 &= 60 \\ \frac{\left(x - \frac{1}{2}\right)^2}{5} + \frac{(y + 1)^2}{3} &= 1 \end{aligned}$$

$a^2 = 5, b^2 = 3$

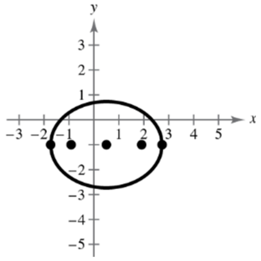
$c^2 = 5 - 3 = 2$

Center: $\left(\dfrac{1}{2}, -1\right)$

Vertices: $\left(\dfrac{1}{2} \pm \sqrt{5}, -1\right)$

Foci: $\left(\dfrac{1}{2} \pm \sqrt{2}, -1\right)$

Eccentricity: $e = \dfrac{\sqrt{2}}{\sqrt{5}} = \dfrac{\sqrt{10}}{5}$



47. $5x^2 + 3y^2 = 15$

$\dfrac{x^2}{3} + \dfrac{y^2}{5} = 1$

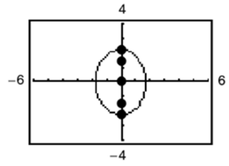
$a = \sqrt{5}, b = \sqrt{3}, c = \sqrt{2}$

Center: $(0, 0)$

Vertices: $\left(0, \pm\sqrt{5}\right)$

Foci: $\left(0, \pm\sqrt{2}\right)$

Eccentricity: $e = \dfrac{\sqrt{10}}{5}$



49.
$$\begin{aligned} x^2 + 9y^2 - 10x + 36y + 52 &= 0 \\ \left(x^2 - 10x + 25\right) + 9\left(y^2 + 4y + 4\right) &= -52 + 25 + 36 \\ (x - 5)^2 + 9(y + 2)^2 &= 9 \\ \frac{(x - 5)^2}{9} + \frac{(y + 2)}{1} &= 1 \end{aligned}$$

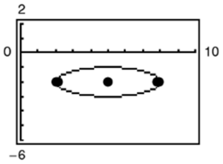
$a = 3, b = 1, c = 2\sqrt{2}$

Center: $(5, -2)$

Vertices: $(8, -2), (2, -2)$

Foci: $\left(5 \pm 2\sqrt{2}, -2\right)$

Eccentricity: $e = \dfrac{2\sqrt{2}}{3}$



51. Vertices: $(\pm 5, 0), a = 5, h = 0, k = 0$

Eccentricity: $e = \dfrac{3}{5} = \dfrac{c}{a}$

$$\frac{3}{5} = \frac{c}{5}$$

$$3 = c$$

$$b^2 = a^2 - c^2 = 25 - 9 = 16$$

Horizontal major axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

53. Foci: $(1, 1)$ and $(1, 13) \Rightarrow c = 6$

Eccentricity: $e = \dfrac{2}{3}$

$$e = \frac{c}{a} \Rightarrow \frac{2}{3} = \frac{6}{a}$$

$$2a = 18$$

$$a = 9$$

$$a^2 = b^2 + c^2$$

$$81 = b^2 + 36$$

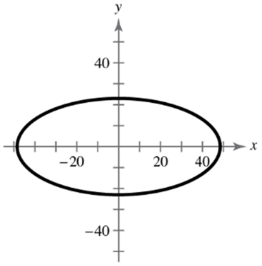
$$45 = b^2$$

The center is the midpoint between the foci, $(1, 7)$, and the major axis is vertical.

$$\frac{(x-1)^2}{b^2} + \frac{(y-7)^2}{a^2} = 1$$

$$\frac{(x-1)^2}{45} + \frac{(y-7)^2}{81} = 1$$

55. (a)



$$\frac{x^2}{2352.25} + \frac{y^2}{23^2} = 1 \text{ or } \frac{x^2}{529} + \frac{y^2}{2352.25} = 1$$

$$a = \frac{97}{2}, b = 23, c = \sqrt{\left(\frac{97}{2}\right)^2 - (23)^2} \approx 4.7$$

(b) Distance between foci: $2(4.7) \approx 85.4$ feet

57. The length of the major axis and minor axis are 280 millimeters and 160 millimeters, respectively. Therefore,

$$2a = 280 \Rightarrow a = 140 \text{ and } 2b = 160 \Rightarrow b = 80.$$

$$a^2 = b^2 + c^2$$

$$140^2 = 80^2 + c^2$$

$$13{,}200 = c^2$$

$$\sqrt{13{,}200} = c$$

$$20\sqrt{33} = c$$

The kidney stone and spark plug are each located at a focus, therefore they are 2c millimeters apart, or $2\left(20\sqrt{33}\right) = 40\sqrt{33} \approx 229.8$ millimeters apart.

59. $a + c = 6378 + 939 = 7317$

$a - c = 6378 + 215 = 6593$

Solving this system for a and c yields

$$a + c = 7317$$

$$a - c = 6593$$

$$2a = 13{,}910$$

$$a = 6955$$

$$6955 + c = 7317$$

$$c = 362$$

Eccentricity: $e = \dfrac{c}{a} = \dfrac{362}{6955} \approx 0.0520$

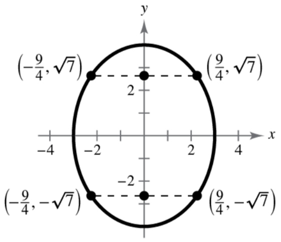
61. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

$a = 4, b = 3, c = \sqrt{7}$

Points on the ellipse: $(\pm 3, 0), (0, \pm 4)$

Length of latus recta: $\dfrac{2b^2}{a} = \dfrac{2(3)^2}{4} = \dfrac{9}{2}$

Additional points: $\left(\pm\dfrac{9}{4}, -\sqrt{7}\right), \left(\pm\dfrac{9}{4}, \sqrt{7}\right)$



63. $5x^2 + 3y^2 = 15$

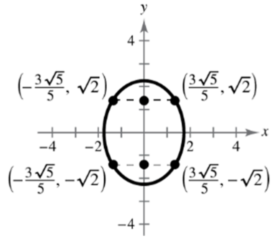
$$\frac{x^2}{3} + \frac{y^2}{5} = 1$$

$a = \sqrt{5}, b = \sqrt{3}, c = \sqrt{2}$

Points on the ellipse: $\left(\pm\sqrt{3}, 0\right), \left(0, \pm\sqrt{5}\right)$

Length of latus recta: $\dfrac{2b^2}{a} = \dfrac{2 \cdot 3}{\sqrt{5}} = \dfrac{6\sqrt{5}}{5}$

Additional points: $\left(\pm\dfrac{3\sqrt{5}}{5}, \pm\sqrt{2}\right)$



65. False. The graph of $\dfrac{x^2}{4} + y^4 = 1$ is not an ellipse.

The degree of y is 4, not 2.

67. *Sample answer:* Foci: $(2, 2), (10, 2) \Rightarrow c = 4$

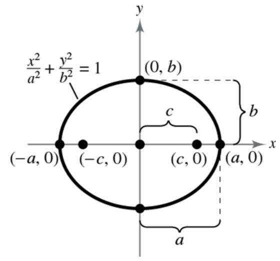
Center: $(6, 2)$

Let $a^2 = 324$ and $b^2 = 308$

So that $c^2 = a^2 - b^2$.

$$\frac{(x-6)^2}{324} + \frac{(y-2)^2}{308} = 1$$

69.



The length of half the major axis is a and the length of half the minor axis is b.

Find the distance between $(0, b)$ and $(c, 0)$ and $(0, b)$ and $(-c, 0)$.

$$d_1 = \sqrt{(0-c)^2 + (b-0)^2} = \sqrt{c^2 + b^2}$$

$$d_2 = \sqrt{(0-(-c))^2 + (b-0)^2} = \sqrt{c^2 + b^2}$$

The sum of the distances from any point on the ellipse to the two foci is constant. Using the vertex $(a, 0)$, the constant sum is $(a + c) + (a - c) = 2a$.

So, the sum of the distances from $(0, b)$ to the two foci is

$$\begin{aligned} \sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} &= 2a \\ 2\sqrt{c^2 + b^2} &= 2a \\ \sqrt{c^2 + b^2} &= a \\ c^2 + b^2 &= a^2 \end{aligned}$$

So, $a^2 = b^2 + c^2$ for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > 0, b > 0$.

71. Right shift 3 units: $y = (x - 3)^2$

73. Given: $A = 73^\circ, C = 40^\circ, a = 11.9$

$$B = 180^\circ - A - C = 180^\circ - 73^\circ - 40^\circ = 67^\circ$$

$$b = \frac{a}{\sin A}(\sin B) = \frac{11.9}{\sin 73^\circ}(\sin 67^\circ) \approx 11.45$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{11.9}{\sin 73^\circ}(\sin 40^\circ) \approx 8.00$$

75. Given: $a = 17.1, b = 13, c = 9.4$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{13^2 + 9.4^2 - 17.1^2}{2 \cdot 13 \cdot 9.4}$$
$$\approx -0.1434 \Rightarrow A \approx 98.25^\circ$$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$
$$\sin B = b\left(\frac{\sin A}{a}\right)$$
$$\sin B \approx 13\left(\frac{\sin 98.25^\circ}{17.1}\right)$$
$$B \approx 48.80^\circ$$
$$C = 180^\circ - A - B \approx 180^\circ - 98.25^\circ - 48.80^\circ = 32.95^\circ$$

77. $\begin{cases} 3x - y = 12 & \text{Equation 1} \\ x + 2y = -10 & \text{Equation 2} \end{cases}$

Multiply Equation 1 by 2 and add to Equation 2:

$$\begin{array}{r} 6x - 2y = 24 \\ \underline{x + 2y = -10} \\ 7x \quad = 14 \Rightarrow x = 2 \end{array}$$

Substitute $x = 2$ into Equation 1:

$3(2) - y = 12 \Rightarrow y = -6$

Solution: $(2, -6)$

79. $\begin{cases} 9x - 3y = 5 & \text{Equation 1} \\ 3x - y = -21 & \text{Equation 2} \end{cases}$

Multiply Equation 2 by -3 and add to Equation 1:

$$\begin{array}{r} 9x - 3y = 5 \\ \underline{-9x + 3y = 63} \\ 0 = 68 \end{array}$$

There is no solution.

Section 10.4 Hyperbolas

1. hyperbola; foci

3. transverse axis; center

5. The asymptotes intersect at the center of the hyperbola (h, k).

7. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

Center: $(0, 0)$

$a = 3, b = 5$

Vertical transverse axis
Matches graph (b).

8. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

Center: $(0, 0)$

$a = 3, b = 5$

Horizontal transverse axis
Matches graph (d).

9. $\dfrac{x^2}{25} - \dfrac{(y + 2)^2}{9} = 1$

Center: $(0, -2)$

$a = 5, b = 3$

Horizontal transverse axis
Matches graph (c).

10. $\dfrac{(y + 4)^2}{25} - \dfrac{(x - 2)^2}{9} = 1$

Center: $(2, -4)$

$a = 5, b = 3$

Vertical transverse axis
Matches graph (a).

11. Vertices: $(0, \pm 2) \Rightarrow a = 2$

Foci: $(0, \pm 4) \Rightarrow c = 4$

$b^2 = c^2 - a^2 = 16 - 4 = 12$

Center: $(0, 0) = (h, k)$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$
$$\frac{y^2}{4} - \frac{x^2}{12} = 1$$

13. Vertices: $(4, 1), (4, 9) \Rightarrow a = 4$

Foci: $(4, 0), (4, 10) \Rightarrow c = 5$

$b^2 = c^2 - a^2 = 25 - 16 = 9$

Center: $(4, 5) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-5)^2}{16} - \frac{(x-4)^2}{9} = 1$$

15. Vertices: $(2, 3), (2, -3) \Rightarrow a = 3$

Passes through the point: $(0, 5)$

Center: $(2, 0) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{b^2} = \frac{y^2}{9} - 1 = \frac{y^2 - 9}{9}$$

$$b^2 = \frac{9(x-2)^2}{y^2 - 9} = \frac{9(-2)^2}{25 - 9}$$

$$= \frac{36}{16} = \frac{9}{4}$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1$$

$$\frac{y^2}{9} - \frac{4(x-2)^2}{9} = 1$$

17. Vertices: $(0, -3), (4, -3) \Rightarrow a = 2$

Center: $(2, -3)$

Passes through: $(-4, 5)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{b^2} = 1$$

$$\frac{(-4-2)^2}{4} - \frac{(5+3)^2}{b^2} = 1$$

$$9 - \frac{64}{b^2} = 1 \Rightarrow b^2 = 8$$

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{8} = 1$$

19. $x^2 - y^2 = 1$

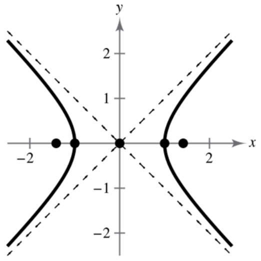
$a = 1, b = 1, c = \sqrt{2}$

Center: $(0, 0)$

Vertices: $(\pm 1, 0)$

Foci: $(\pm\sqrt{2}, 0)$

Asymptotes: $y = \pm x$



21. $\dfrac{y^2}{36} - \dfrac{x^2}{100} = 1$

$a = 6, b = 10$

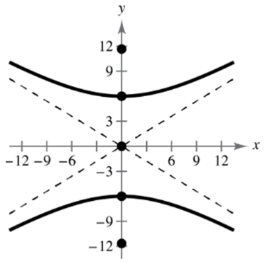
$c^2 = a^2 + b^2 = 136 \Rightarrow c = 2\sqrt{34}$

Center: $(0, 0)$

Vertices: $(0, \pm 6)$

Foci: $(0, \pm 2\sqrt{34})$

Asymptotes: $y = \pm\dfrac{3}{5}x$



23. $2y^2 - \dfrac{x^2}{2} = 2$

$y^2 - \dfrac{x^2}{4} = 1$

$a = 1, b = 2,$

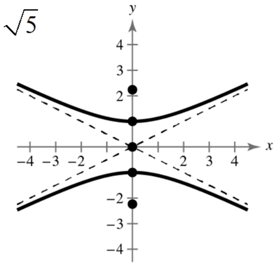
$c^2 = a^2 + b^2 = 5 \Rightarrow c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $(0, \pm\sqrt{5})$

Asymptotes: $y = \pm\dfrac{1}{2}x$



25. $\dfrac{(y+6)^2}{1/9} - \dfrac{(x-2)^2}{1/4} = 1$

$a = \dfrac{1}{3}, b = \dfrac{1}{2},$

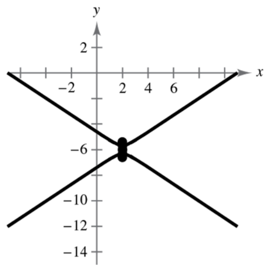
$c = \dfrac{\sqrt{13}}{6}$

Center: $(2, -6)$

Vertices: $\left(2, -\dfrac{17}{3}\right), \left(2, -\dfrac{19}{3}\right)$

Foci: $\left(2, -6 \pm \dfrac{\sqrt{13}}{6}\right)$

Asymptotes: $y = -6 \pm \dfrac{2}{3}(x - 2)$



27.
$$\begin{aligned}9x^2 - y^2 - 36x - 6y + 18 &= 0\\ 9\left(x^2 - 4x + 4\right) - \left(y^2 + 6y + 9\right) &= -18 + 36 - 9\\ 9(x - 2)^2 - (y + 3)^2 &= 9\\ \frac{(x - 2)^2}{1} - \frac{(y + 3)^2}{9} &= 1\end{aligned}$$

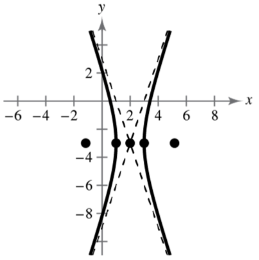
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $\left(2 \pm \sqrt{10}, -3\right)$

Asymptotes:
$y = -3 \pm 3(x - 2)$



29.
$$\begin{aligned}4x^2 - y^2 + 8x + 2y - 1 &= 0\\ 4\left(x^2 + 2x + 1\right) - \left(y^2 - 2y + 1\right) &= 1 + 4 - 1\\ 4(x + 1)^2 - (y - 1)^2 &= 4\\ \frac{(x + 1)^2}{1} - \frac{(y - 1)^2}{4} &= 1\end{aligned}$$

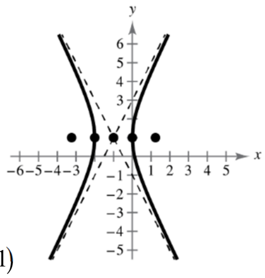
$a = 1, b = 2, c = \sqrt{5}$

Center: $(-1, 1)$

Vertices: $(-2, 1), (0, 1)$

Foci: $\left(-1 \pm \sqrt{5}, 1\right)$

Asymptotes: $y = 1 \pm 2(x + 1)$



31. $2x^2 - 3y^2 = 6$

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

$a = \sqrt{3}, b = \sqrt{2}, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $\left(\pm\sqrt{3}, 0\right)$

Foci: $\left(\pm\sqrt{5}, 0\right)$

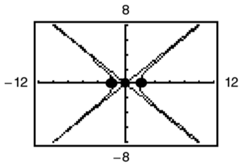
Asymptotes: $y = \pm\sqrt{\dfrac{2}{3}}x = \pm\dfrac{\sqrt{6}}{3}x$

To use a graphing utility, solve for y first.

$$y^2 = \frac{2x^2 - 6}{3}$$

$$\left.\begin{aligned}y_1 &= \sqrt{\frac{2x^2 - 6}{3}}\\ y_2 &= -\sqrt{\frac{2x^2 - 6}{3}}\end{aligned}\right\}\text{Hyperbola}$$

$$\left.\begin{aligned}y_3 &= \frac{\sqrt{6}}{3}x\\ y_4 &= -\frac{\sqrt{6}}{3}x\end{aligned}\right\}\text{Asymptotes}$$



33. $25y^2 - 9x^2 = 225$

$$\frac{y^2}{9} - \frac{x^2}{25} = 1$$

$a = 3, b = 5,$

$c^2 = a^2 + b^2 = 9 + 25 = 34 \Rightarrow c = \sqrt{34}$

Center: $(0, 0)$

Vertices: $(0, \pm 3)$

Foci: $\left(0, \pm\sqrt{34}\right)$

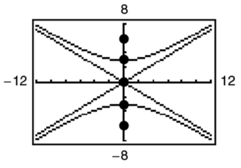
Asymptotes: $y = \pm\dfrac{3}{5}x$

To use a graphing utility, solve for y first.

$$y^2 = \frac{225 + 9x^2}{25}$$

$$\left.\begin{aligned}y_1 &= \sqrt{\frac{9x^2 + 225}{25}}\\ y_2 &= -\sqrt{\frac{9x^2 + 225}{25}}\end{aligned}\right\}\text{Hyperbola}$$

$$\left.\begin{aligned}y_3 &= \frac{3}{5}x\\ y_4 &= -\frac{3}{5}x\end{aligned}\right\}\text{Asymptotes}$$



35. $$9y^2 - x^2 + 2x + 54y + 62 = 0$$

$$9\left(y^2 + 6y + 9\right) - \left(x^2 - 2x + 1\right) = -62 - 1 + 81$$

$$9(y + 3)^2 - (x - 1)^2 = 18$$

$$\frac{(y + 3)^2}{2} - \frac{(x - 1)^2}{18} = 1$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $\left(1, -3 \pm \sqrt{2}\right)$

Foci: $\left(1, -3 \pm 2\sqrt{5}\right)$

Asymptotes: $y = -3 \pm \frac{1}{3}(x - 1)$

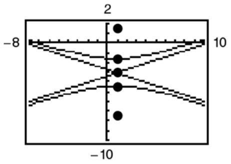
To use a graphing utility, solve for y first.

$$9(y + 3)^2 = 18 + (x - 1)^2$$

$$y = -3 \pm \sqrt{\frac{18 + (x - 1)^2}{9}}$$

$$\left.\begin{aligned} y_1 &= -3 + \frac{1}{3}\sqrt{18 + (x - 1)^2} \\ y_2 &= -3 - \frac{1}{3}\sqrt{18 + (x - 1)^2} \end{aligned}\right\} \text{Hyperbola}$$

$$\left.\begin{aligned} y_3 &= -3 + \frac{1}{3}(x - 1) \\ y_4 &= -3 - \frac{1}{3}(x - 1) \end{aligned}\right\} \text{Asymptotes}$$



37. Vertices: $(\pm 1, 0) \Rightarrow a = 1$

Asymptotes: $y = \pm 5x \Rightarrow \frac{b}{a} = 5, b = 5$

Center: $(0, 0) = (h, k)$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

$$\frac{x^2}{1} - \frac{y^2}{25} = 1$$

39. Foci: $(0, \pm 8) \Rightarrow c = 8$

Asymptotes: $y = \pm 4x \Rightarrow \frac{a}{b} = 4 \Rightarrow a = 4b$

Center: $(0, 0) = (h, k)$

$$c^2 = a^2 + b^2 \Rightarrow 64 = 16b^2 + b^2$$

$$\frac{64}{17} = b^2 \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

$$\frac{17y^2}{1024} - \frac{17x^2}{64} = 1$$

41. Vertices: $(1, 2), (3, 2) \Rightarrow a = 1$

Asymptotes: $y = x, y = 4 - x$

$$\frac{b}{a} = 1 \Rightarrow \frac{b}{1} = 1 \Rightarrow b = 1$$

Center: $(2, 2) = (h, k)$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x - 2)^2}{1} - \frac{(y - 2)^2}{1} = 1$$

43. Vertices: $(3, 0), (3, 4) \Rightarrow a = 2$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

$$\frac{a}{b} = \frac{2}{3} \Rightarrow b = 3$$

Center: $(3, 2) = (h, k)$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

$$\frac{(y - 2)^2}{4} - \frac{(x - 3)^2}{9} = 1$$

45. Foci: $(-1, -1), (9, -1) \Rightarrow c = 5$

Asymptotes: $y = \frac{3}{4}x - 4,\ y = -\frac{3}{4}x + 2$

$$\frac{b}{a} = \frac{3}{4} \Rightarrow b = 3, a = 4$$

Center: $(4, -1) = (h, k)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

47. $9x^2 + 4y^2 - 18x + 16y - 119 = 0$

$A = 9, C = 4$

$AC = (9)(4) = 36 > 0 \Rightarrow$ Ellipse

49. $y^2 - 4x^2 + 4x - 2y - 4 = 0$

$A = -4, C = 1$

$AC = (-4)(1) = -4 < 0 \Rightarrow$ Hyperbola

51. $4x^2 + 25y^2 + 16x + 250y + 541 = 0$

$A = 4, C = 25$

$AC = (4)(25) = 100 > 0 \Rightarrow$ Ellipse

53. $25x^2 - 10x - 200y - 119 = 0$

$A = 25, C = 0$

$AC = 25(0) = 0 \Rightarrow$ Parabola

55. $100x^2 + 100y^2 - 100x + 400y + 409 = 0$

$A = 100, C = 100$

$A = C \Rightarrow$ Circle

57.
$$2c = 4 \text{ mi} = 21{,}120 \text{ ft}$$
$$c = 10{,}560 \text{ ft}$$

$(1100 \text{ ft/s})(18 \text{ s}) = 19{,}800$ ft

The lightning occurred 19,800 feet further from B than from A:

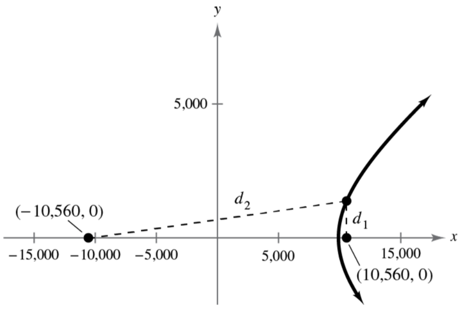
$$d_2 - d_1 = 2a = 19{,}800 \text{ ft}$$
$$a = 9900 \text{ ft}$$

$$b^2 = c^2 - a^2 = (10{,}560)^2 - (9900)^2$$

$$b^2 = 13{,}503{,}600$$

$$\frac{x^2}{(9900)^2} - \frac{y^2}{13{,}503{,}600} = 1$$

$$\frac{x^2}{98{,}010{,}000} - \frac{y^2}{13{,}503{,}600} = 1$$



59. (a) Foci: $(\pm 150, 0) \Rightarrow c = 150$

Center: $(0, 0) = (h, k)$

$$\frac{d_2}{186{,}000} - \frac{d_1}{186{,}000} = 0.001 \Rightarrow 2a = 186, a = 93$$

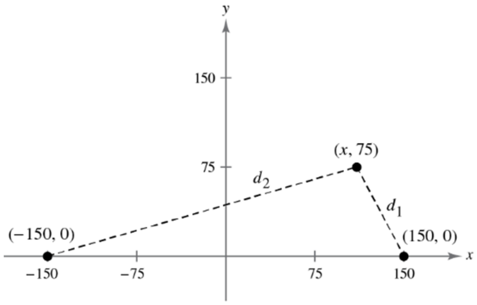
$$b^2 = c^2 - a^2 = 150^2 - 93^2 = 13{,}851$$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right) \approx 12{,}161$$

$$x \approx 110.3 \text{ miles}$$

(b) $c - a = 150 - 93 = 57$ miles



(c) Using the asymptote with positive slope,

$$y = k \pm \frac{b}{a}(x - h)$$

$$y = \frac{\sqrt{13{,}851}}{\sqrt{8694}}x$$

$$y = \frac{27\sqrt{19}}{93}x$$

61. True. For a hyperbola, $c^2 = a^2 + b^2$ or

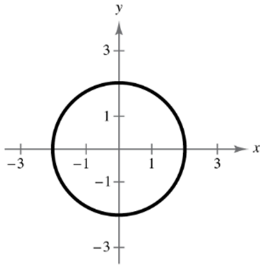
$$e^2 = \frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

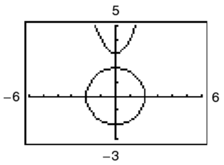
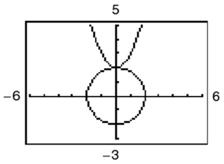
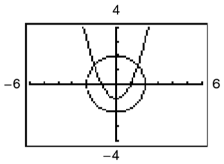
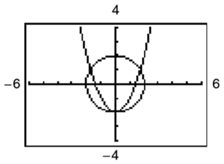
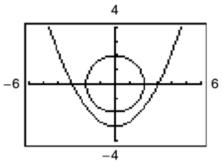
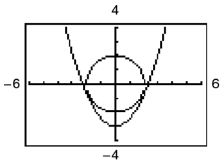
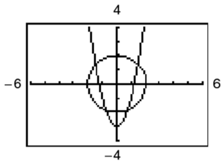
The larger the ratio of b to a, the larger the eccentricity $e = c/a$ of the hyperbola.

63. False. The graph is two intersecting lines.

$$\begin{aligned} x^2 - y^2 + 4x - 4y &= 0 \\ (x^2 + 4x + 4) - (y^2 + 4y + 4) &= 4 - 4 \\ (x - 2)^2 - (y + 2)^2 &= 0 \\ (x - 2)^2 &= (y + 2)^2 \\ x - 2 &= \pm(y - 2) \\ y &= x \text{ and } y = -x + 4 \end{aligned}$$

65.



Value of C	Possible number of points of intersection
$C > 2$	
$C = 2$	
$-2 < C < 2$	
$C = -2$	
$C < -2$	 or  or 

For $C \le -2$, analyze the two curves to determine the number of points of intersection.

$$C = -2:\ x^2 + y^2 = 4 \text{ and } y = x^2 - 2$$
$$x^2 = y + 2$$

Substitute: $(y + 2) + y^2 = 4$

$$y^2 + y - 2 = 0$$
$$(y + 2)(y - 1) = 0$$
$$y = -2, 1$$

$$\begin{aligned} x^2 &= y + 2 \\ x^2 &= -2 + 2 \\ x^2 &= 0 \\ x &= 0 \\ &(0, -2) \end{aligned} \qquad \begin{aligned} x^2 &= y + 2 \\ x^2 &= 1 + 2 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \\ &\left(-\sqrt{3}, 1\right), \left(\sqrt{3}, 1\right) \end{aligned}$$

There are three points of intersection when $C = -2$.

$C < -2$: $x^2 + y^2 = 4$ and $y = x^2 + C$

$$x^2 = y - C$$

Substitute: $(y - C) + y^2 = 4$

$$y^2 + y - 4 - C = 0$$
$$y = \frac{-1 \pm \sqrt{(1)^2 - (4)(1)(-C - 4)}}{2}$$
$$y = \frac{-1 \pm \sqrt{1 + 4(C + 4)}}{2}$$

If $1 + 4(C + 4) < 0$, there are no real solutions (no points of intersection):

$$1 + 4C + 16 < 0$$
$$4C < -17$$
$$C < \frac{-17}{4}, \text{ no points of intersection}$$

If $1 + 4(C + 4) = 0$, there is one real solution (two points of intersection):

$$1 + 4C + 16 = 0$$
$$4C = -17$$
$$C = \frac{-17}{4}, \text{ two points of intersection}$$

If $1 + 4(C + 4) > 0$, there are two real solutions (four points of intersection):

$$1 + 4C + 16 > 0$$
$$4C > -17$$
$$C > \frac{-17}{4}, \text{ (but } C < -2\text{), four points of intersection}$$

Summary:

a. no points of intersection: $C > 2$ or $C < \frac{-17}{4}$

b. one point of intersection: $C = 2$

c. two points of intersection: $-2 < C < 2$ or $C = \frac{-17}{4}$

d. three points of intersection: $C = -2$

e. four points of intersection: $\frac{-17}{4} < C < -2$

67. Because the transverse axis is vertical, $\frac{(y+5)^2}{9} - \frac{(x-3)^2}{4} = 1$, where $a = 3, b = 2, h = 3,$ and $k = -5$ the equations of the asymptotes should be $y = k \pm \frac{a}{b}(x-h)$

$$y = -5 \pm \frac{3}{2}(x-3)$$

$$y = \frac{3}{2}x - \frac{19}{2} \text{ and } y = -\frac{3}{2}x - \frac{1}{2}.$$

69.
$$\begin{aligned} x^2 - 2x &= 15 \\ x^2 - 2x + 1 &= 15 + 1 \\ (x-1)^2 &= 16 \\ x - 1 &= \pm 4 \\ x &= 1 \pm 4 \\ x &= 5, -3 \end{aligned}$$

71.
$$\begin{aligned} x^2 &= 4(4x - 13) \\ x^2 &= 16x - 52 \\ x^2 - 16x &= -52 \\ x^2 - 16x + 64 &= -52 + 64 \\ (x-8)^2 &= 12 \\ x - 8 &= \pm\sqrt{12} = \pm 2\sqrt{3} \\ x &= 8 \pm 2\sqrt{3} \end{aligned}$$

73.
$$\begin{aligned} 9x^2 - 42x + 38 &= 0 \\ 9x^2 - 42x + 49 &= -38 + 49 \\ (3x-7)^2 &= 11 \\ 3x - 7 &= \pm\sqrt{11} \\ 3x &= 7 \pm \sqrt{11} \\ x &= \frac{7 \pm \sqrt{11}}{3} \end{aligned}$$

75.
$$\begin{aligned} x^2 + 9x + 2 &= -5 \\ x^2 + 9x + 7 &= 0 \\ x &= \frac{-9 \pm \sqrt{81 - 4(7)}}{2} \\ x &= \frac{-9 \pm \sqrt{53}}{2} \end{aligned}$$

77.
$$\begin{aligned} -x^2 - 20x - 3 &= 10 \\ x^2 + 20x + 13 &= 0 \\ x &= \frac{-20 \pm \sqrt{400 - 4(13)}}{2} \\ x &= \frac{-20 \pm \sqrt{348}}{2} \\ x &= -10 \pm \sqrt{87} \end{aligned}$$

79.
$$\begin{aligned} 8x^2 + 6x + 8 &= 14 \\ 8x^2 + 6x - 6 &= 0 \\ 4x^2 + 3x - 3 &= 0 \\ x &= \frac{-3 \pm \sqrt{9 - 4(4)(-3)}}{8} \\ x &= \frac{-3 \pm \sqrt{57}}{8} \end{aligned}$$

81.
$$\begin{aligned} \cot 2\theta &= \frac{1}{\tan 2\theta} \\ &= \frac{1 - \tan^2\theta}{2\tan\theta} \\ &= \frac{1}{2}\left(\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}\right) \\ &= \frac{1}{2}(\cot\theta - \tan\theta) \end{aligned}$$

83.
$$\begin{aligned} 2\tan\theta &= 1 - \tan^2\theta \\ \frac{2\sin\theta}{\cos\theta} &= 1 - \frac{\sin^2\theta}{\cos^2\theta} \\ 2\sin\theta\cos\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= \cos 2\theta \\ \tan 2\theta &= 1 \\ 2\theta &= \frac{\pi}{4} + n\pi \\ \theta &= \frac{\pi}{8} + \frac{n\pi}{2} \\ \theta &= \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8} \end{aligned}$$

85.

$$\begin{aligned}\cot 4\theta &= -\cos 2\theta\\ 2\cos^2 2\theta - 1 &= -\cos 2\theta\\ 2\cos^2 2\theta + \cos 2\theta - 1 &= 0\\ (2\cos 2\theta - 1)(\cos 2\theta + 1) &= 0\end{aligned}$$

$$\cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3} + 2n\pi \text{ and } 2\theta = \frac{5\pi}{3} + 2n\pi$$

$$\Rightarrow \theta = \frac{\pi}{6} + n\pi \text{ and } \theta = \frac{5\pi}{6} + n\pi$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$$

$$\cos 2\theta = -1 \Rightarrow 2\theta = \pi + n\pi \Rightarrow \theta = \frac{\pi}{2} + \frac{n\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Solutions: $\theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

Section 10.5 Rotation of Conics

1. rotation; axes

3. $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$

5. $\theta = 90°$; Point: $(2, 0)$

$$\begin{aligned}x &= x'\cos\theta - y'\sin\theta & y &= x'\sin\theta + y'\cos\theta\\ 2 &= x'\cos 90° - y'\sin 90° & 0 &= x'\sin 90° + y'\cos 90°\\ 2 &= -y' & 0 &= x'\\ y' &= -2 & &\end{aligned}$$

So, $(x', y') = (0, -2)$.

7. $\theta = 30°$; Point: $(1, 3)$

$$\begin{aligned}x &= x'\cos\theta - y'\sin\theta\\ y &= x'\sin\theta + y'\cos\theta\end{aligned} \Rightarrow \begin{cases}1 = x'\cos 30° - y'\sin 30°\\ 3 = x'\sin 30° + y'\cos 30°\end{cases}$$

Solving the system yields $(x', y') = \left(\frac{3 + \sqrt{3}}{2}, \frac{3\sqrt{3} - 1}{2}\right)$.

9. $\theta = 45°$; Point: $(2, 1)$

$$\begin{aligned}x &= x'\cos\theta - y'\sin\theta\\ y &= x'\sin\theta + y'\cos\theta\end{aligned} \Rightarrow \begin{cases}2 = x'\cos 45° - y'\sin 45°\\ 1 = x'\sin 45° + y'\cos 45°\end{cases}$$

Solving the system yields $(x', y') = \left(\frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

11. $\theta = 60°$; Point: $(-1, 2)$

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \Rightarrow \begin{cases} -1 = x'\cos 60° - y'\sin 60° \\ 2 = x'\sin 60° + y'\cos 60° \end{cases}$$

Solving the system yields $(x', y') = \left(\dfrac{-1 + 2\sqrt{3}}{2}, \dfrac{2 + \sqrt{3}}{2}\right)$.

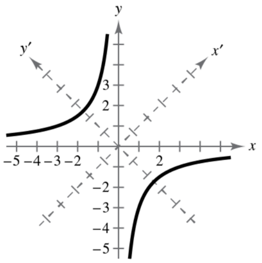
13. $xy + 3 = 0, A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} x &= x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \\ &= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} y &= x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} \\ &= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{x' + y'}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} xy + 3 &= 0 \\ \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 3 &= 0 \\ \frac{(x')^2}{2} - \frac{(y')^2}{2} &= -3 \\ \frac{(y')^2}{6} - \frac{(x')^2}{6} &= 1 \end{aligned}$$

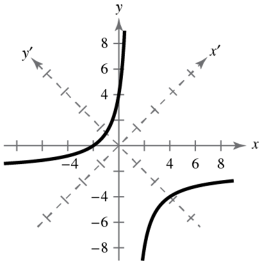


15. $xy + 2x - y + 4 = 0$

$A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} x &= x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned} \qquad \begin{aligned} y &= x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} \\ &= \frac{x' + y'}{\sqrt{2}} \end{aligned}$$



$$\begin{aligned} xy + 2x - y + 4 &= 0 \\ \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 2\left(\frac{x' - y'}{\sqrt{2}}\right) - \left(\frac{x' + y'}{\sqrt{2}}\right) + 4 &= 0 \\ \frac{(x')^2}{2} - \frac{(y')^2}{2} + \frac{2x'}{\sqrt{2}} - \frac{2y'}{\sqrt{2}} - \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} + 4 &= 0 \end{aligned}$$

$$\left[(x')^2 + \sqrt{2}x' + \left(\frac{\sqrt{2}}{2}\right)^2\right] - \left[(y')^2 + 3\sqrt{2}y' + \left(\frac{3\sqrt{2}}{2}\right)^2\right] = -8 + \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{3\sqrt{2}}{2}\right)^2$$

$$\left(x' + \frac{\sqrt{2}}{2}\right)^2 - \left(y' + \frac{3\sqrt{2}}{2}\right)^2 = -12$$

$$\frac{\left(y' + \frac{3\sqrt{2}}{2}\right)^2}{12} - \frac{\left(x' + \frac{\sqrt{2}}{2}\right)^2}{12} = 1$$

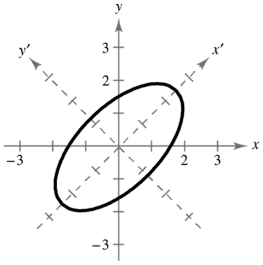
17. $5x^2 - 6xy + 5y^2 - 12 = 0$

$A = 5, B = -6, C = 5$

$$\cot 2\theta = \frac{A - C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$



$$5x^2 - 6xy + 5y^2 - 12 = 0$$

$$5\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 6\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 5\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 12 = 0$$

$$\frac{5(x')^2}{2} - 5x'y' + \frac{5(y')^2}{2} - 3(x')^2 + 3(y')^2 + \frac{5(x')^2}{2} + 5x'y' + \frac{5(y')^2}{2} - 12 = 0$$

$$2(x')^2 + 8(y')^2 = 12$$

$$\frac{(x')^2}{6} + \frac{(y')^2}{3/2} = 1$$

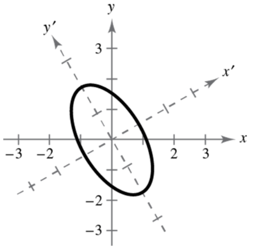
19. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

$A = 13, B = 6\sqrt{3}, C = 7$

$$\cot 2\theta = \frac{A - C}{B} = \frac{1}{\sqrt{3}} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$$

$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

$$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}$$



$$13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$$

$$13\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 7\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$

$$\frac{39(x')^2}{4} - \frac{13\sqrt{3}x'y'}{2} + \frac{13(y')^2}{4} + \frac{18(x')^2}{4} + \frac{18\sqrt{3}x'y'}{4} - \frac{6\sqrt{3}x'y'}{4}$$

$$-\frac{18(y')^2}{4} + \frac{7(x')^2}{4} + \frac{7\sqrt{3}x'y'}{2} + \frac{21(y')^2}{4} - 16 = 0$$

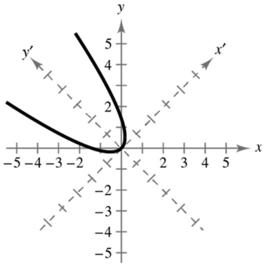
$$16(x')^2 + 4(y')^2 = 16$$

$$\frac{(x')^2}{1} + \frac{(y')^2}{4} = 1$$

21. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0, A = 1, B = 2, C = 1$

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 1}{2} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}} \qquad y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$$



$$x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 + \sqrt{2}\left(\frac{x' - y'}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{x' + y'}{\sqrt{2}}\right) = 0$$

$$\frac{(x')^2}{2} - x'y' + \frac{(y')^2}{2} + (x')^2 - (y')^2 + \frac{(x')^2}{2} + x'y' + \frac{(y')^2}{2} + x' - y' - x' - y' = 0$$

$$2(x')^2 - 2y' = 0$$

$$2(x')^2 = 2y'$$

$$(x')^2 = y'$$

23. $9x^2 + 24xy + 16y^2 + 19x - 130y = 0$

$A = 9, B = 24, C = 16$

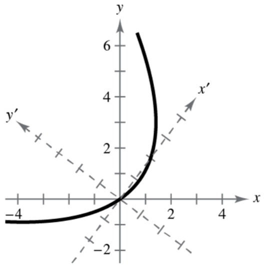
$$\cot 2\theta = \frac{A - C}{B} = -\frac{7}{24} \Rightarrow \theta \approx 53.13°$$

$$\cos 2\theta = -\frac{7}{25}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \left(-\frac{7}{25}\right)}{2}} = \frac{4}{5}$$

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \left(-\frac{7}{25}\right)}{2}} = \frac{3}{5}$$

$$x = x'\cos\theta - y'\sin\theta = x'\left(\frac{3}{5}\right) - y'\left(\frac{4}{5}\right) = \frac{3x' - 4y'}{5} \qquad y = x'\sin\theta + y'\cos\theta = x'\left(\frac{4}{5}\right) + y'\left(\frac{3}{5}\right) = \frac{4x' + 3y'}{5}$$



$$9x^2 + 24xy + 16y^2 + 90x - 130y = 0$$

$$9\left(\frac{3x' - 4y'}{5}\right)^2 + 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 16\left(\frac{4x' + 3y'}{5}\right)^2 + 90\left(\frac{3x' - 4y'}{5}\right) - 130\left(\frac{4x' + 3y'}{5}\right) = 0$$

$$\frac{81(x')^2}{25} - \frac{216x'y'}{25} + \frac{144(y')^2}{25} + \frac{288(x')^2}{25} - \frac{168x'y'}{25} - \frac{288(y')^2}{25} + \frac{256(x')^2}{25} + \frac{384x'y'}{25} + \frac{144(y')^2}{25}$$
$$+ 54x' - 72y' - 104x' - 78y' = 0$$

$$25(x')^2 - 50x' - 150y' = 0$$

$$(x')^2 - 2x' = 6y'$$

$$(x')^2 - 2x' + 1 = 6y' + 1$$

$$(x' - 1)^2 = 6\left(y' + \frac{1}{6}\right)$$

25. $x^2 - 4xy + 2y^2 = 6$

$A = 1, B = -4, C = 2$

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 2}{-4} = \frac{1}{4}$$

$$\frac{1}{\tan 2\theta} = \frac{1}{4}$$

$$\tan 2\theta = 4$$

$$2\theta \approx 75.96$$

$$\theta \approx 37.98°$$

To graph conic with a graphing calculator, solve for y in terms of x.

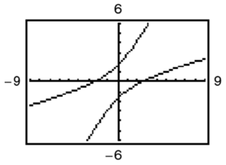
$$x^2 - 4xy + 2y^2 = 6$$

$$y^2 - 2xy + x^2 = 3 - \frac{x^2}{2} + x^2$$

$$(y - x)^2 = 3 + \frac{x^2}{2}$$

$$y - x = \pm\sqrt{3 + \frac{x^2}{2}}$$

$$y = x \pm \sqrt{3 + \frac{x^2}{2}}$$



Enter $y_1 = x + \sqrt{3 + \frac{x^2}{2}}$ and $y_2 = x - \sqrt{3 + \frac{x^2}{2}}$.

27. $14x^2 + 16xy + 9y^2 = 44$

$A = 14, B = 16, C = 9$

$$\cot 2\theta = \frac{A - C}{B} = \frac{14 - 9}{16} = \frac{5}{16}$$

$$\tan 2\theta = \frac{16}{5}$$

$$2\theta \approx 72.65°$$

$$\theta \approx 36.32°$$

Solve for y in terms of x using the Quadratic Formula.

$$(9)y^2 + (16x)y + (14x^2 - 44) = 0$$

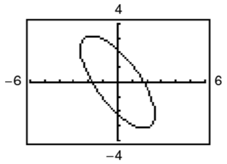
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(16x) \pm \sqrt{(16x)^2 - 4(9)(14x^2 - 44)}}{2(9)}$$

$$y = \frac{-16x \pm \sqrt{-248x^2 + 1548}}{18}$$

Use $y_1 = \dfrac{-16x + \sqrt{-248x^2 + 1548}}{18}$

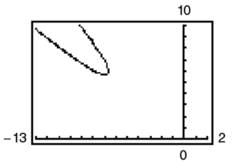
and $y_2 = \dfrac{-16x - \sqrt{-248x^2 + 1548}}{18}$



29. $2x^2 + 4xy + 2y^2 + \sqrt{26}x + 3y = -15$

$A = 2, B = 4, C = 2$

$\cot 2\theta = \dfrac{A - C}{B} = 0 \Rightarrow 2\theta = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}$ or 45°



Solve for y in terms of x using the Quadratic Formula.

$2y^2 + (4x + 3)y + \left(2x^2 + \sqrt{26}x + 15\right) = 0$

$$y = \frac{-(4x + 3) \pm \sqrt{(4x + 3)^2 - 4(2)\left(2x^2 + \sqrt{26}x + 15\right)}}{2(2)}$$

$$= \frac{-(4x + 3) \pm \sqrt{(4x + 3)^2 - 8\left(2x^2 + \sqrt{26}x + 15\right)}}{4}$$

Enter $y_1 = \dfrac{-(4x + 3) + \sqrt{(4x + 3)^2 - 8\left(2x^2 + \sqrt{26}x + 15\right)}}{4}$ and

$y_2 = \dfrac{-(4x + 3) - \sqrt{(4x + 3)^2 - 8\left(2x^2 + \sqrt{26}x + 15\right)}}{4}.$

31. $xy + 2 = 0$

$B^2 - 4AC = 1 \Rightarrow$ The graph is a hyperbola.

$\cot 2\theta = \dfrac{A - C}{B} = 0 \Rightarrow \theta = 45°$

Matches graph (e).

32. $x^2 - xy + 3y^2 - 5 = 0$

$A = 1, B = -1, C = 3$

$B^2 - 4AC = (-1)^2 - 4(1)(3) = -11$

The graph is an ellipse.

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 3}{-1} = 2 \Rightarrow \theta \approx 13.28°$

Matches graph (a).

33. $3x^2 + 2xy + y^2 - 10 = 0$

$B^2 - 4AC = (2)^2 - 4(3)(1) = -8 \Rightarrow$

The graph is an ellipse or circle.

$\cot 2\theta = \dfrac{A - C}{B} = 1 \Rightarrow \theta = 22.5°$

Matches graph (d).

34. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

$A = 1, B = -4, C = 4$

$B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$

The graph is a parabola.

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 4}{-4} = \dfrac{3}{4} \Rightarrow \theta \approx 26.57°$

Matches graph (c).

35. $x^2 + 2xy + y^2 = 0$

$(x + y)^2 = 0$

$x + y = 0$

$y = -x$

The graph is a line. Matches graph (f).

36. $-2x^2 + 3xy + 2y^2 + 3 = 0$

$B^2 - 4AC = (3)^2 - 4(-2)(2) = 25 \Rightarrow$

The graph is a hyperbola.

$\cot 2\theta = \dfrac{A - C}{B} = -\dfrac{4}{3} \Rightarrow \theta \approx -18.43°$

Matches graph (b).

37. (a) $16x^2 - 8xy + y^2 - 10x + 5y = 0$

$B^2 - 4AC = (-8)^2 - 4(16)(1) = 0$

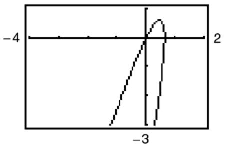
The graph is a parabola.

(b) $y^2 + (-8x + 5)y + \left(16x^2 - 10x\right) = 0$

$$y = \frac{-(-8x + 5) \pm \sqrt{(-8x + 5)^2 - 4(1)\left(16x^2 - 10x\right)}}{2(1)}$$

$$= \frac{(8x - 5) \pm \sqrt{(8x - 5)^2 - 4\left(16x^2 - 10x\right)}}{2}$$

(c)



39. (a) $12x^2 - 6xy + 7y^2 - 45 = 0$

$B^2 - 4AC = (-6)^2 - 4(12)(7) = -300 < 0$

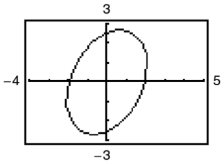
The graph is an ellipse.

(b) $7y^2 + (-6x)y + (12x^2 - 45) = 0$

$$y = \frac{-(-6x) \pm \sqrt{(-6x)^2 - 4(7)(12x^2 - 45)}}{2(7)}$$

$$= \frac{6x \pm \sqrt{36x^2 - 28(12x^2 - 45)}}{14}$$

(c)



41. (a) $x^2 - 6xy - 5y^2 + 4x - 22 = 0$

$B^2 - 4AC = (-6)^2 - 4(1)(-5) = 56 > 0$

The graph is a hyperbola.

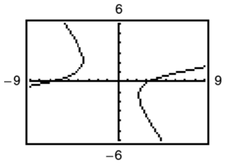
(b) $-5y^2 + (-6x)y + (x^2 + 4x - 22) = 0$

$$y = \frac{-(-6x) \pm \sqrt{(-6x)^2 - 4(-5)(x^2 + 4x - 22)}}{2(-5)}$$

$$= \frac{6x \pm \sqrt{36x^2 + 20(x^2 + 4x - 22)}}{-10}$$

$$= \frac{-6x \pm \sqrt{36x^2 + 20(x^2 + 4x - 22)}}{10}$$

(c)



43. (a) $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$

$B^2 - 4AC = (4)^2 - 4(1)(4) = 0$

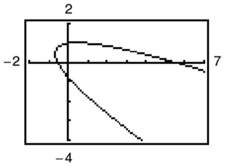
The graph is a parabola.

(b) $4y^2 + (4x - 1)y + (x^2 - 5x - 3) = 0$

$$y = \frac{-(4x - 1) \pm \sqrt{(4x - 1)^2 - 4(4)(x^2 - 5x - 3)}}{2(4)}$$

$$= \frac{-(4x - 1) \pm \sqrt{(4x - 1)^2 - 16(x^2 - 5x - 3)}}{8}$$

(c)

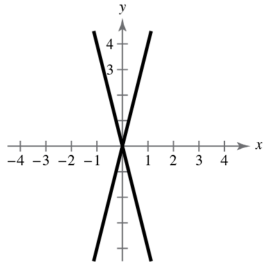


45. $y^2 - 16x^2 = 0$

$y^2 = 16x^2$

$y = \pm 4x$

Two intersecting lines



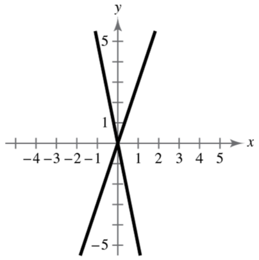
47. $15x^2 - 2xy - y^2 = 0$

$(5x - y)(3x + y) = 0$

$5x - y = 0 \quad 3x + y = 0$

$y = 5x \qquad y = -3x$

Two intersecting lines



49. $x^2 - 2xy + y^2 = 0$

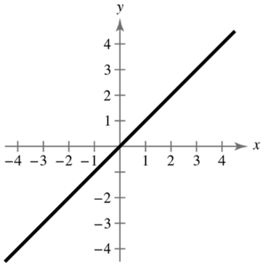
$y^2 - 2xy + x^2 = x^2 - x^2$

$(y - x)^2 = 0$

$y - x = 0$

$y = x$

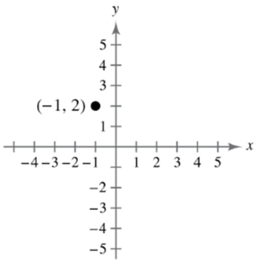
Line



51.

$$\begin{aligned} x^2 + y^2 + 2x - 4y + 5 &= 0 \\ x^2 + 2x + 1 + y^2 - 4y + 4 &= -5 + 1 + 4 \\ (x+1)^2 + (y-2)^2 &= 0 \end{aligned}$$

Point $(-1, 2)$



53. $\begin{cases} x^2 + y^2 - 4 = 0 & \text{Equation 1} \\ xy - 2 = 0 & \text{Equation 2} \end{cases}$

Solve Equation 2 for y:

$$y = \frac{2}{x}$$

Substitute $y = \dfrac{2}{x}$ for y in Equation 1 and solve for x:

$$\begin{aligned} x^2 + \frac{4}{x^2} &= 4 \\ x^4 + 4 &= 4x^2 \\ x^4 - 4x^2 + 4 &= 0 \\ \left(x^2 - 2\right)^2 &= 0 \\ x &= \pm\sqrt{2} \end{aligned}$$

Points of intersection: $\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right)$

55. (a) Because $A = 1$, $B = -2$ and $c = 1$ you have $\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1-1}{-2} = 0 \Rightarrow 2\theta = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}$

which implies that

$$\begin{aligned} x &= x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' + y'}{\sqrt{2}}. \end{aligned}$$

The equation in the $x'y'$-system is obtained by $x^2 - 2xy - 27\sqrt{2}x + y^2 + 9\sqrt{2}y + 378 = 0$.

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) - 27\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 + 9\sqrt{2}\left(\frac{x'+y'}{\sqrt{2}}\right) + 378 = 0$$

$$\frac{(x')^2}{2} - \frac{2x'y'}{\sqrt{2}} + \frac{(y')^2}{2} - (x')^2 + (y')^2 - 27x' + 27y' + \frac{(x')^2}{2} + \frac{2x'y'}{\sqrt{2}} + \frac{(y')^2}{2} + 9x' + 9y' + 378 = 0$$

$$\begin{aligned} 2(y')^2 + 36y' - 18x' + 378 &= 0 \\ (y')^2 + 18y' + 81 &= 9x' - 189 + 81 \\ (y' + 9)^2 &= 9(x' - 12) \\ (y' + 9)^2 &= 4(9/4)(x' - 12) \end{aligned}$$

(b) Since $p = 9/4 = 2.25$, the distance from the vertex to the receiver is 2.25 feet.

57. $x^2 + xy + ky^2 + 6x + 10 = 0$

$B^2 - 4AC = 1^2 - 4(1)(k) = 1 - 4k > 0 \Rightarrow -4k > -1 \Rightarrow k < \frac{1}{4}$

True. For the graph to be a hyperbola, the discriminant must be greater than zero.

59. $$\begin{aligned} r^2 = x^2 + y^2 &= (x' \cos\theta - y' \sin\theta)^2 + (y' \cos\theta + x' \sin\theta)^2 \\ &= (x')^2 \cos^2\theta - 2x'y' \cos\theta \sin\theta + (y')^2 \sin^2\theta + (y')^2 \cos^2\theta + 2x'y' \cos\theta \sin\theta + (x')^2 \sin^2\theta \\ &= (x')^2(\cos^2\theta + \sin^2\theta) + (y')^2(\sin^2\theta + \cos^2\theta) = (x')^2 + (y')^2 \end{aligned}$$

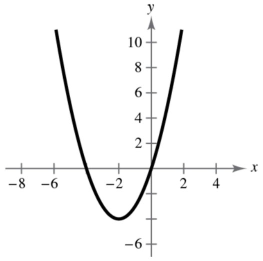
So, $(x')^2 + (y')^2 = r^2$.

61. $y = x^2 + 4x$

$y = (-x)^2 + 4(-x) \Rightarrow y = x^2 - 4x \Rightarrow$ No y-axis symmetry

$(-y) = x^2 + 4x \Rightarrow -y = x^2 + 4x \Rightarrow y = -x^2 - 4x \Rightarrow$ No x-axis symmetry

$(-y) = (-x)^2 + 4(-x) \Rightarrow -y = x^2 - 4x \Rightarrow y = -x^2 + 4x \Rightarrow$ No origin symmetry



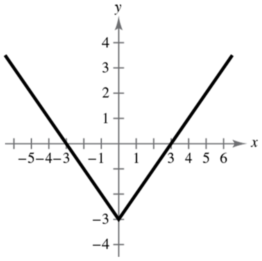
Intercepts: $(0, 0)$, $(-4, 0)$

63. $y = |x| - 3$

$y = |-x| - 3 \Rightarrow y = |x| - 3 \Rightarrow$ y-axis symmetry

$-y = |x| - 3 \Rightarrow y = -|x| + 3 \Rightarrow$ No x-axis symmetry

$-y = |-x| - 3 \Rightarrow y = -|x| + 3 \Rightarrow$ No origin symmetry



Intercepts: $(0, -3)$, $(3, 0)$, $(-3, 0)$

65. $\dfrac{1 - \sin^2 x}{\cos x \sin x} = \dfrac{\cos^2 x}{\cos x \sin x} = \dfrac{\cos x}{\sin x} = \cot x$

67. $$\begin{aligned} \sin x \tan\left(\frac{\pi}{2} - x\right) \sec x &= \sin x \cot x \sec x \\ &= \sin x \cdot \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\cos x}\right) \\ &= 1 \end{aligned}$$

69. $$\begin{cases} x \qquad - 3z = 0 & \text{Equation 1} \\ 2x + y - 7z = -3 & \text{Equation 2} \end{cases}$$

$$\begin{cases} x \qquad - 3z = 0 & \\ \qquad y - z = -3 & -2\text{ Eq. 1} + \text{Eq. 2} \end{cases}$$

Let $z = a$. Then $y = a - 3$ and $x = 3a$.

Solution: $(3a, a - 3, a)$, where a is any real number

Section 10.6 Parametric Equations

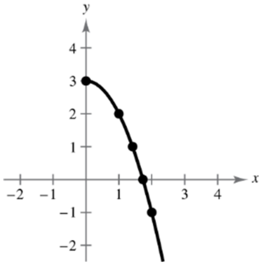
1. plane curve

3. Given a set of parametric equations, the process of finding a corresponding rectangular equation is called eliminating the parameter.

5. (a) $x = \sqrt{t}, y = 3 - t$

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	3	2	1	0	−1

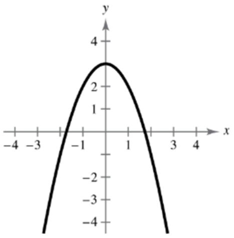
(b)



(c) $x = \sqrt{t} \quad \Rightarrow x^2 = t$

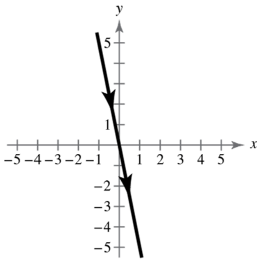
$y = 3 - t \quad \Rightarrow y = 3 - x^2$

The graph of the parametric equations only shows the right half of the parabola, whereas the rectangular equation yields the entire parabola.



7. $x = t, y = -5t$

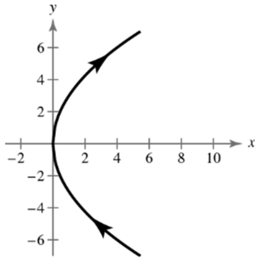
t	−2	−1	0	1	2
x	−2	−1	0	1	2
y	10	5	0	−5	−10



The curve is traced from left to right.

9. $x = t^2, y = 3t$

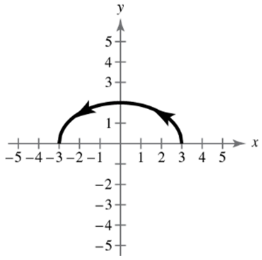
t	−2	−1	0	1	2
x	4	1	0	1	4
y	−6	−3	0	3	6



The curve is traced clockwise.

11. $x = 3\cos\theta, y = 2\sin^2\theta, 0 \le \theta \le \pi$

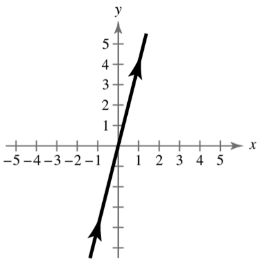
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	3	$\frac{3\sqrt{2}}{2}$	0	$-\frac{3\sqrt{2}}{2}$	−3
y	0	1	2	1	0



The curve is traced from right to left.

13. (a) $x = t, y = 4t$

t	−2	−1	0	1	2
x	−2	−1	0	1	2
y	−8	−4	0	4	8

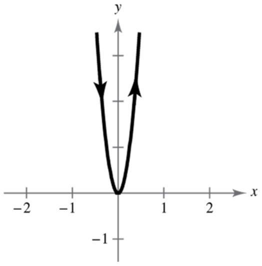


(b) $x = t \quad \Rightarrow t = x$

$y = 4t \Rightarrow y = 4x$

15. (a) $x = \frac{1}{4}t,\ y = t^2$

t	-2	-1	0	1	2
x	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
y	4	1	0	1	4

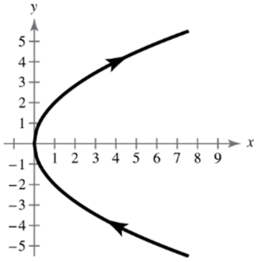


(b) $x = \frac{1}{4}t \Rightarrow t = 4x$

$y = t^2 \Rightarrow y = 16x^2$

17. (a) $x = t^2,\ y = -2t$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	4	2	0	-2	-4

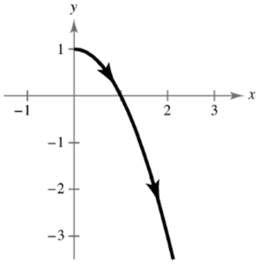


(b) $x = t^2 \Rightarrow t = \pm\sqrt{x}$

$y = -2t = \pm 2\sqrt{x}$

19. (a) $x = \sqrt{t},\ y = 1 - t$

t	0	1	2	3
x	0	1	$\sqrt{2}$	$\sqrt{3}$
y	1	0	-1	-2

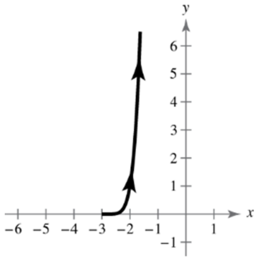


(b) $x = \sqrt{t} \Rightarrow x^2 = t,\ t \geq 0$

$y = 1 - t = 1 - x^2,\ x \geq 0$

21. (a) $x = \sqrt{t} - 3,\ y = t^3$

t	0	1	2	3	4
x	-3	-2	$\sqrt{2} - 3$	$\sqrt{3} - 3$	-1
y	0	1	8	27	64

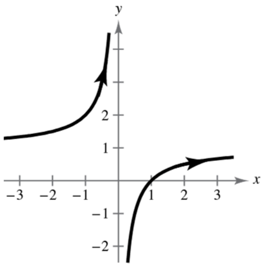


(b) $x = \sqrt{t} - 3 \Rightarrow t = (x + 3)^2$

$y = t^3 \Rightarrow y = \left[(x + 3)^2\right]^3 = (x + 3)^6,\ x \geq -3$

23. (a) $x = t + 1,\ y = \dfrac{t}{t + 1}$

t	-3	-2	0	1	2
x	-2	-1	1	2	3
y	$\frac{3}{2}$	2	0	$\frac{1}{2}$	$\frac{2}{3}$

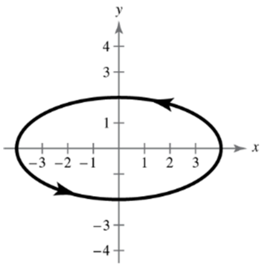


(b) $x = t + 1 \Rightarrow t = x - 1$

$y = \dfrac{t}{t + 1} \Rightarrow y = \dfrac{x - 1}{x}$

25. (a) $x = 4\cos\theta,\ y = 2\sin\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	4	0	–4	0	4
y	0	2	0	–2	0



(b) $x = 4\cos\theta \Rightarrow \left(\dfrac{x}{4}\right)^2 = \cos^2\theta$

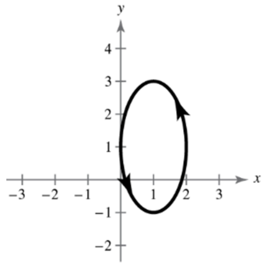
$$y = 2\sin\theta \Rightarrow \left(\frac{y}{2}\right)^2 = \sin^2\theta$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

27. (a) $x = 1 + \cos\theta,\ y = 1 + 2\sin\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	2	1	0	1	2
y	1	3	1	–1	1



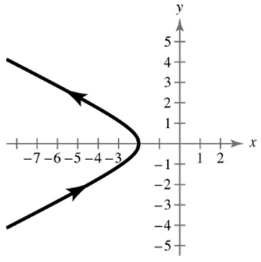
(b) $x = 1 + \cos\theta \Rightarrow (x - 1)^2 = \cos^2\theta$

$$y = 1 + 2\sin\theta \Rightarrow \left(\frac{y-1}{2}\right)^2 = \sin^2\theta$$

$$\frac{(x-1)^2}{1} + \frac{(y-1)^2}{4} = 1$$

29. (a) $x = 2\sec\theta,\ y = \tan\theta,\ \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$

θ	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
x	undefined	$-2\sqrt{2}$	-2	$-2\sqrt{2}$	undefined
y	undefined	-1	0	1	undefined



(b) $x = 2\sec\theta \Rightarrow \sec\theta = \dfrac{x}{2}$

$$y = \tan\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

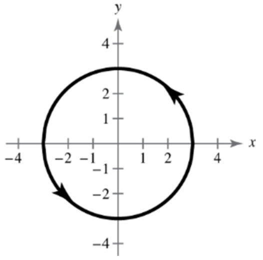
$$1 + y^2 = \left(\frac{x}{2}\right)^2$$

$$1 + y^2 = \frac{x^2}{4}$$

$$\frac{x^2}{4} - y^2 = 1,\ x \le -2$$

31. (a) $x = 3\cos\theta,\ y = 3\sin\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	3	$3\sqrt{2}/2$	0	$-3\sqrt{2}/2$	-3	$-3\sqrt{2}/2$	0	$3\sqrt{2}/2$	3
y	0	$3\sqrt{2}/2$	3	$3\sqrt{2}/2$	0	$-3\sqrt{2}/2$	-3	$-3\sqrt{2}/2$	0



(b) $x = 3\cos\theta \Rightarrow \cos\theta = \dfrac{x}{3}$

$y = 3\sin\theta \Rightarrow \sin\theta = \dfrac{x}{3}$

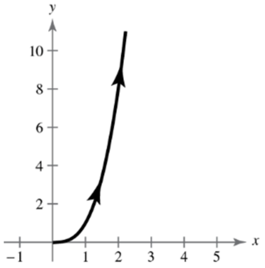
$\sin^2\theta + \cos^2\theta = 1$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{x}{3}\right)^2 = 1$

$x^2 + y^2 = 9$

33. (a) $x = e^t,\ y = e^{3t}$

t	–3	–2	–1	0	1
x	0.0498	0.1353	0.3679	1	2.7183
y	0.0001	0.0024	0.0498	1	20.0855

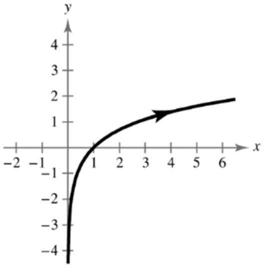


(b) $x = e^t$

$y = e^{3t} = \left(e^t\right)^3 \Rightarrow y = x^3, x > 0$

35. (a) $x = t^3,\ y = 3\ln t$

t	$\frac{1}{2}$	1	2	3	4
x	$\frac{1}{8}$	1	8	27	64
y	–2.0794	0	2.0794	3.2958	4.1589



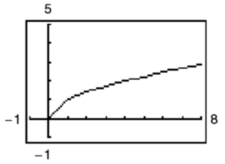
(b) $x = t^3 \qquad \Rightarrow x^{1/3} = t$

$y = 3\ln t \Rightarrow y = \ln t^3$

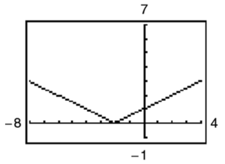
$y = \ln\left(x^{1/3}\right)^3$

$y = \ln x$

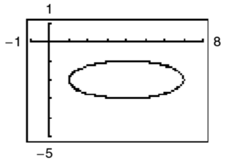
37. $x = t,\ y = \sqrt{t}$



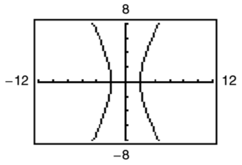
39. $x = 2t,\ y = \left|t + 1\right|$



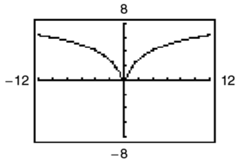
41. $x = 4 + 3\cos\theta,\ y = -2 + \sin\theta$



43. $x = 2\csc\theta,\ y = 4\cot\theta$



45. $x = \dfrac{t}{2},\ y = \ln\left(t^2 + 1\right)$



47. By eliminating the parameter, each curve becomes $y = 2x + 1$.

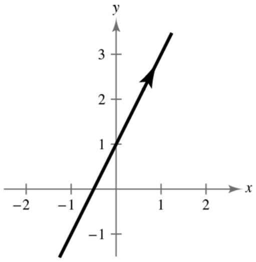
(a) $x = t$

$y = 2t + 1$

There are no restrictions on x and y.

Domain: $(-\infty, \infty)$

Orientation: Left to right



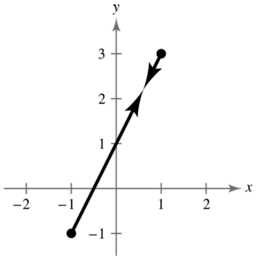
(b) $x = \cos\theta \quad \Rightarrow -1 \le x \le 1$

$y = 2\cos\theta + 1 \quad \Rightarrow -1 \le y \le 3$

The graph oscillates.

Domain: $[-1, 1]$

Orientation: Depends on θ

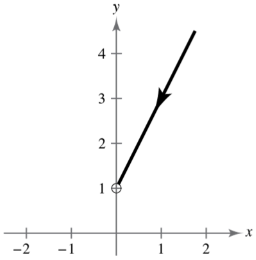


(c) $x = e^{-t} \quad \Rightarrow x > 0$

$y = 2e^{-t} + 1 \quad \Rightarrow y > 1$

Domain: $(0, \infty)$

Orientation: Downward or right to left

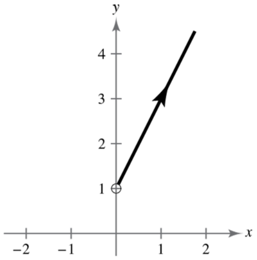


(d) $x = e^{t} \quad \Rightarrow x > 0$

$y = 2e^{t} + 1 \quad \Rightarrow y > 1$

Domain: $(0, \infty)$

Orientation: Upward or left to right



49. $x = x_1 + t(x_2 - x_1),\ y = y_1 + t(y_2 - y_1)$

$$\frac{x - x_1}{x_2 - x_1} = t$$

$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = m(x - x_1)$$

51. $x = h + a\cos\theta,\ y = k + b\sin\theta$

$$\frac{x - h}{a} = \cos\theta,\ \frac{y - k}{b} = \sin\theta$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

53. Line through $(0, 0)$ and $(3, 6)$

From Exercise 49:

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) & y &= y_1 + t(y_2 - y_1) \\ &= 0 + t(3 - 0) & &= 0 + t(6 - 0) \\ &= 3t & &= 6t \end{aligned}$$

55. Circle with center $(3, 2)$; radius 4

From Exercise 50:

$x = 3 + 4\cos\theta$

$y = 2 + 4\sin\theta$

57. Ellipse

Vertices: $(\pm 5, 0) \Rightarrow (h, k) = (0, 0)$ and $a = 5$

Foci: $(\pm 4, 0) \Rightarrow c = 4$

$b^2 = a^2 - c^2 \Rightarrow 25 - 16 = 9 \Rightarrow b = 3$

From Exercise 51:

$x = h + a\cos\theta = 5\cos\theta$

$y = k + b\sin\theta = 3\sin\theta$

59. Hyperbola

Vertices: $(1, 0), (9, 0) \Rightarrow (h, k) = (5, 0)$ and $a = 4$

Foci: $(0, 0), (10, 0) \Rightarrow c = 5$

$c^2 = a^2 + b^2 \Rightarrow 25 = 16 + b^2 \Rightarrow b = 3$

From Exercise 52:

$x = 5 + 4\sec\theta$

$y = 3\tan\theta$

61. Line segment between $(0, 0)$ and $(-5, 2)$.

$x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$

$x = 0 + t(-5 - 0)$ and $y = 0_1 + t(2 - 0)$

$x = -5t$

$y = 2t$

$0 \le t \le 1$

63. Left branch of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$.

$a = 3, c = 5$

Center: $(0, 0)$, $h = 0, k = 0$.

$b^2 = c^2 - a^2 = 25 - 9 = 16$

$x = h + a \sec \theta \Rightarrow x = 0 + 3 \sec \theta$

$y = k + b \tan \theta \Rightarrow y = 0 + 4 \tan \theta$

$x = 3 \sec \theta$

$y = 4 \tan \theta$

$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

65. $y = 3x - 2$

(a) $t = x \Rightarrow x = t$ and $y = 3t - 2$

(b) $t = 2 - x \Rightarrow x = -t + 2$ and

$y = 3(-t + 2) - 2 = -3t + 4$

67. $x = 2y + 1$

(a) $t = x \Rightarrow x = t$ and

$t = 2y + 1 \Rightarrow y = \frac{1}{2}t - \frac{1}{2}$

(b) $t = 2 - x \Rightarrow x = -t + 2$ and

$-t + 2 = 2y + 1 \Rightarrow y = -\frac{1}{2}t + \frac{1}{2}$

69. $y = x^2 + 1$

(a) $t = x \Rightarrow x = t$ and $y = t^2 + 1$

(b) $t = 2 - x \Rightarrow x = -t + 2$ and

$y = (-t + 2)^2 + 1 = t^2 - 4t + 5$

71. $y = 1 - 2x^2$

(a) $t = x \Rightarrow x = t$

$\Rightarrow y = 1 - 2t^2$

(b) $t = 2 - x \Rightarrow x = -t + 2$

$\Rightarrow y = 1 - 2(-t + 2)^2$

$= -2t^2 + 8t - 7$

73. $y = \frac{1}{x}$

(a) $t = x \Rightarrow x = t$ and $y = \frac{1}{t}$

(b) $t = 2 - x \Rightarrow x = -t + 2$ and

$y = \frac{1}{-t + 2} = \frac{-1}{t - 2}$

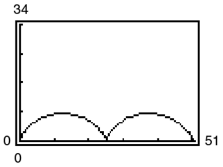
75. $y = e^x$

(a) $t = x \Rightarrow x = t$ and $y = e^t$

(b) $t = 2 - x \Rightarrow x = -t + 2$ and $y = e^{-t+2}$

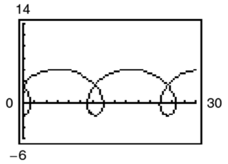
77. $x = 4(\theta - \sin \theta)$

$y = 4(1 - \cos \theta)$



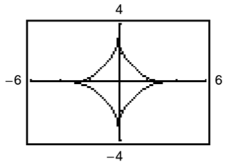
79. $x = 2\theta - 4 \sin \theta$

$y = 2 - 4 \cos \theta$



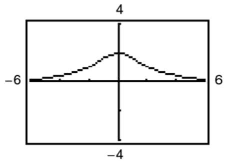
81. $x = 3 \cos^3 \theta$

$y = 3 \sin^3 \theta$



83. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$



85. $x = 2 \cos \theta \Rightarrow -2 \le x \le 2$

$y = \sin 2\theta \Rightarrow -1 \le y \le 1$

Matches graph (b).

Domain: $[-2, 2]$

Range: $[-1, 1]$

86. $x = 4 \cos^3 \theta \Rightarrow -4 \le x \le 4$

$y = 6 \sin^3 \theta \Rightarrow -6 \le y \le 6$

Matches graph (c).

Domain: $[-4, 4]$

Range: $[-6, 6]$

87. $x = \frac{1}{2}(\cos\theta + \theta\sin\theta)$

$y = \frac{1}{2}(\sin\theta - \theta\cos\theta)$

Matches graph (d).

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

88. $x = \frac{1}{2}\cot\theta \Rightarrow -\infty < x < \infty$

$y = 4\sin\theta\cos\theta \Rightarrow -2 \le y \le 2$

Matches graph (a).

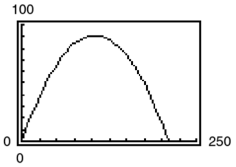
Domain: $(-\infty, \infty)$

Range: $[-2, 2]$

89. $x = (v_0\cos\theta)t$ and $y = h + (v_0\sin\theta)t - 16t^2$

(a) $\theta = 60°, v_0 = 88$ ft/sec

$x = (88\cos 60°)t$ and $y = (88\sin 60°)t - 16t^2$

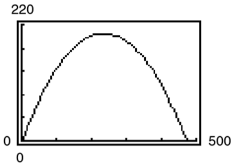


Maximum height: 90.7 feet

Range: 209.6 feet

(b) $\theta = 60°, v_0 = 132$ ft/sec

$x = (132\cos 60°)t$ and $y = (132\sin 60°)t - 16t^2$

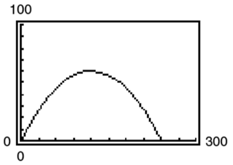


Maximum height: 204.2 feet

Range: 471.6 feet

(c) $\theta = 45°, v_0 = 88$ ft/sec

$x = (88\cos 45°)t$ and $y = (88\sin 45°)t - 16t^2$

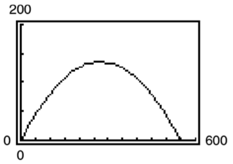


Maximum height: 60.5 ft

Range: 242.0 ft

(d) $\theta = 45°, v_0 = 132$ ft/sec

$x = (132\cos 45°)t$ and $y = (132\sin 45°)t - 16t^2$



Maximum height: 136.1 ft

Range: 544.5 ft

91. (a) 100 miles per hour $= 100\left(\frac{5280}{3600}\right)$ ft/sec $= \frac{440}{3}$ ft/sec

$x = \left(\frac{440}{3}\cos\theta\right)t \approx (146.67\cos\theta)t$

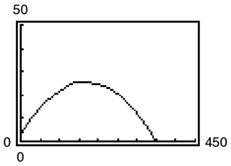
$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \approx 3 + (146.67\sin\theta)t - 16t^2$

(b) For $\theta = 15°$:

$x = \left(\frac{440}{3}\cos 15°\right)t \approx 141.7t$

$y = 3 + \left(\frac{440}{3}\sin 15°\right)t - 16t^2 \approx 3 + 38.0t - 16t^2$

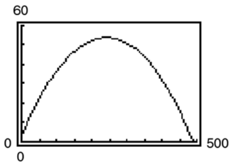
The ball hits the ground inside the ballpark, so it is not a home run.



(c) For $\theta = 23°$:

$x = \left(\frac{440}{3}\cos 23°\right)t \approx 135.0t$

$y = 3 + \left(\frac{440}{3}\sin 23°\right)t - 16t^2 \approx 3 + 57.3t - 16t^2$



The ball easily clears the 7-foot fence at 408 feet so it is a home run.

(d) Find θ so that $y = 7$ when $x = 408$ by graphing the parametric equations for θ values between 15° and 23°. This occurs when $\theta \approx 19.3°$.

93. (a) $x = (\cos 35°)v_0 t$

$y = 7 + (\sin 35°)v_0 t - 16t^2$

(b) If the ball is caught at time t_1, then:

$90 = (\cos 35°)v_0 t_1$

$4 = 7 + (\sin 35°)v_0 t_1 - 16t_1^2$

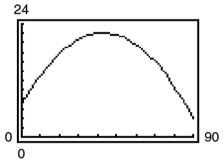
$$v_0 t_1 = \frac{90}{\cos 35°} \Rightarrow -3 = (\sin 35°)\frac{90}{\cos 35°} - 16t_1^2$$

$$\Rightarrow 16t_1^2 = 90 \tan 35° + 3$$

$$\Rightarrow t_1 \approx 2.03 \text{ seconds}$$

$$\Rightarrow \text{v}_0 = \frac{90}{t_1 \cos 35°} \approx 54.09 \text{ ft/sec}$$

(c)



Maximum height ≈ 22 feet

(d) From part (b), $t_1 \approx 2.03$ seconds.

95. When the circle has rolled θ radians, the center is at $(a\theta, a)$.

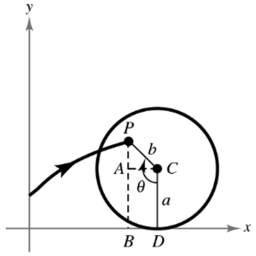
$$\sin \theta = \sin(180° - \theta)$$

$$= \frac{|AC|}{b} = \frac{|BD|}{b} \Rightarrow |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180° - \theta)$$

$$= \frac{|AP|}{-b} \Rightarrow |AP| = -b \cos \theta$$

So, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



97. True

$x = t$

$y = t^2 + 1 \Rightarrow y = x^2 + 1$

$x = 3t$

$y = 9t^2 + 1 \Rightarrow y = x^2 + 1$

99. False. It is possible for both x and y to be functions of t, but y cannot be a function of x. For example, consider the parametric equations $x = 3 \cos t$ and $y = 3 \sin t$. Both x and y are functions of t. However, after eliminating the parameter and finding the rectangular equation $\frac{x^2}{9} + \frac{y^2}{9} = 1$, you can see that y is not a function of x.

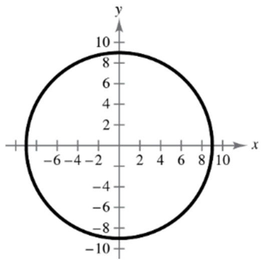
101. Plotting each set of coordinates (x, y) in the order of increasing value of t shows the direction, or orientation, of the curve.

103. The use of parametric equations is useful when graphing two functions simultaneously on the same coordinate system. For example, this is useful when tracking the path of an object so the position and the time associated with that position can be determined.

105. $x^2 + y^2 = 81 = 9^2$

Center: $(0, 0)$

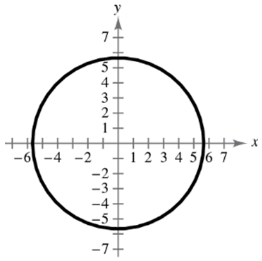
Radius: $r = 9$



107. $x^2 + y^2 = 32 = (4\sqrt{2})^2$

Center: $(0, 0)$

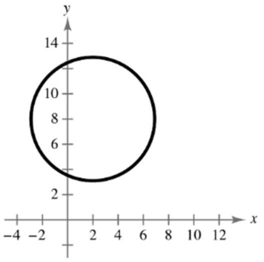
Radius: $r = 4\sqrt{2}$



109. $(x - 2)^2 + (y - 8)^2 = 24 = (2\sqrt{6})^2$

Center: $(2, 8)$

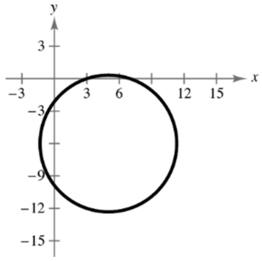
Radius: $r = 2\sqrt{6}$



111.
$$x^2 - 10x + y^2 + 12y + 21 = 0$$
$$(x^2 - 10x + 25) + (y^2 + 12y + 36) = -21 + 25 + 36$$
$$(x-5)^2 + (y+6)^2 = 40 = \left(2\sqrt{10}\right)^2$$

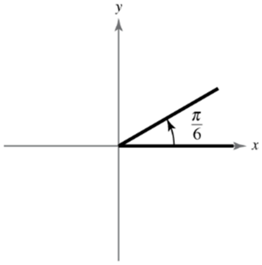
Center: $(5, -6)$

Radius: $r = 2\sqrt{10}$



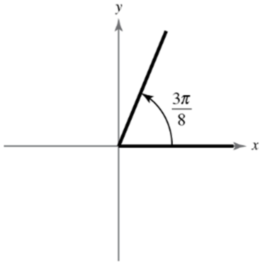
113. Coterminal angles for $\theta = \frac{\pi}{6}$: $\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$

$$\frac{\pi}{6} - 2\pi = \frac{-11\pi}{6}$$



115. Coterminal angles for $\theta = \frac{3\pi}{8}$: $\frac{3\pi}{8} + 2\pi = \frac{19\pi}{8}$

$$\frac{3\pi}{8} - 2\pi = \frac{-13\pi}{8}$$



117. $\sin\theta = -\frac{\sqrt{3}}{2}$

$$\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$$

119. $\tan\theta = \sqrt{3}$

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$$

Section 10.7 Polar Coordinates

1. pole

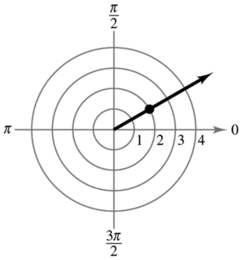
3. The rectangular coordinates (x, y) and the polar coordinates (r, θ) are related by the following equation:

$$x = r\cos\theta \qquad \tan\theta = \frac{y}{x}$$
$$\text{and}$$
$$y = r\sin\theta \qquad r^2 = x^2 + y^2$$

5. Polar coordinates: $\left(2, \frac{5\pi}{6}\right)$

Additional representations:

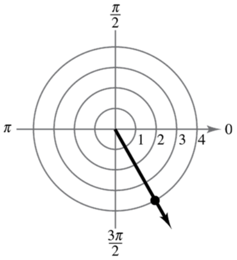
$$\left(-2, \frac{\pi}{6} - \pi\right) = \left(-2, -\frac{5\pi}{6}\right)$$
$$\left(2, \frac{\pi}{6} - 2\pi\right) = \left(2, -\frac{11\pi}{6}\right)$$
$$\left(-2, \frac{\pi}{6} + \pi\right) = \left(-2, \frac{7\pi}{6}\right)$$



7. Polar coordinates: $\left(4, -\frac{\pi}{3}\right)$

Additional representations:

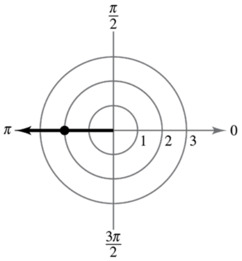
$$\left(4, -\frac{\pi}{3} + 2\pi\right) = \left(4, \frac{5\pi}{3}\right)$$
$$\left(-4, -\frac{\pi}{3} - \pi\right) = \left(-4, -\frac{4\pi}{3}\right)$$
$$\left(-4, -\frac{\pi}{3} + \pi\right) = \left(-4, \frac{2\pi}{3}\right)$$



9. Polar coordinates: $(2, 3\pi)$

Additional representations:

$$(2, 3\pi - 2\pi) = (2, \pi)$$
$$(-2, 3\pi - 2\pi) = (-2, \pi)$$
$$(-2, 3\pi - 3\pi) = (-2, 0)$$



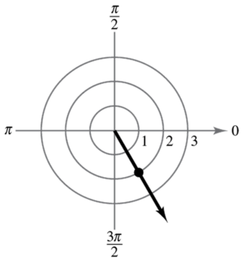
11. Polar coordinates: $\left(-2, \frac{2\pi}{3}\right)$

Additional representations:

$$\left(-2, \frac{2\pi}{3} - 2\pi\right) = \left(-2 -\frac{4\pi}{3}\right)$$

$$\left(2, \frac{2\pi}{3} + \pi\right) = \left(2, \frac{5\pi}{3}\right)$$

$$\left(-2, \frac{2\pi}{3} - \pi\right) = \left(-2, -\frac{\pi}{3}\right)$$



13. Polar coordinates: $\left(0, \frac{7\pi}{6}\right)$

Additional representations:

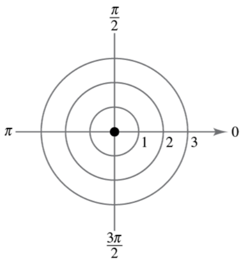
$$\left(0, \frac{7\pi}{6} - \pi\right) = \left(0, \frac{\pi}{6}\right)$$

$$\left(0, \frac{7\pi}{6} - 2\pi\right) = \left(0, -\frac{5\pi}{6}\right)$$

$$\left(0, \frac{7\pi}{6} - 3\pi\right) = \left(0, -\frac{11\pi}{6}\right)$$

or $(0, \theta)$ for any θ,

$-2\pi < \theta < 2\pi$



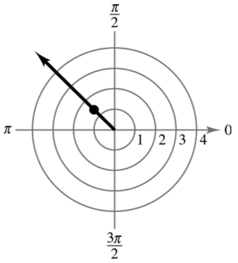
15. Polar coordinates: $\left(\sqrt{2}, 2.36\right)$

Additional representations:

$$\left(\sqrt{2}, 2.36 - 2\pi\right) \approx \left(\sqrt{2}, -3.92\right)$$

$$\left(-\sqrt{2}, 2.36 - \pi\right) \approx \left(-\sqrt{2}, -0.78\right)$$

$$\left(-\sqrt{2}, 2.36 + \pi\right) \approx \left(-\sqrt{2}, 5.50\right)$$



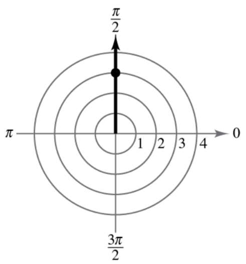
17. Polar coordinates: $(-3, -1.57)$

Additional representations:

$(3, 1.57)$

$(-3, 4.71)$

$(3, -4.71)$



19. Polar coordinates: $(0, \pi) = (r, \theta)$

$$x = 0 \cos \pi = (0)(-1) = 0$$

$$y = 0 \sin \pi = (0)(0) = 0$$

Rectangular coordinates: $(0, 0)$

21. Polar coordinates: $\left(3, \frac{\pi}{2}\right)$

$$x = 3 \cos \frac{\pi}{2} = 0$$

$$y = 3 \sin \frac{\pi}{2} = 3$$

Rectangular coordinates: $(0, 3)$

23. Polar coordinates: $\left(2, \frac{3\pi}{4}\right)$

$$x = 2 \cos \frac{3\pi}{4} = -\sqrt{2}$$

$$y = 2 \sin \frac{3\pi}{4} = \sqrt{2}$$

Rectangular coordinates: $\left(-\sqrt{2}, \sqrt{2}\right)$

25. Polar coordinates: $\left(-2, \frac{7\pi}{6}\right) = (r, \theta)$

$$x = r \cos \theta = -2 \cos \frac{7\pi}{6} = \sqrt{3}$$

$$y = r \sin \theta = -2 \sin \frac{7\pi}{6} = 1$$

Rectangular coordinates: $\left(\sqrt{3}, 1\right)$

27. Polar coordinates: $\left(-3, -\frac{\pi}{3}\right)$

$$x = r \cos \theta = (-3) \cos\left(-\frac{\pi}{3}\right) = (-3)\left(\frac{1}{2}\right) = -\frac{3}{2}$$

$$y = r \sin \theta = (-3) \sin\left(-\frac{\pi}{3}\right) = (-3)\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$

Rectangular coordinates: $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

29. Polar coordinates: $\left(2, \frac{7\pi}{8}\right) = (r, \theta)$

$$x = r \cos \theta = 2 \cos \frac{7\pi}{8} \approx -1.85$$

$$y = r \sin \theta = 2 \sin \frac{7\pi}{8} \approx 0.77$$

Rectangular coordinates: $(-1.85, 0.77)$

31. Polar coordinates: $\left(1, \frac{5\pi}{12}\right)$

$$x = \cos\frac{5\pi}{12} \approx 0.26$$
$$y = \sin\frac{5\pi}{12} \approx 0.97$$

Rectangular coordinates: $(0.26, 0.97)$

33. Polar coordinates: $(-2.5, 1.1) = (r, \theta)$

$$x = r\cos\theta = -2.5\cos 1.1 \approx -1.13$$
$$y = r\sin\theta = -2.5\sin 1.1 \approx -2.23$$

Rectangular coordinates: $(-1.13, -2.23)$

35. Polar coordinates: $(2.5, -2.9) = (r, \theta)$

$$x = r\cos\theta = 2.5\cos(-2.9) \approx -2.43$$
$$y = r\sin\theta = 2.5\sin(-2.9) \approx -0.60$$

Rectangular coordinates: $(-2.43, -0.60)$

37. Polar coordinates: $(-3.1, 7.92) = (r, \theta)$

$$x = r\cos\theta = -3.1\cos 7.92 \approx 0.20$$
$$y = r\sin\theta = -3.1\sin 7.92 \approx -3.09$$

Rectangular coordinates: $(0.20, -3.09)$

39. Rectangular coordinates: $(1, 1)$

$$r = \pm\sqrt{2}, \tan\theta = 1, \theta = \frac{\pi}{4}$$

Polar coordinates: $\left(\sqrt{2}, \frac{\pi}{4}\right)$

41. Rectangular coordinates: $(-3, -3)$

$$r = 3\sqrt{2}, \tan\theta = 1, \theta = \frac{5\pi}{4}$$

Polar coordinates: $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$

43. Rectangular coordinates: $(3, 0)$

$$r = \sqrt{9 + 0} = 3, \tan\theta = 0, \theta = 0$$

Polar coordinates: $(3, 0)$

45. Rectangular coordinates: $(0, -5)$

$$r = 5, \tan\theta \text{ undefined}, \theta = \frac{\pi}{2}$$

Polar coordinates: $\left(5, \frac{3\pi}{2}\right)$

47. Rectangular coordinates: $\left(-\sqrt{3}, -\sqrt{3}\right)$

$$r = \pm\sqrt{3 + 3} = \pm\sqrt{6}, \tan\theta = 1, \theta = \frac{5\pi}{4}$$

Polar coordinates: $\left(\sqrt{6}, \frac{5\pi}{4}\right)$

49. Rectangular coordinates: $\left(\sqrt{3}, -1\right)$

$$r = \sqrt{3 + 1} = 2, \tan\theta = -\frac{1}{\sqrt{3}}, \theta = \frac{11\pi}{6}$$

Polar coordinates: $\left(2, \frac{11\pi}{6}\right)$

51. Rectangular coordinates: $(3, -2)$

$$R \blacktriangleright Pr(3, -2) \approx 3.61 = r$$
$$R \blacktriangleright P\theta(3, -2) \approx 5.70 = \theta$$

Polar coordinates: $(3.61, 5.70)$

53. Rectangular coordinates: $(-5, 2)$

$$R \blacktriangleright Pr(-5, -2) \approx 5.39 = r$$
$$R \blacktriangleright P\theta(-5, -2) \approx 2.76 = \theta$$

Polar coordinates: $(5.39, 2.76)$

55. Rectangular coordinates: $\left(-\sqrt{3}, -4\right)$

$$R \blacktriangleright \mathrm{Pr}\left(-\sqrt{3}, -4\right) \approx 4.36 = r$$
$$R \blacktriangleright P\theta\left(-\sqrt{3}, -4\right) \approx 4.30 = \theta$$

Polar coordinates: $(4.36, 4.30)$

57. Rectangular coordinates: $\left(\frac{5}{2}, \frac{4}{3}\right)$

$$R \blacktriangleright Pr\left(\frac{5}{2}, \frac{4}{3}\right) \approx 2.83 = r$$
$$R \blacktriangleright P\theta\left(\frac{5}{2}, \frac{4}{3}\right) \approx 0.49 = \theta$$

Polar coordinates: $(2.83, 0.49)$

59. $x^2 + y^2 = 9$

$$r = 3$$

61.
$$\begin{aligned} y &= x \\ r\cos\theta &= r\sin\theta \\ 1 &= \tan\theta \\ \theta &= \frac{\pi}{4} \end{aligned}$$

63.
$$\begin{aligned} x &= 10 \\ r\cos\theta &= 10 \\ r &= 10\sec\theta \end{aligned}$$

65.
$$\begin{aligned} 3x - y + 2 &= 0 \\ 3r\cos\theta - r\sin\theta + 2 &= 0 \\ r(3\cos\theta - \sin\theta) &= -2 \\ r &= \frac{-2}{3\cos\theta - \sin\theta} \end{aligned}$$

67.
$$\begin{aligned} xy &= 16 \\ (r\cos\theta)(r\sin\theta) &= 16 \\ r^2 &= 16\sec\theta\csc\theta = 32\csc 2\theta \end{aligned}$$

69.
$$\begin{aligned} x &= a \\ r\cos\theta &= a \\ r &= a\sec\theta \end{aligned}$$

71.
$$\begin{aligned} x^2 + y^2 &= a^2 \\ r^2 &= a^2 \\ r &= a \end{aligned}$$

73.
$$\begin{aligned} x^2 + y^2 - 2ax &= 0 \\ r^2 - 2a\,r\cos\theta &= 0 \\ r(r - 2a\cos\theta) &= 0 \\ r - 2a\cos\theta &= 0 \\ r &= 2a\cos\theta \end{aligned}$$

75.
$$\begin{aligned} (x^2 + y^2)^2 &= x^2 - y^2 \\ (r^2)^2 &= x^2 - y^2 \\ r^4 &= x^2 - y^2 \\ r^2 &= \frac{x^2}{r^2} - \frac{y^2}{r^2} \\ r^2 &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\ r^2 &= \cos^2\theta - \sin^2\theta \\ r^2 &= \cos 2\theta \end{aligned}$$

77.
$$\begin{aligned} y^3 &= x^2 \\ (r\sin\theta)^3 &= (r\cos\theta)^2 \\ r^3\sin^3\theta &= r^2\cos^2\theta \\ \frac{r\sin^3\theta}{\cos^2\theta} &= 1 \\ r\sin\theta\tan^2\theta &= 1 \\ r &= \csc\theta\cot^2\theta \end{aligned}$$

79.
$$\begin{aligned} r &= 5 \\ r^2 &= 25 \\ x^2 + y^2 &= 25 \end{aligned}$$

81.
$$\begin{aligned} \theta &= \frac{2\pi}{3} \\ \tan\theta &= \tan\frac{2\pi}{3} \\ \frac{y}{x} &= -\sqrt{3} \\ y &= -\sqrt{3}x \\ \sqrt{3}x + y &= 0 \end{aligned}$$

83.
$$\begin{aligned} r &= 4\csc\theta \\ r\sin\theta &= 4 \\ y &= 4 \end{aligned}$$

85.
$$\begin{aligned} r &= -2\cos\theta \\ r^2 &= -2r\cos\theta \\ x^2 + y^2 &= -2x \\ x^2 + y^2 + 2x &= 0 \end{aligned}$$

87.
$$\begin{aligned} r^2 &= \cos\theta \\ r^3 &= r\cos\theta \\ \left(\pm\sqrt{x^2 + y^2}\right)^3 &= x \\ \pm(x^2 + y^2)^{3/2} &= x \\ (x^2 + y^2)^3 &= x^2 \\ x^2 + y^2 &= x^{2/3} \\ x^2 + y^2 - x^{2/3} &= 0 \end{aligned}$$

89.

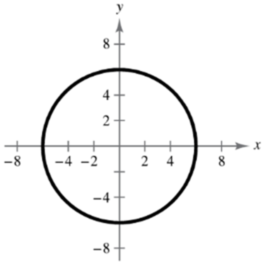
$$\begin{aligned}
r &= 2\sin 3\theta\\
r &= 2\sin(\theta + 2\theta)\\
r &= 2[\sin\theta\cos 2\theta + \cos\theta\sin 2\theta]\\
r &= 2\left[\sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)\right]\\
r &= 2\left[\sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta\right]\\
r &= 2\left[\sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)\right]\\
r &= 2(3\sin\theta - 4\sin^3\theta)\\
r^4 &= 6r^3\sin\theta - 8r^3\sin^3\theta\\
(x^2 + y^2)^2 &= 6(x^2 + y^2)y - 8y^3\\
(x^2 + y^2)^2 &= 6x^2y - 2y^3
\end{aligned}$$

91.

$$\begin{aligned}
r &= \frac{6}{2 - 3\sin\theta}\\
r(2 - 3\sin\theta) &= 6\\
2r &= 6 + 3r\sin\theta\\
2\left(\pm\sqrt{x^2 + y^2}\right) &= 6 + 3y\\
4(x^2 + y^2) &= (6 + 3y)^2\\
4x^2 + 4y^2 &= 36 + 36y + 9y^2\\
4x^2 - 5y^2 - 36y - 36 &= 0
\end{aligned}$$

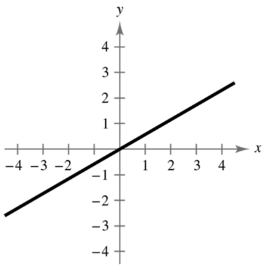
93. The graph of the polar equation consists of all points that are six units from the pole.

$$\begin{aligned}
r &= 6\\
r^2 &= 36\\
x^2 + y^2 &= 36
\end{aligned}$$



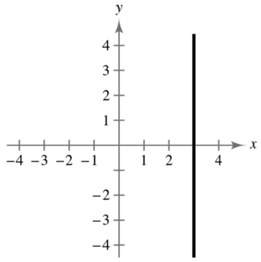
95. The graph of the polar equation consists of all points that make an angle of $\pi/6$ with the polar axis.

$$\begin{aligned}
\theta &= \frac{\pi}{6}\\
\tan\theta &= \tan\frac{\pi}{6}\\
\frac{y}{x} &= \frac{\sqrt{3}}{3}\\
y &= \frac{\sqrt{3}}{3}x\\
3y &= \sqrt{3}x\\
-\sqrt{3}x + 3y &= 0
\end{aligned}$$



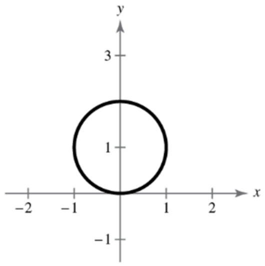
97. The graph of the polar equation is not evident by simple inspection. Convert to rectangular form first.

$$\begin{aligned}
r &= 3\sec\theta\\
r\cos\theta &= 3\\
x &= 3\\
x - 3 &= 0
\end{aligned}$$

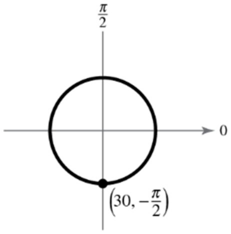


99. The graph of the polar equation consists of all points on the circle with radius 1 and center $(0, 1)$.

$$\begin{aligned}
r &= 2\sin\theta\\
r^2 &= 2r\sin\theta\\
x^2 + y^2 &= 2y\\
x^2 + y^2 - 2y &= 0\\
x^2 + y^2 - 2y + 1 &= 1\\
x^2 + (y - 1)^2 &= 1
\end{aligned}$$



101. (a)



Since the passengers enter a car at the point $(r, \theta) = \left(30, -\frac{\pi}{2}\right)$, $r = 30$ is the polar equation for the model.

(b) Since it takes 45 seconds for the Ferris wheel to complete one revolution clockwise, after 15 seconds a passenger car makes one-third of one revolution or an angle of $\frac{2\pi}{3}$ radians.

Because $\theta = -\frac{\pi}{2} - \frac{2\pi}{3} = -\frac{7\pi}{6}$, the passenger car is at $\left(30, -\frac{7\pi}{6}\right) = \left(30, \frac{5\pi}{6}\right)$.

(c) Polar coordinates: $\left(30, \frac{5\pi}{6}\right)$

$$x = r\cos\theta = 30\cos\frac{5\pi}{6} = 15\sqrt{3} \approx 25.98$$

$$y = r\sin\theta = 30\sin\frac{5\pi}{6} = 15$$

Rectangular coordinates: (25.98, 15)

The car is about 25.98 feet to the left of the center and 15 feet above the center.

103. True. Because r is a directed distance, then the point (r, θ) can be represented as $(r, \theta \pm 2n\pi)$.

105. Rectangular coordinates: $\left(1, -\sqrt{3}\right)$

$r = \sqrt{1 + 3} = \sqrt{4} = 2$, $\tan\theta = \frac{-\sqrt{3}}{1}$, $\theta = -\frac{\pi}{3}$, the point lies in Quadrant IV.

Polar coordinates: $(r, \theta)\left(2, -\frac{\pi}{3}\right)$ or $\left(2, \frac{5\pi}{3}\right)$

107.

$$x^2 + y^2 = 25$$
$$(x - 3)^2 + y^2 = 4$$

Subtract equations:

$$\begin{aligned} x^2 - (x - 3)^2 &= 25 - 4 \\ x^2 - \left(x^2 - 6x + 9\right) &= 21 \\ 6x - 9 &= 21 \\ 6x &= 30 \\ x &= 5 \Rightarrow y = 0 \end{aligned}$$

Solution: (5, 0)

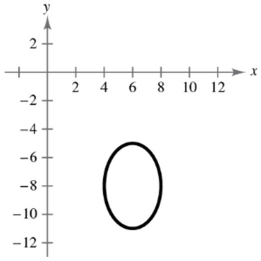
109. $x = 2\sin\theta + 6$

$y = 3\cos\theta - 8$

$$\frac{(x - 6)}{2} = \sin\theta$$

$$\frac{(y + 8)}{3} = \cos\theta$$

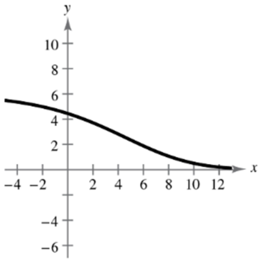
$$\sin^2\theta + \cos^2\theta = \frac{(x - 6)^2}{4} + \frac{(y + 8)^2}{9} = 1$$



The curve is traced clockwise.

111. $x = 4\ln(t-1) + 9$

$y = 6t^{-3}$



The curve is traced from left to right.

Section 10.8 Graphs of Polar Equations

1. lemniscate

3. The graph of a polar equation is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$ its because replacing (r, θ) with $(r, \pi - \theta)$ or $(-r, -\theta)$ yields an equivalent equation.

5. $r = 3\cos\theta$

Circle

7. $r = 3(1 - 2\cos\theta)$

Limaçon with inner loop

9. $r = 6\cos 2\theta$

Rose curve with 4 petals

11. $r = 6 + 3\cos\theta$

$\theta = \dfrac{\pi}{2}$: $-r = 6 + 3\cos(-\theta)$

$-r = 6 + 3\cos\theta$

Not an equivalent equation

Polar axis: $r = 6 + 3\cos(-\theta)$

$r = 6 + 3\cos\theta$

Equivalent equation

Pole: $-r = 6 + 3\cos\theta$

Not an equivalent equation

Answer: Symmetric with respect to polar axis.

13. $r = \dfrac{2}{1 + \sin\theta}$

$\theta = \dfrac{\pi}{2}$: $r = \dfrac{2}{1 + \sin(\pi - \theta)}$

$r = \dfrac{2}{1 + \sin\pi\cos\theta - \cos\pi\sin\theta}$

$r = \dfrac{2}{1 + \sin\theta}$

Equivalent equation

Polar axis: $r = \dfrac{2}{1 + \sin(-\theta)}$

$r = \dfrac{2}{1 - \sin\theta}$

Not an equivalent equation

Pole: $-r = \dfrac{2}{1 + \sin\theta}$

Answer: Symmetric with respect to $\theta = \pi/2$

15. $r^2 = 36\cos 2\theta$

$\theta = \dfrac{\pi}{2}$: $(-r)^2 = 36\cos 2(-\theta)$

$r^2 = 36\cos 2\theta$

Equivalent equation

Polar axis: $r^2 = 36\cos 2(-\theta)$

$r^2 = 36\cos 2\theta$

Equivalent equation

Pole: $(-r)^2 = 36\cos 2\theta$

$r^2 = 36\cos 2\theta$

Equivalent equation

Answer: Symmetric with respect to $\theta = \dfrac{\pi}{2}$, the polar axis, and the pole

17. $|r| = |10 - 10\sin\theta| = 10|1 - \sin\theta| \le 10(2) = 20$

$$|1 - \sin\theta| = 2$$

$$1 - \sin\theta = 2 \quad \text{or} \quad 1 - \sin\theta = -2$$

$$\sin\theta = -1 \qquad \sin\theta = 3$$

$$\theta = \frac{3\pi}{2} \qquad \text{Not possible}$$

Maximum: $|r| = 20$ when $\theta = \dfrac{3\pi}{2}$

$$0 = 10(1 - \sin\theta)$$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

Zero: $r = 0$ when $\theta = \dfrac{\pi}{2}$

19. $|r| = |4\cos 3\theta| = 4|\cos 3\theta| \le 4$

$$|\cos 3\theta| = 1$$

$$\cos 3\theta = \pm 1$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$$

Maximum: $|r| = 4$ when $\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$$0 = 4\cos 3\theta$$

$$\cos 3\theta = 0$$

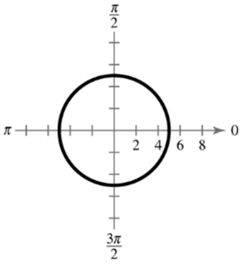
$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

Zero: $r = 0$ when $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

21. $r = 5$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$, polar axis, pole

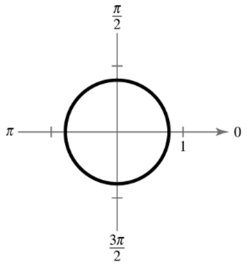
Circle with radius 5



23. $r = \dfrac{\pi}{4}$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$, polar axis, pole

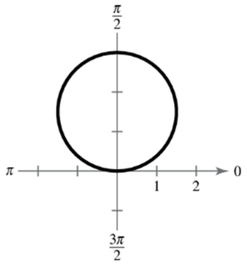
Circle with radius $\dfrac{\pi}{4}$



25. $r = 3\sin\theta$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

Circle with radius $\dfrac{3}{2}$



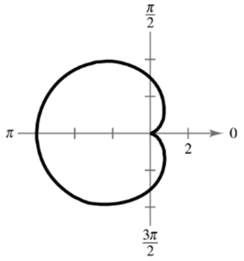
27. $r = 3(1 - \cos\theta)$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{3}{3} = 1 \Rightarrow$ Cardioid

$|r| = 6$ when $\theta = \pi$

$r = 0$ when $\theta = 0$



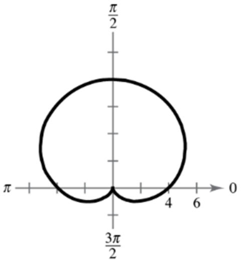
29. $r = 4(1 + \sin\theta)$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

$\dfrac{a}{b} = \dfrac{4}{4} = 1 \Rightarrow$ Cardioid

$|r| = 8$ when $\theta = \dfrac{\pi}{2}$

$r = 0$ when $\theta = \dfrac{3\pi}{2}$

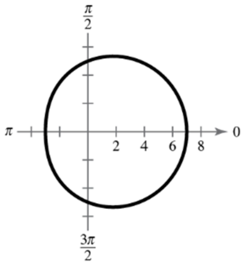


31. $r = 5 + 2\cos\theta$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{5}{2} > 1 \Rightarrow$ Dimpled limaçon

$|r| = 7$ when $\theta = 0$



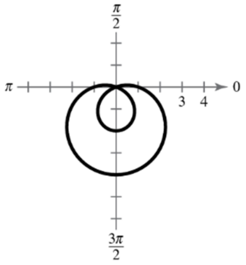
33. $r = 1 - 3\sin\theta$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

$\dfrac{a}{b} = \dfrac{1}{3} < 1 \Rightarrow$ Limaçon with inner loop

$|r| = 4$ when $\theta = \dfrac{3\pi}{2}$

$r = 0$ when $\theta = 0.3398, 2.802$



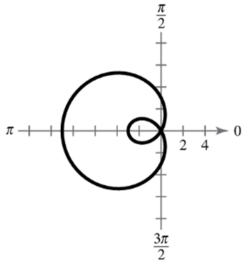
35. $r = 3 - 6\cos\theta$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{1}{2} < 1 \Rightarrow$ Limaçon with inner loop

$|r| = 9$ when $\theta = \pi$

$r = 0$ when $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$



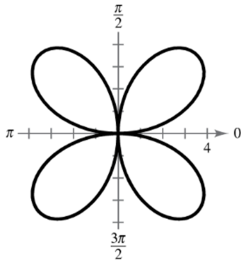
37. $r = 5\sin 2\theta$

Symmetric with respect to $\theta = \pi/2$, the polar axis, and the pole

Rose curve $(n = 2)$ with 4 petals

$|r| = 5$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$r = 0$ when $\theta = 0, \dfrac{\pi}{2}, \pi$



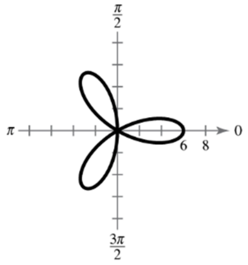
39. $r = 6\cos 3\theta$

Symmetric with respect to polar axis

Rose curve $(n = 3)$ with three petals

$|r| = 6$ when $\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi$

$r = 0$ when $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

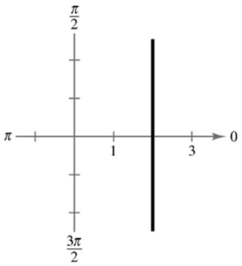


41. $r = 2\sec\theta$

$r = \dfrac{2}{\cos\theta}$

$r\cos\theta = 2$

$x = 2 \Rightarrow$ Line

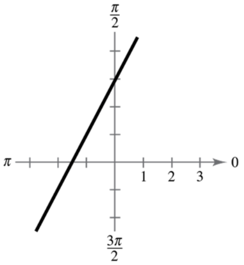


43. $r = \dfrac{3}{\sin\theta - 2\cos\theta}$

$r(\sin\theta - 2\cos\theta) = 3$

$y - 2x = 3$

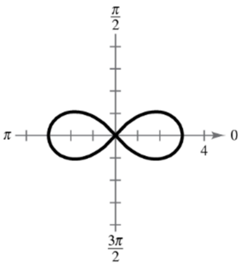
$y = 2x + 3 \Rightarrow$ Line



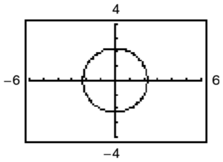
45. $r^2 = 9\cos 2\theta$

Symmetric with respect to the polar axis, $\theta = \pi/2$, and the pole

Lemniscate



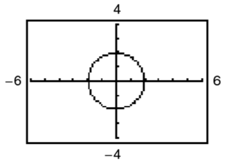
47. $r = \dfrac{9}{4}$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = −6
Xmax = 6
Xscl = 1
Ymin = −4
Ymax = 4
Yscl = 1

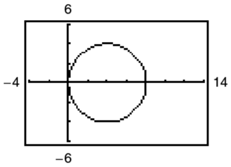
49. $r = \dfrac{5\pi}{8}$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = −6
Xmax = 6
Xscl = 1
Ymin = −4
Ymax = 4
Yscl = 1

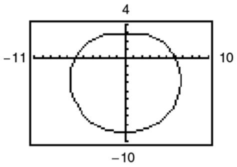
51. $r = 8\cos\theta$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = −4
Xmax = 14
Xscl = 2
Ymin = −6
Ymax = 6
Yscl = 2

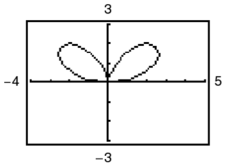
53. $r = 3(2 - \sin\theta)$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = −11
Xmax = 10
Xscl = 1
Ymin = −10
Ymax = 4
Yscl = 1

55. $r = 8 \sin\theta \cos^2\theta$

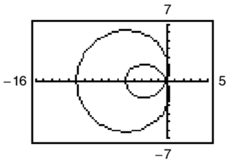


$0 \le \theta \le 2\pi$

$\theta\text{min} = 0$
$\theta\text{max} = 2\pi$
$\theta\text{step} = \pi/24$
$\text{Xmin} = -4$
$\text{Xmax} = 5$
$\text{Xscl} = 1$
$\text{Ymin} = -3$
$\text{Ymax} = 3$
$\text{Yscl} = 1$

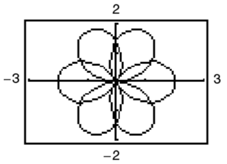
57. $r = 3 - 8\cos\theta$

$0 \le \theta < 2\pi$



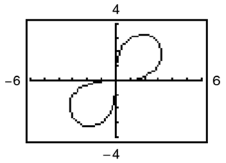
59. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



61. $r^2 = 9\sin 2\theta$

$0 \le \theta < \pi$



63.

$$\begin{aligned}
r &= 2 - \sec\theta = 2 - \frac{1}{\cos\theta}\\
r\cos\theta &= 2\cos\theta - 1\\
r(r\cos\theta) &= 2r\cos\theta - r\\
\left(\pm\sqrt{x^2+y^2}\right)x &= 2x - \left(\pm\sqrt{x^2+y^2}\right)\\
\left(\pm\sqrt{x^2+y^2}\right)(x+1) &= 2x\\
\left(\pm\sqrt{x^2+y^2}\right) &= \frac{2x}{x+1}\\
x^2+y^2 &= \frac{4x^2}{(x+1)^2}\\
y^2 &= \frac{4x^2}{(x+1)^2} - x^2 = \frac{4x^2 - x^2(x+1)^2}{(x+1)^2} = \frac{4x^2 - x^2(x^2+2x+1)}{(x+1)^2}\\
&= \frac{-x^4 - 2x^3 + 3x^2}{(x+1)^2} = \frac{-x^2(x^2+2x-3)}{(x+1)^2}\\
y &= \pm\sqrt{\frac{x^2(3-2x-x^2)}{(x+1)^2}} = \pm\left|\frac{x}{x+1}\right|\sqrt{3-2x-x^2}
\end{aligned}$$

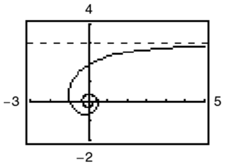
The graph has an asymptote at $x = -1$.

65. $r = \dfrac{3}{\theta}$

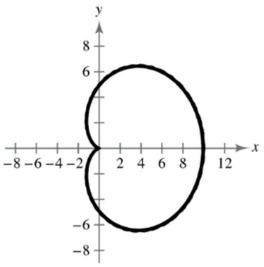
$\theta = \dfrac{3}{r} = \dfrac{3\sin\theta}{r\sin\theta} = \dfrac{3\sin\theta}{y}$

$y = \dfrac{3\sin\theta}{\theta}$

As $\theta \to 0$, $y \to 3$



67. (a)



The graph is a cardioid.

(b) Since $|r|$ is at a maximum when $r = 10$ at $\theta = 0$ radians, the microphone is most sensitive to sound when $\theta = 0$.

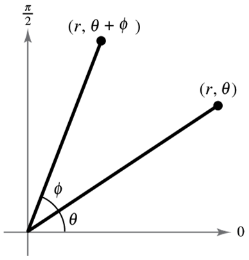
69. True. The equation is of the form $r = a \sin n\theta$, where n is odd, so it has five petals.

71. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$.

If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$.

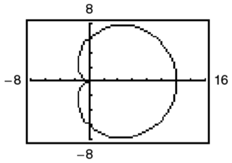
That is, $g(\theta_1 + \phi) = r_1 = f(\theta_1)$. Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$,

you see that $g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi)$.

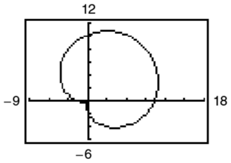


73. $r = 6[1 + \cos(\theta - \phi)]$

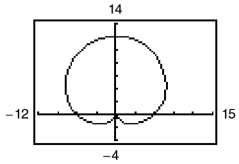
(a)



(b)



(c)



The angle ϕ has the effect of rotating the graph by the angle ϕ. For part (c), $r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right] = 6(1 + \sin\theta)$.

75. (a)
$$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{\pi}{4}\right) \\ &= 2 - \left[\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right] \\ &= 2 - \frac{\sqrt{2}}{2}(\sin\theta - \cos\theta) \end{aligned}$$

(b)
$$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{\pi}{2}\right) \\ &= 2 - \left[\sin\theta\cos\frac{\pi}{2} - \cos\theta\sin\frac{\pi}{2}\right] \\ &= 2 + \cos\theta \end{aligned}$$

(c)
$$\begin{aligned} r &= 2 - \sin(\theta - \pi) \\ &= 2 - [\sin\theta\cos\pi - \cos\theta\sin\pi] \\ &= 2 + \sin\theta \end{aligned}$$

(d)
$$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{3\pi}{2}\right) \\ &= 2 - \left[\sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}\right] \\ &= 2 - \cos\theta \end{aligned}$$

77. $16x^2 + 25y^2 = 400$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

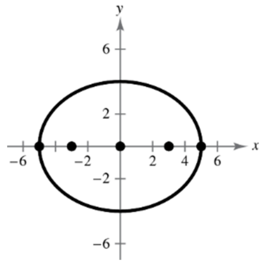
Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

$c^2 = a^2 - b^2 = 25 - 16 = 9 \Rightarrow c = 3$

Foci: $(\pm 3, 0)$

Eccentricity $= e = \frac{c}{a} = \frac{3}{5}$



79. $25y^2 - 9x^2 + 18x - 234 = 0$

$-9x^2 + 25y^2 + 18x - 234 = 0$

$AC = (-9)(25) < 0 \Rightarrow$ Hyperbola

81. $36x^2 + 432x + 25y^2 - 250y + 1021 = 0$

$36x^2 + 25y^2 + 432x - 250y + 1021 = 0$

$AC = (36)(25) > 0 \Rightarrow$ Ellipse

Section 10.9 Polar Equations of Conics

1. conic

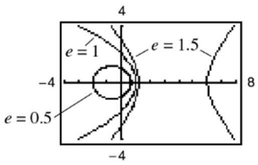
3. An equation of the form $r = \dfrac{ep}{1 + e\cos\theta}$ corresponds to a conic with a vertical directrix.

5. $r = \dfrac{2e}{1 + e\cos\theta}$

$e = 1: r = \dfrac{2}{1 + \cos\theta} \Rightarrow$ parabola

$e = 0.5: r = \dfrac{1}{1 + 0.5\cos\theta} \Rightarrow$ ellipse

$e = 1.5: r = \dfrac{3}{1 + 1.5\cos\theta} \Rightarrow$ hyperbola

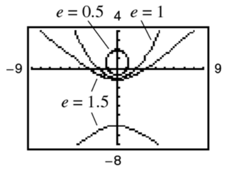


7. $r = \dfrac{2e}{1 - e\sin\theta}$

$e = 1: r = \dfrac{2}{1 - \sin\theta} \Rightarrow$ parabola

$e = 0.5: r = \dfrac{1}{1 - 0.5\sin\theta} \Rightarrow$ ellipse

$e = 1.5: r = \dfrac{3}{1 - 1.5\sin\theta} \Rightarrow$ hyperbola



9. $r = \dfrac{4}{1 - \cos\theta}$

$e = 1 \Rightarrow$ Parabola

Vertical directrix to the left of the pole

Matches graph (c).

10. $r = \dfrac{3}{2 + \cos\theta}$

$e = \dfrac{1}{2} \Rightarrow$ Ellipse

Vertical directrix to the right of the pole

Matches graph (d).

11. $r = \dfrac{4}{1 + \sin\theta}$

$e = 1 \Rightarrow$ Parabola

Horizontal directrix above the pole

Matches graph (a).

12. $r = \dfrac{4}{1 - 3\sin\theta}$

$e = 3 \Rightarrow$ Hyperbola

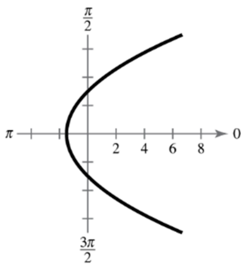
Horizontal directrix below pole

Matches graph (b).

13. $r = \dfrac{3}{1 - \cos\theta}$

$e = 1 \Rightarrow$ Parabola

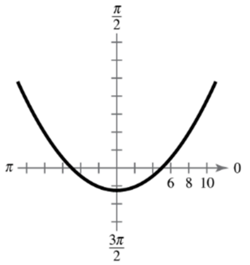
Vertex: $\left(\dfrac{3}{2}, \pi\right)$



15. $r = \dfrac{5}{1 - \sin\theta}$

$e = 1,$ the graph is a parabola.

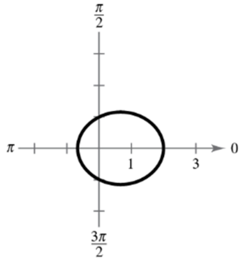
Vertex: $\left(\dfrac{5}{2}, -\dfrac{\pi}{2}\right)$



17. $r = \dfrac{2}{2 - \cos\theta} = \dfrac{1}{1 - (1/2)\cos\theta}$

$e = \dfrac{1}{2} < 1$, the graph is an ellipse.

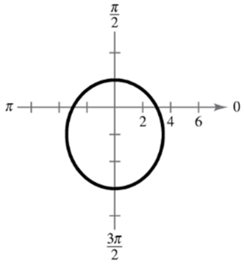
Vertices: $(2, 0), \left(\dfrac{2}{3}, \pi\right)$



19. $r = \dfrac{6}{2 + \sin\theta} = \dfrac{3}{1 + (1/2)\sin\theta}$

$e = \dfrac{1}{2} < 1$, the graph is an ellipse.

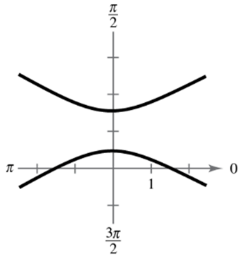
Vertices: $\left(2, \dfrac{\pi}{2}\right), \left(6, \dfrac{3\pi}{2}\right)$



21. $r = \dfrac{3}{2 + 4\sin\theta} = \dfrac{3/2}{1 + 2\sin\theta}$

$e = 2 > 1$, the graph is a hyperbola.

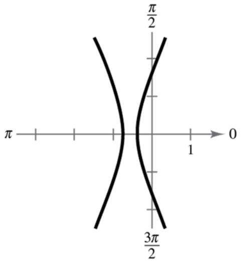
Vertices: $\left(\dfrac{1}{2}, \dfrac{\pi}{2}\right), \left(-\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$



23. $r = \dfrac{3}{2 - 6\cos\theta} = \dfrac{3/2}{1 - 3\cos\theta}$

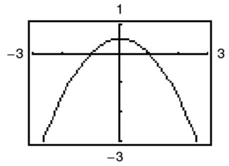
$e = 3 > 1$, the graph is a hyperbola.

Vertices: $\left(-\dfrac{3}{4}, 0\right), \left(\dfrac{3}{8}, \pi\right)$



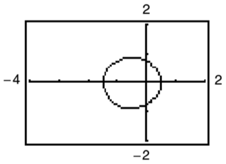
25. $r = \dfrac{-1}{1 - \sin\theta}$

$e = 1 \Rightarrow$ Parabola



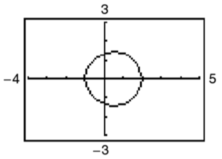
27. $r = \dfrac{3}{-4 + 2\cos\theta}$

$e = \dfrac{1}{2} \Rightarrow$ Ellipse



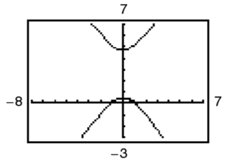
29. $r = \dfrac{4}{3 - \cos\theta}$

$e = \dfrac{1}{3} \Rightarrow$ Ellipse



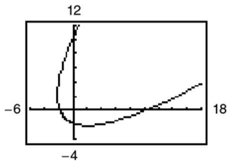
31. $r = \dfrac{14}{14 + 17\sin\theta} = \dfrac{1}{1 + (17/14)\sin\theta}$

$e = \dfrac{17}{14} > 1 \Rightarrow$ Hyperbola

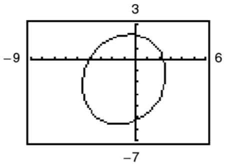


33. $r = \dfrac{3}{1 - \cos(\theta - \pi/4)}$

Rotate the graph in Exercise 13 through the angle $\pi/4$.



35. $r = \dfrac{6}{2 + \sin\left(\theta + \dfrac{\pi}{6}\right)}$



Rotate the graph in Exercise 19 through the angle $-\pi/6$.

37. Parabola: $e = 1$

Directrix: $x = -1$

Vertical directrix to the left of the pole

$$r = \frac{1(1)}{1 - 1\cos\theta} = \frac{1}{1 - \cos\theta}$$

39. Ellipse: $e = \dfrac{1}{2}$

Directrix: $x = 3$

$p = 3$

Vertical directrix to the right of the pole

$$r = \frac{(1/2)(3)}{1 + (1/2)\cos\theta} = \frac{\frac{3}{2}}{1 + \frac{1}{2}\cos\theta} = \frac{3}{2 + \cos\theta}$$

41. Hyperbola: $e = 2$

Directrix: $x = 1$

$p = 1$

Vertical directrix to the right of the pole

$$r = \frac{2(1)}{1 + 2\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

43. Parabola

Vertex: $(2, 0) \Rightarrow e = 1, p = 4$

Vertical directrix to the right of the pole

$$r = \frac{1(4)}{1 + 1\cos\theta} = \frac{4}{1 + \cos\theta}$$

45. Parabola

Vertex: $(5, \pi) \Rightarrow e = 1, p = 10$

Vertical directrix to the left of the pole

$$r = \frac{1(10)}{1 - 1\cos\theta} = \frac{10}{1 - \cos\theta}$$

47. Ellipse: Vertices $(2, 0), (10, \pi)$

Center: $(4, \pi); c = 4, a = 6, e = \dfrac{2}{3}$

Vertical directrix to the right of the pole

$$r = \frac{(2/3)p}{1 + (2/3)\cos\theta} = \frac{2p}{3 + 2\cos\theta}$$

$$2 = \frac{2p}{3 + 2\cos 0}$$

$$p = 5$$

$$r = \frac{2(5)}{3 + 2\cos\theta} = \frac{10}{3 + 2\cos\theta}$$

49. Ellipse: Vertices $(20, 0), (4, \pi)$

Center: $(8, 0); c = 8, a = 12, e = \dfrac{2}{3}$

Vertical directrix to the left of the pole

$$r = \frac{(2/3)p}{1 - (2/3)\cos\theta} = \frac{2p}{3 - 2\cos\theta}$$

$$20 = \frac{2p}{3 - 2\cos 0}$$

$$p = 10$$

$$r = \frac{2(10)}{3 - 2\cos\theta} = \frac{20}{3 - 2\cos\theta}$$

51. Hyperbola: Vertices $\left(1, \dfrac{3\pi}{2}\right), \left(9, \dfrac{3\pi}{2}\right)$

Center: $\left(5, \dfrac{3\pi}{2}\right); c = 5, a = 4, e = \dfrac{5}{4}$

Horizontal directrix below the pole

$$r = \frac{(5/4)p}{1 - (5/4)\sin\theta} = \frac{5p}{4 - 5\sin\theta}$$

$$1 = \frac{5p}{4 - 5\sin(3\pi/2)}$$

$$p = \frac{9}{5}$$

$$r = \frac{5(9/5)}{4 - 5\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

53. When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,
$$a(1 + e) = \frac{ep}{1 - e\cos 0}$$
$$a(1 + e)(1 - e) = ep$$
$$a(1 - e^2) = ep.$$

So, $r = \dfrac{ep}{1 - e\cos\theta} = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$

55. Earth:

(a) $r = \dfrac{\left[1 - (0.0167)^2(9.2957 \times 10^7)\right]}{1 - 0.0167\cos\theta} \approx \dfrac{9.2931 \times 10^7}{1 - 0.0167\cos\theta}$

(b) Perihelion distance: $r = 9.2957 \times 10^7(1 - 0.0167) \approx 9.1405 \times 10^7$ miles

Aphelion distance: $r = 9.2957 \times 10^7(1 + 0.0167) \approx 9.4509 \times 10^7$ miles

57. Venus:

(a) $r = \dfrac{\left[1 - (0.0067)^2(1.0821 \times 10^8)\right]}{1 - 0.0067\cos\theta} \approx \dfrac{1.0821 \times 10^8}{1 - 0.0067\cos\theta}$

(b) Perihelion distance: $r = 1.0821 \times 10^8(1 - 0.0067) \approx 1.0748 \times 10^8$ kilometers

Aphelion distance: $r = 1.0821 \times 10^8(1 + 0.0067) \approx 1.0894 \times 10^8$ kilometers

59. Mars:

(a) $r = \dfrac{\left[1 - (0.0935)^2(1.4162 \times 10^8)\right]}{1 - 0.0935\cos\theta} \approx \dfrac{1.4038 \times 10^8}{1 - 0.0935\cos\theta}$

(b) Perihelion distance: $r = 1.4162 \times 10^8(1 - 0.0935) \approx 1.2838 \times 10^8$ miles

Aphelion distance: $r = 1.4162 \times 10^8(1 + 0.0935) \approx 1.5486 \times 10^8$ miles

61.
$$r = \frac{3}{2 + \sin\theta}$$
$$= \frac{3/2}{1 + 1/2\sin\theta}$$

Because $e = \frac{1}{2}$, the equation represents an ellipse.

63. True. The graphs represent the same hyperbola, although the graphs are not traced out in the same order as θ goes from 0 to 2π.

65. False.

$$r = \frac{6}{3 - 2\cos\theta} = \frac{2}{1 - (2/3)\cos\theta}$$

$$e = \frac{2}{3} < 1$$

The graph is an ellipse.

67.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2(1 - \cos^2\theta)}{b^2} = 1$$
$$r^2b^2\cos^2\theta + r^2a^2 - r^2a^2\cos^2\theta = a^2b^2$$
$$r^2(b^2 - a^2)\cos^2\theta + r^2a^2 = a^2b^2$$

Since $b^2 - a^2 = -c^2$, we have:

$$-r^2c^2\cos^2\theta + r^2a^2 = a^2b^2$$
$$-r^2\left(\frac{c}{a}\right)^2\cos^2\theta + r^2 = b^2, e = \frac{c}{a}$$
$$-r^2e^2\cos^2\theta + r^2 = b^2$$
$$r^2(1 - e^2\cos^2\theta) = b^2$$
$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}$$

69. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

$a = 13, b = 12, c = 5, e = \dfrac{5}{13}$

$r^2 = \dfrac{144}{1 - (25/169)\cos^2\theta} = \dfrac{24{,}336}{169 - 25\cos^2\theta}$

71. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

$a = 3, b = 4, c = 5, e = \dfrac{5}{3}$

$r^2 = \dfrac{-16}{1 - (25/9)\cos^2\theta} = \dfrac{144}{25\cos^2\theta - 9}$

73. If *e* remains fixed and *p* changes, then the lengths of both the major axis and the minor axis change.

For example, graph $r = \dfrac{5}{1 - (2/3)\sin\theta}$, with $e = \dfrac{2}{3}$ and $p = \dfrac{15}{2}$, and graph $r = \dfrac{6\frac{2}{3}}{1 - (2/3)\sin\theta}$, with $e = \dfrac{2}{3}$ and $p = 10$ on the same set of coordinate axes.

The first ellipse has a major axis of length 18 and a minor axis of length $6\sqrt{5}$, and the second ellipse has a major axis of length 21.6 and a minor axis of length $7.2\sqrt{5}$.

75.

$$\begin{aligned} 9x^2 + 42x + 32 &= 0 \\ 9x^2 + 42x + 49 &= -32 + 49 \\ (3x + 7)^2 &= 17 \\ 3x + 7 &= \pm\sqrt{17} \\ x &= \frac{-7 \pm \sqrt{17}}{3} \end{aligned}$$

77.

$$\begin{aligned} 9y^2 &= 299 - 30y \\ 9y^2 + 30y + 25 &= 299 + 25 \\ (3y + 5)^2 &= 324 \\ 3y + 5 &= \pm 18 \\ y &= \tfrac{1}{3}(-5 \pm 18) \\ y &= -\tfrac{23}{3}, \tfrac{13}{3} \end{aligned}$$

79. $-x^2 - y^2 - 8x + 20y - 7 = 0 \Rightarrow (x + 4)^2 + (y - 10)^2 = 123$: Circle

$\underline{x^2 + 9y^2 + 8x + 4y + 7 = 0} \Rightarrow (x + 4)^2 + 9\left(y + \tfrac{2}{9}\right)^2 = \tfrac{85}{9} \Rightarrow \dfrac{(x + 4)}{\frac{85}{9}} + \dfrac{\left(y + \frac{2}{9}\right)^2}{\frac{85}{81}} = 1$: Ellipse

$$\begin{aligned} 8y^2 + 24y &= 0 \\ 8y(y + 3) &= 0 \\ y &= 0 \text{ or } y = -3 \end{aligned}$$

$$\begin{aligned} \text{When } y = 0\text{: } x^2 + 9(0)^2 + 8x + 4(0) + 7 &= 0 \\ x^2 + 8x + 7 &= 0 \\ (x + 7)(x + 1) &= 0 \\ x = -7 \text{ or } x &= -1 \\ \text{When } y = -3\text{: } x^2 + 9(-3)^2 + 8x + 4(-3) + 7 &= 0 \\ x^2 + 8x + 76 &= 0 \end{aligned}$$

No real solutions

The points of intersection are $(-7, 0)$ and $(-1, 0)$.

81. $y = 7 + x - 0.02x^2$

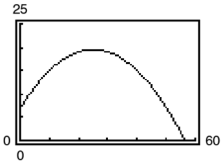
(a) $h = 7$

$$h + (v_0 \sin\theta)t - 16t^2 = 7 + (v_0 \cos\theta)t - 0.02\left[(v_0\cos\theta)t\right]^2$$

$$v_0 \sin\theta = v_0\cos\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = 45°,\ \cos\theta = \frac{\sqrt{2}}{2}$$

$$-16 = -0.02\,v_0^2\cos^2\theta \Rightarrow v_0^2 = 1600 \Rightarrow v_0 = 40$$

(b)



(c) Maximum height: 19.5 feet

Range: 56.2 feet

Review Exercises for Chapter 10

1. Points: $(-1, 2)$ and $(2, 5)$

$$m = \frac{5-2}{2-(-1)} = \frac{3}{3} = 1$$

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}\text{ radian} = 45°$$

3. $5x + 2y + 4 = 0$

$$2y = -5x - 4$$

$$y = -\frac{5}{2}x - 2$$

$$m = -\frac{5}{2}$$

$$\tan\theta = -\frac{5}{2}$$

$$\theta = \arctan\left(-\frac{5}{2}\right) \approx \pi - 1.1902 = 1.9513\text{ radians or }111.8°$$

5. $4x + y = 2 \Rightarrow y = -4x + 2 \Rightarrow m_1 = -4$

$-5x + y = -1 \Rightarrow y = 5x - 1 \Rightarrow m_2 = 5$

$$\tan\theta = \left|\frac{5-(-4)}{1+(-4)(5)}\right| = \frac{9}{19}$$

$$\theta = \arctan\frac{9}{19} \approx 0.4424\text{ radian} \approx 25.35°$$

7. $2x - 7y = 8 \Rightarrow y = \frac{2}{7}x - \frac{8}{7} \Rightarrow m_1 = \frac{2}{7}$

$0.4x + y = 0 \Rightarrow y = -0.4x \Rightarrow m_2 = -0.4$

$$\tan\theta = \left|\frac{-0.4-(2/7)}{1+(2/7)(-0.4)}\right| = \frac{24}{31}$$

$$\theta = \arctan\left(\frac{24}{31}\right) \approx 0.6588\text{ radian} \approx 37.7°$$

9. $(4, 3) \Rightarrow x_1 = 4, y_1 = 3$

$2x - y - 1 = 0 \Rightarrow A = 2, B = -1, C = -1$

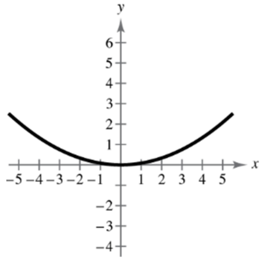
$$d = \frac{|(2)(4) + (-1)(3) + (-1)|}{\sqrt{(2)^2 + (-1)^2}} = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$$

11. Hyperbola

13. Vertex: $(0, 0) = (h, k)$

Focus: $(0, 3) \Rightarrow p = 3$

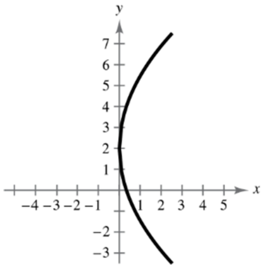
$$(x - h)^2 = 4p(y - k)$$
$$(x - 0)^2 = 4(3)(y - 0)$$
$$x^2 = 12y$$



15. Vertex: $(0, 2) = (h, k)$

Focus: $x = -3 \Rightarrow p = 3$

$$(y - k)^2 = 4p(x - h)$$
$$(y - 2)^2 = 12x$$



17. $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y \Rightarrow p = \frac{1}{8}$

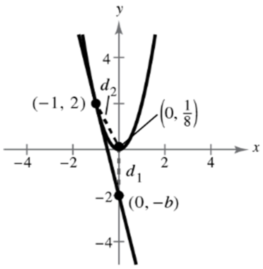
Focus: $\left(0, \frac{1}{8}\right)$

$$d_1 = b + \frac{1}{8}$$
$$d_2 = \sqrt{(-1 - 0)^2 + \left(2 - \frac{1}{8}\right)^2}$$
$$= \sqrt{1 + \frac{225}{64}} = \frac{17}{8}$$
$$d_1 = d_2$$
$$b + \frac{1}{8} = \frac{17}{8}$$
$$b = 2$$

slope $m = \dfrac{-2 - 2}{0 + 1} = -4.$

Point-slope: $y - 2 = -4(x + 1)$

Tangent line: $y = -4x - 2$



19. Parabola

Opens downward

Vertex: $(0, 10)$

$$(x - h)^2 = 4p(y - k)$$
$$x^2 = 4p(y - 10)$$

Solution points: $(\pm 3, 8)$

$$9 = 4p(8 - 10)$$
$$9 = -8p$$
$$-\tfrac{9}{8} = p$$
$$x^2 = 4\left(-\tfrac{9}{8}\right)(y - 10)$$
$$x^2 = -\tfrac{9}{2}(y - 10)$$

To find the x-intercepts, let $y = 0$.

$$x^2 = 45$$
$$x = \pm\sqrt{45} = \pm 3\sqrt{5}$$

At the base, the archway is $2\left(3\sqrt{5}\right) = 6\sqrt{5} \approx 13.4$ meters wide.

21. Vertices: $(2, 0), (2, 16) \Rightarrow a = 8$

Center: $(2, 8) = (h, k)$

Minor axis of length $6 \Rightarrow b = 3$

Ends of minor axis: $(2, 3), (2, 13)$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$
$$\frac{(x - 2)^2}{9} + \frac{(y - 8)^2}{64} = 1$$

23. Vertices: $(0, 1), (4, 1) \Rightarrow a = 2, (h, k) = (2, 1)$

Endpoints of minor axis: $(2, 0), (2, 2) \Rightarrow b = 1$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$
$$\frac{(x - 2)^2}{4} + (y - 1)^2 = 1$$

25. $2a = 10 \Rightarrow a = 5$

$$b = 4$$
$$c^2 = a^2 - b^2 = 25 - 16 = 9 \Rightarrow c = 3$$

The foci occur 3 feet from the center of the arch on a line connecting the tops of the pillars.

27. $\dfrac{(x+2)^2}{64} + \dfrac{(y-5)^2}{36} = 1$

$a = 8, b = 6$

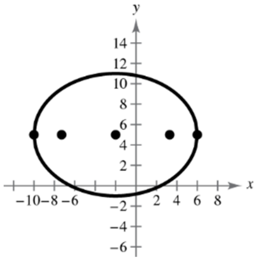
$c^2 = a^2 - b^2 = 64 - 36 = 28 \Rightarrow c = 2\sqrt{7}$

Center: $(-2, 5)$

Vertices: $(-10, 5), (6, 5)$

Foci: $\left(-2 \pm 2\sqrt{7}, 5\right)$

Eccentricity: $e = \dfrac{2\sqrt{7}}{8} = \dfrac{\sqrt{7}}{4}$



29.
$$\begin{aligned} 16x^2 + 9y^2 - 32x + 72y + 16 &= 0 \\ 16(x^2 - 2x + 1) + 9(y^2 + 8y + 16) &= -16 + 16 + 144 \\ 16(x-1)^2 + 9(y+4)^2 &= 144 \\ \frac{(x-1)^2}{9} + \frac{(y+4)^2}{16} &= 1 \end{aligned}$$

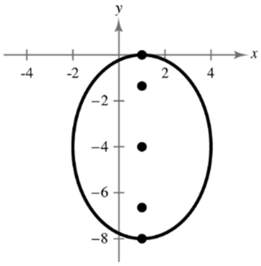
$a = 4, b = 3, c = \sqrt{7}$

Center: $(1, -4)$

Vertices: $(1, 0)$, and $(1, -8)$

Foci: $\left(1, -4 \pm \sqrt{7}\right)$

Eccentricity: $e = \dfrac{\sqrt{7}}{4}$



31. Vertices: $(0, \pm 6) \Rightarrow a = 6, (h, k) = (0, 0)$

Foci: $(0, \pm 8) \Rightarrow c = 8$

$$\begin{aligned} b^2 = c^2 - a^2 = 64 - 36 &= 28 \\ \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1 \\ \frac{y^2}{36} - \frac{x^2}{28} &= 1 \end{aligned}$$

33. Foci: $(\pm 5, 0) \Rightarrow c = 5, (h, k) = (0, 0)$

Asymptotes:

$y = \pm\dfrac{3}{4}x \Rightarrow y = \pm\dfrac{b}{a}x \Rightarrow b = 3, a = 4$

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

$\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1 \Rightarrow \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

35. $\dfrac{(x-4)^2}{49} - \dfrac{(y+2)^2}{25} = 1$

$a = 7, b = 5$

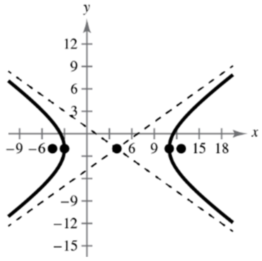
$c^2 = a^2 + b^2 = 49 + 25 = 74 \Rightarrow c = \sqrt{74}$

Center: $(4, -2)$

Vertices: $(11, -2), (-3, -2)$

Foci: $\left(4 \pm \sqrt{74}, -2\right)$

Asymptotes: $y = -2 \pm \dfrac{5}{7}(x - 4)$



37.
$$\begin{aligned} 9x^2 - 16y^2 - 18x - 32y - 151 &= 0 \\ 9(x^2 - 2x + 1) - 16(y^2 + 2y + 1) &= 151 + 9 - 16 \\ 9(x-1)^2 - 16(y+1)^2 &= 144 \\ \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} &= 1 \end{aligned}$$

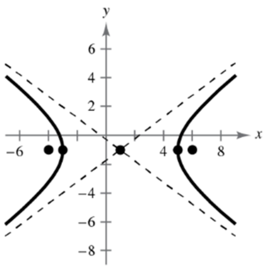
$a = 4, b = 3, c = 5$

Center: $(1, -1)$

Vertices: $(5, -1)$ and $(-3, -1)$

Foci: $(6, -1)$ and $(-4, -1)$

Asymptotes: $y = -1 \pm \dfrac{3}{4}(x - 1)$

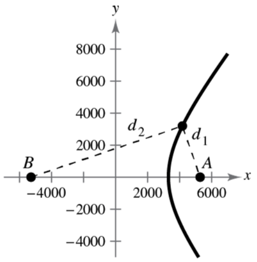


39. Since the microphones were two miles apart, 2 miles $\cdot$ 5280 $\Rightarrow c = 5280$. Since sound travels at 1100 feet per second, $|d_1 - d_2| = 2a = 6600 \Rightarrow a = 3300$.

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 5280^2 - 3300^2 \\ &= 27{,}878{,}400 - 10{,}890{,}000 \\ &= 16{,}988{,}400 \end{aligned}$$

Since the explosion took place 6600 feet farther from B than from A. The locus of all points that are 6600 feet closer to A than to B is one branch of the hyperbola of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, so it follows that

$$\frac{x^2}{10{,}890{,}000} - \frac{y^2}{16{,}988{,}400} = 1.$$



41. $5x^2 - 2y^2 + 10x - 4y + 17 = 0$

$AC = 5(-2) = -10 < 0$

Hyperbola

43. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

$A = 3, C = 2$

$AC = 3(2) = 6 > 0$

Ellipse

45. $xy + 5 = 0$

$A = C = 0, B = 1$

$B^2 - 4AC = 1^2 - 4(0)(0) = 1 > 0 \Rightarrow$ Hyperbola

$$\cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

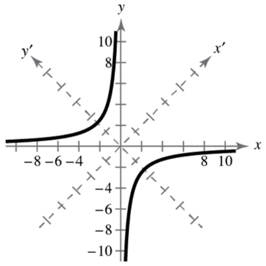
$$x = x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$$

$$y = x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 5 = 0$$

$$\frac{(x')^2 - (y')^2}{2} = -5$$

$$\frac{(y')^2}{10} - \frac{(x')^2}{10} = 1$$

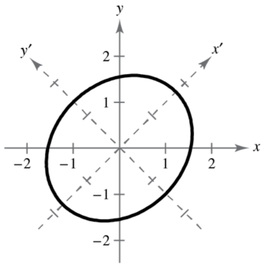


47. $5x^2 - 2xy + 5y^2 - 12 = 0$

$A = C = 5, B = -2$

$B^2 - 4AC = (-2)^2 - 4(5)(5) = -96 < 0$

The graph is an ellipse.



$\cot 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

$x = x' \cos \frac{\pi}{4} - y' \sin \frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$

$y = x' \sin \frac{\pi}{4} + y' \cos \frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$

$$5\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 5\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 12 = 0$$

$$\frac{5}{2}\left[(x')^2 - 2(x'y') + (y')^2\right] - \left[(x')^2 - (y')^2\right] + \frac{5}{2}\left[(x')^2 + 2(x'y') + (y')^2\right] = 12$$

$$4(x')^2 + 6(y')^2 = 12$$

$$\frac{(x')^2}{3} + \frac{(y')^2}{2} = 1$$

49. (a) $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$

$B^2 - 4AC = (-24)^2 - 4(16)(9) = 0$

The graph is a parabola.

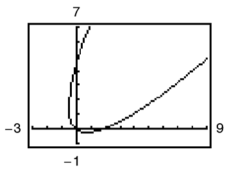
(b) To use a graphing utility, we need to solve for y in terms of x.

$9y^2 + (-24x - 40)y + (16x^2 - 30x) = 0$

$$y = \frac{-(-24x - 40) \pm \sqrt{(-24x - 40)^2 - 4(9)(16x^2 - 30x)}}{2(9)}$$

$$= \frac{(24x + 40) \pm \sqrt{(24x + 40)^2 - 36(16x^2 - 30x)}}{18}$$

(c)



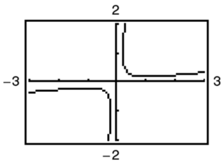
51. (a) $x^2 - 10xy + y^2 + 1 = 0$

Since $B^2 - 4AC = (-10)^2 - 4(1)(1) > 0 \Rightarrow$ Hyperbola

(b) Use the Quadratic Formula to solve for y in terms of x: $y^2 - 10xy + x^2 + 1 = 0$

$$y = \frac{10x \pm \sqrt{100x^2 - 4(x^2 + 1)}}{2}$$

(c)

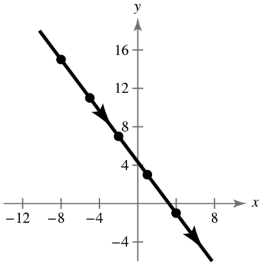


53. $x = 3t - 2, y = 7 - 4t$

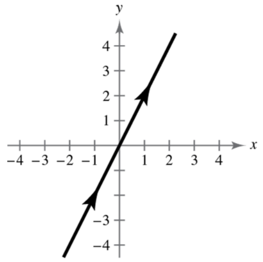
(a)

t	-2	-1	0	1	2
x	-8	-5	-2	1	4
y	15	11	7	3	-1

(b)



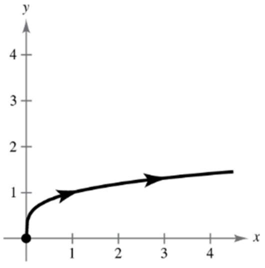
55. (a)



(b) $x = 2t \Rightarrow \dfrac{x}{2} = t$

$y = 4t \Rightarrow y = 4\left(\dfrac{x}{2}\right) = 2x$

57. (a)



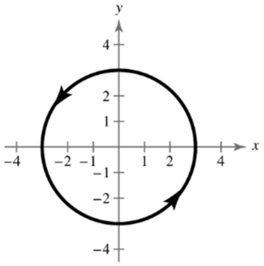
(b) $x = t^2, x \geq 0$

$y = \sqrt{t} \Rightarrow y^2 = t$

$x = \left(y^2\right)^2 \Rightarrow x$

$= y^4 \Rightarrow y = \sqrt[4]{x}$

59. (a)



(b) $x = 3\cos\theta, y = 3\sin\theta$

$\left(\dfrac{x}{3}\right)^2 = \cos^2\theta, \left(\dfrac{y}{3}\right)^2 = \sin^2\theta$

$x^2 + y^2 = 9$

61. $y = 2x + 3$

(a) $t = x \Rightarrow x = t$

$y = 2x + 3 = 2t + 3$

(b) $t = x + 1 \Rightarrow x = t - 1$

$y = 2x + 3 = 2(t - 1) + 3 = 2t + 1$

(c) $t = 3 - x \Rightarrow x = 3 - t$

$y = 2x + 3 = 2(3 - t) + 3 = 9 - 2t$

63. $y = x^2 + 3$

(a) $t = x \Rightarrow x = t$

$y = x^2 + 3 = t^2 + 3$

(b) $t = x + 1 \Rightarrow x = t - 1$

$y = x^2 + 3 = (t - 1)^2 + 3 = t^2 - 2t + 4$

(c) $t = 3 - x \Rightarrow x = 3 - t$

$y = x^2 + 3 = (3 - t)^2 + 3 = t^2 - 6t + 12$

65. $y = 1 - 4x^2$

(a) $t = x \Rightarrow x = t$

$y = 1 - 4x^2 = 1 - 4t^2$

(b) $t = x + 1 \Rightarrow x = t - 1$

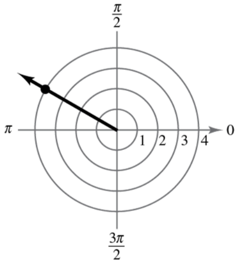
$y = 1 - 4x^2 = 1 - 4(t - 1)^2 = -4t^2 + 8t - 3$

(c) $t = 3 - x \Rightarrow x = 3 - t$

$y = 1 - 4x^2 = 1 - 4(3 - t)^2 = -4t^2 + 24t - 35$

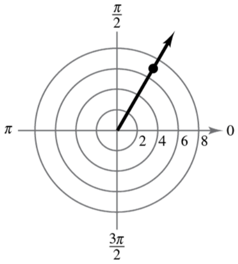
67. Polar coordinates: $\left(4, \frac{5\pi}{6}\right)$

Additional polar representations:
$\left(4, -\frac{7\pi}{6}\right), \left(-4, -\frac{\pi}{6}\right), \left(-4, \frac{11\pi}{6}\right)$



69. Polar coordinates: $(-7, 4.19)$

Additional polar representations: $(7, 1.05), (-7, -2.09)$
$(7, -5.23)$



71. Polar coordinates: $\left(0, \frac{\pi}{2}\right) = (r, \theta)$

$$x = r\cos\theta = 0\cos\frac{\pi}{2} = 0$$

$$y = r\sin\theta = 0\sin\frac{\pi}{2} = 0$$

Rectangular coordinates: $(0, 0)$

73. Polar coordinates: $\left(-1, \frac{\pi}{3}\right)$

$$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

Rectangular coordinates: $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

75. Rectangular coordinates: $(3, 3)$

$$r = \sqrt{(3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$$

$$\tan\theta = 1, \theta = \frac{\pi}{4}$$

Polar coordinates: $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

77. Rectangular coordinates: $\left(-\sqrt{5}, \sqrt{5}\right)$

$$r = \sqrt{\left(-\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2} = \sqrt{10}$$

$$\tan\theta = -1, \theta = \frac{3\pi}{4}$$

Polar coordinates: $\left(\sqrt{10}, \frac{3\pi}{4}\right)$

79.
$$\begin{aligned} x^2 + y^2 &= 81 \\ r^2 &= 81 \\ r &= 9 \end{aligned}$$

81.
$$\begin{aligned} x &= 5 \\ r\cos\theta &= 5 \\ r &= \frac{5}{\cos\theta} \\ r &= 5\sec\theta \end{aligned}$$

83.
$$\begin{aligned} xy &= 5 \\ (r\cos\theta)(r\sin\theta) &= 5 \\ r^2 &= \frac{5}{\sin\theta\cos\theta} \\ &= \frac{10}{\sin 2\theta} = 10\csc 2\theta \end{aligned}$$

85.
$$\begin{aligned} r &= 4 \\ r^2 &= 16 \\ x^2 + y^2 &= 16 \end{aligned}$$

87.
$$\begin{aligned} r &= 3\cos\theta \\ r^2 &= 3r\cos\theta \\ x^2 + y^2 &= 3x \end{aligned}$$

89.
$$\begin{aligned} r^2 &= \sin\theta \\ r^3 &= r\sin\theta \\ \left(\pm\sqrt{x^2 + y^2}\right)^3 &= y \\ \left(x^2 + y^2\right)^3 &= y^2 \\ x^2 + y^2 &= y^{2/3} \end{aligned}$$

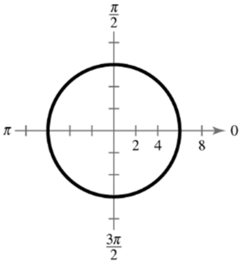
91. $r = 6$

Circle of radius 6 centered at the pole

Symmetric with respect to $\theta = \pi/2$, the polar axis and the pole

Maximum value of $|r| = 6$, for all values of θ

Zeros: None



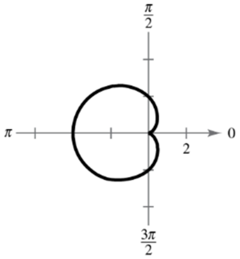
93. $r = -2(1 + \cos\theta)$

Symmetric with respect to the polar axis

Maximum value of $|r| = 4$ when $\theta = 0$

Zeros: $r = 0$ when $\theta = \pi$

$\frac{a}{b} = \frac{2}{2} = 1 \Rightarrow$ Cardioid



95. $r = 4\sin 2\theta$

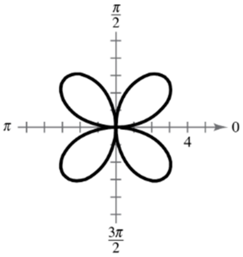
Rose curve $(n = 2)$ with 4 petals

Symmetric with respect to $\theta = \pi/2$, the polar axis, and the pole

Maximum value of $|r| = 4$

when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Zeros: $r = 0$ when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$



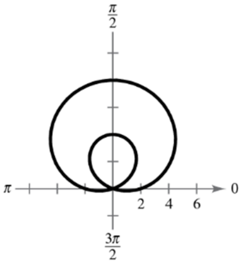
97. $r = 2 + 6\sin\theta$

Limaçon with inner loop

$r = f(\sin\theta) \Rightarrow \theta = \frac{\pi}{2}$ symmetry

Maximum value: $|r| = 8$ when $\theta = \frac{\pi}{2}$

Zeros: $2 + 6\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{3} \Rightarrow \theta \approx 3.4814, 5.9433$



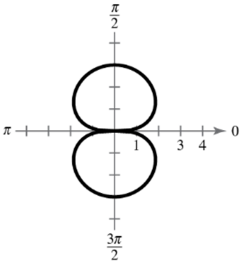
99. $r^2 = 9\sin\theta$

$r = \pm 3\sqrt{\sin\theta}$

Symmetric with respect to polar axis, $\theta = \frac{\pi}{2}$, and the pole.

Maximum value of $|r| = 3$ when $\theta = \frac{\pi}{2}$

Zeros: $r = 0$ when $\theta = 0$ and π

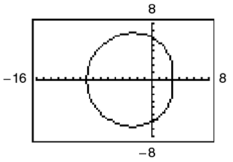


101. $r = 3(2 - \cos\theta)$

$= 6 - 3\cos\theta$

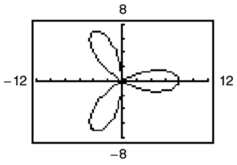
$\frac{a}{b} = \frac{6}{3} = 2$

Convex limaçon



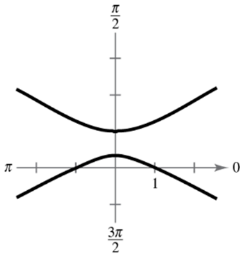
103. $r = 8\cos 3\theta$

Rose curve $(n = 3)$ with three petals



105. $r = \dfrac{1}{1 + 2\sin\theta}, e = 2$

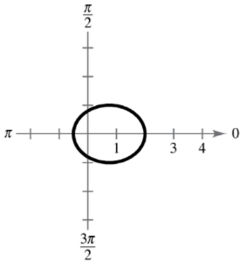
Hyperbola symmetric with respect to $\theta = \dfrac{\pi}{2}$ and having vertices at $\left(\dfrac{1}{3}, \dfrac{\pi}{2}\right)$ and $\left(-1, \dfrac{3\pi}{2}\right)$



107. $r = \dfrac{4}{5 - 3\cos\theta}$

$r = \dfrac{4/5}{1 - (3/5)\cos\theta}, e = \dfrac{3}{5}$

Ellipse symmetric with respect to the polar axis and having vertices at (2, 0) and $(1/2, \pi)$.



109. Parabola: $r = \dfrac{ep}{1 - e\cos\theta}, e = 1$

Vertex: $(2, \pi)$

Focus: $(0, 0) \Rightarrow p = 4$

$r = \dfrac{4}{1 - \cos\theta}$

111. Ellipse: $r = \dfrac{ep}{1 - e\cos\theta}$

Vertices: $(5, 0), (1, \pi) \Rightarrow a = 3$

One focus: $(0, 0) \Rightarrow c = 2$

$e = \dfrac{c}{a} = \dfrac{2}{3}, p = \dfrac{5}{2}$

$r = \dfrac{(2/3)(5/2)}{1 - (2/3)\cos\theta} = \dfrac{5/3}{1 - (2/3)\cos\theta} = \dfrac{5}{3 - 2\cos\theta}$

113.

$$\begin{aligned} a + c &= 122{,}800 + 4000 &&\Rightarrow & a + c &= 126{,}800 \\ a - c &= 119 + 4000 &&\Rightarrow & a - c &= 4{,}119 \\ & && & 2a &= 130{,}919 \\ & && & a &= 65{,}459.5 \\ & && & c &= 61{,}340.5 \end{aligned}$$

$e = \dfrac{c}{a} = \dfrac{61{,}340.5}{65{,}459.5} \approx 0.937$

$r = \dfrac{ep}{1 - e\cos\theta} \approx \dfrac{0.937p}{1 - 0.937\cos\theta}$

$r = 126{,}800$ when $\theta = 0$

$126{,}800 = \dfrac{ep}{1 - e\cos 0}$

$ep = 126{,}800\left(1 - \dfrac{61{,}340.5}{65{,}459.5}\right) \approx 7978.81$

So, $r \approx \dfrac{7978.81}{1 - 0.937\cos\theta}$.

When

$\theta = \dfrac{\pi}{3}, r \approx \dfrac{7978.81}{1 - 0.937\cos(\pi/3)} \approx 15{,}011.87$ miles.

The distance from the surface of Earth and the satellite is $15{,}011.87 - 4000 \approx 11{,}011.87$ miles.

115. False.

$\dfrac{x^2}{4} - y^4 = 1$ is a fourth-degree equation.

The equation of a hyperbola is a second degree equation.

117. False.

$(r, \theta), (r, \theta + 2\pi), (-r, \theta + \pi)$, etc.

All represent the same point.

119. (a) $x^2 + y^2 = 25$

$r = 5$

The graphs are the same. They are both circles centered at $(0, 0)$ with a radius of 5.

(b) $x - y = 0 \Rightarrow y = x$

$\theta = \dfrac{\pi}{4}$

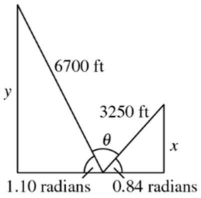
The graphs are the same. They are both lines with slope 1 and intercept $(0, 0)$.

Problem Solving for Chapter 10

1. (a) $\theta = \pi - 1.10 - 0.84 \approx 1.2016$ radians

(b) $\sin 0.84 = \dfrac{x}{3250} \Rightarrow x = 3250 \sin 0.84 \approx 2420$ feet

$\sin 1.10 = \dfrac{y}{6700} \Rightarrow y = 6700 \sin 1.10 \approx 5971$ feet

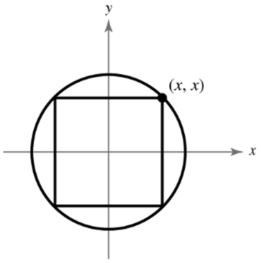


3. Let (x, x) be the corner of the square in Quadrant I.

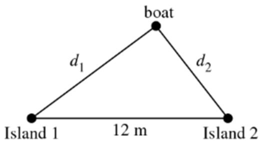
$A = 4x^2$

$\dfrac{x^2}{a^2} + \dfrac{x^2}{b^2} = 1 \Rightarrow x^2 = \dfrac{a^2b^2}{a^2 + b^2}$

So, $A = \dfrac{4a^2b^2}{a^2 + b^2}$.

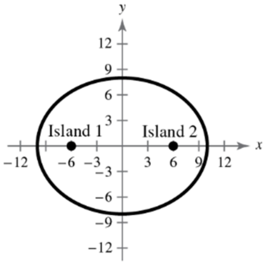


5. (a)



Because $d_1 + d_2 \leq 20$, by definition, the outer bound that the boat can travel is an ellipse. The islands are the foci.

(b)



Island 1 is located at $(-6, 0)$ and Island 2 is located at $(6, 0)$.

(c) $d_1 + d_2 = 2a = 20 \Rightarrow a = 10$

The boat traveled 20 miles.

The vertex is $(10, 0)$.

(d) $c = 6, a = 10 \Rightarrow b^2 = a^2 - c^2 = 64$

$\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$

7. $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Assume that the conic is *not* degenerate.

(a) $A = C, A \neq 0$. Complete the square with respect to x and y, to write the standard equation of a circle.

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

$$\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + \left(y^2 + \frac{E}{A}y + \frac{E^2}{4A^2}\right) = -\frac{F}{A} + \frac{D^2}{4A^2} + \frac{E^2}{4A^2}$$

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}$$

$$(x - h)^2 + (y - k)^2 = r^2$$

This is a circle with center $\left(-\frac{D}{2A}, -\frac{E}{2A}\right)$ and radius $\frac{\sqrt{D^2 + E^2 - 4AF}}{2|A|}$.

(b) $A = 0$ or $C = 0$ (but not both).

Case 1: Let $C = 0$. Complete the square with respect to x to write the standard equation of a parabola with horizontal axis.

$$Ax^2 + Dx + Ey + F = 0$$

$$x^2 + \frac{D}{A}x = -\frac{E}{A}y - \frac{F}{A}$$

$$x^2 + \frac{D}{A}x + \frac{D^2}{4A^2} = -\frac{E}{A}y - \frac{F}{A} + \frac{D^2}{4A^2}$$

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A}\left(y + \frac{F}{E} - \frac{D^2}{4AE}\right)$$

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A}\left(y + \left(\frac{4AF}{E} - \frac{D^2}{4AE}\right)\right)$$

$$(x - h)^2 = 4p(y - k)$$

This is a parabola with vertex $\left(-\frac{D}{2A}, \frac{D^2 - 4AF}{4AE}\right)$.

Case 2: $A = 0$ yields a similar result when you complete the square with respect to y to have a parabola with vertical axis.

(c) $AC > 0 \Rightarrow A$ and C are either both positive or are both negative, if that is the case, move the terms to the other side of the equation so that they are both positive.

Complete the square with respect to x and to y to write the standard equation of an ellipse.

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C}$$

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}$$

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\left(\frac{CD^2 + AE^2 - 4ACF}{4A^2C}\right)} + \frac{\left(y + \frac{E}{2C}\right)^2}{\left(\frac{CD^2 + AE^2 - 4ACF}{4AC^2}\right)} = 1$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Because A and C are both positive, $4A^2C$ and $4AC^2$ are both positive. $CD^2 + AE^2 - 4ACF$ must be positive or the conic is degenerate. So, we have an ellipse with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$. The values of $\frac{CD^2 + AE^2 - 4ACF}{4A^2C}$ and $\frac{CD^2 + AE^2 - 4ACF}{4AC^2}$ will determine if the major axis is vertical or horizontal.

(d) $AC < 0 \Rightarrow A$ and C have opposite signs. Let's assume that A is positive and C is negative.

If A is negative and C is positive, move the terms to the other side of the equation. From part (c) above completing the square with respect to x and to y yields the standard equation of the hyperbola.

$$\frac{\left(x+\frac{D}{2A}\right)^2}{\left(\frac{CD^2+AE^2-4ACF}{4A^2C}\right)}+\frac{\left(y+\frac{E}{2C}\right)^2}{\left(\frac{CD^2+AE^2-4ACF}{4AC^2}\right)}=1$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

Because $A > 0$ and $C < 0$, the first denominator is positive if $CD^2 + AE^2 - 4ACF < 0$ and is negative if $CD^2 + AE^2 - 4ACF > 0$, since $4A^2C$ is negative. Recall in the first sentence we assumed A is positive and C is negative. The second denominator would have the *opposite* sign because $4AC^2 > 0$. So, we have a hyperbola with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

9. At the point $(a, 0)$, the difference of the distances to the foci $(\pm c, 0)$ is $(c + a) - (c - a) = 2a$. Let (x, y) be a point on the hyperbola.

$$\begin{aligned}
2a &= \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\\
2a+\sqrt{(x-c)^2+y^2} &= \sqrt{(x+c)^2+y^2}\\
4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 &= (x+c)^2+y^2\\
4a\sqrt{(x-c)^2+y^2} &= 4cx-4a^2\\
a\sqrt{(x-c)^2+y^2} &= cx-a^2\\
a^2\left(x^2-2cx+c^2+y^2\right) &= c^2x^2-2a^2cx+a^4\\
a^2\left(c^2-a^2\right) &= \left(c^2-a^2\right)x^2-a^2y^2\\
1 &= \frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}
\end{aligned}$$

Thus, $c^2 = a^2 + b^2$.

11. To change the orientation, replace t with $-t$.

$x = \cos(-t) = \cos t$

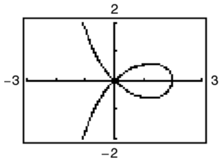
$y = 2\sin(-t) = -2\sin t$

13. (a)

$$\begin{aligned} r &= 2\cos 2\theta \sec\theta \\ r &= \frac{2\left(\cos^2\theta - \sin^2\theta\right)}{\cos\theta} \\ \sqrt{x^2+y^2} &= \frac{2\left[\left(\dfrac{x}{\sqrt{x^2+y^2}}\right)^2 - \left(\dfrac{y}{\sqrt{x^2+y^2}}\right)^2\right]}{\dfrac{x}{\sqrt{x^2+y^2}}} \\ \frac{x}{\sqrt{x^2+y^2}} \cdot \sqrt{x^2+y^2} &= 2\left[\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}\right] \\ x &= 2\left[\frac{x^2-y^2}{x^2+y^2}\right] \\ x\left(x^2+y^2\right) &= 2\left(x^2-y^2\right) \\ x^3 + xy^2 &= 2x^2 - 2y^2 \\ 2y^2 + xy^2 &= 2x^2 - x^3 \\ y^2(2+x) &= x^2(2-x) \\ y^2 &= x^2\left(\frac{2-x}{2+x}\right) \end{aligned}$$

(b) $x = \dfrac{2-2t^2}{1+t^2}$ and $y = \dfrac{t\left(2-2t^2\right)}{1+t^2}$

(c)



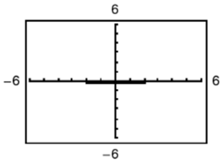
15. $x = (a-b)\cos t + b\cos\left(\dfrac{a-b}{b}t\right)$

$y = (a-b)\sin t - b\sin\left(\dfrac{a-b}{b}t\right)$

(a) $a = 2, b = 1$

$x = \cos t + \cos t = 2\cos t$

$y = \sin t - \sin t = 0$

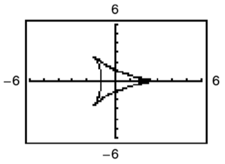


The graph oscillates between −2 and 2 on the x-axis.

(b) $a = 3, b = 1$

$x = 2\cos t + \cos 2t$

$y = 2\sin t - \sin 2t$

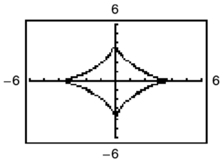


The graph is a three-sided figure with counterclockwise orientation.

(c) $a = 4, b = 1$

$x = 3\cos t + \cos 3t$

$y = 3\sin t - \sin 3t$

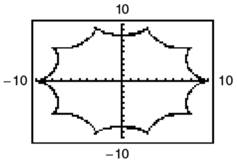


The graph is a four-sided figure with counterclockwise orientation.

(d) $a = 10, b = 1$

$x = 9\cos t + \cos 9t$

$y = 9\sin t - \sin 9t$

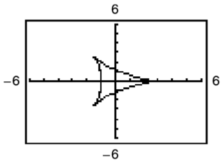


The graph is a ten-sided figure with counterclockwise orientation.

(e) $a = 3, b = 2$

$$x = \cos t + 2\cos\frac{t}{2}$$

$$y = \sin t - 2\sin\frac{t}{2}$$

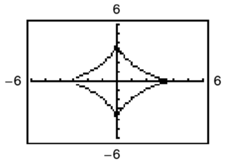


The graph looks the same as the graph in part (b), but is oriented clockwise instead of counterclockwise.

(f) $a = 4, b = 3$

$$x = \cos t + 3\cos\frac{t}{3}$$

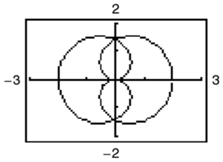
$$y = \sin t - 3\sin\frac{t}{3}$$



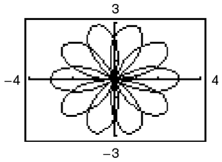
The graph is the same as the graph in part (c), but is oriented clockwise instead of counterclockwise.

17. *Sample answer:*

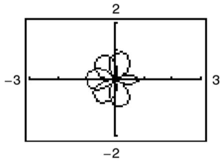
$r = 2\cos\left(\frac{1}{2}\theta\right)$



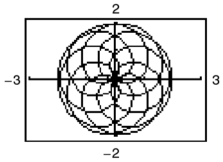
$r = 3\sin\left(\frac{5\theta}{2}\right)$



$r = -\cos\left(\sqrt{2}\theta\right), -2\pi \le \theta \le 2\pi$



$r = -2\sin\left(\frac{4\theta}{7}\right)$



If n is a rational number, then the curve has a finite number of petals. If n is an irrational number, then the curve has an infinite number of petals.

Practice Test for Chapter 10

1. Find the angle, θ, between the lines $3x + 4y = 12$ and $4x - 3y = 12$.

2. Find the distance between the point $(5, -9)$ and the line $3x - 7y = 21$.

3. Find the vertex, focus and directrix of the parabola $x^2 - 6x - 4y + 1 = 0$.

4. Find an equation of the parabola with its vertex at $(2, -5)$ and focus at $(2, -6)$.

5. Find the center, foci, vertices, and eccentricity of the ellipse $x^2 + 4y^2 - 2x + 32y + 61 = 0$.

6. Find an equation of the ellipse with vertices $(0, \pm 6)$ and eccentricity $e = \frac{1}{2}$.

7. Find the center, vertices, foci, and asymptotes of the hyperbola $16y^2 - x^2 - 6x - 128y + 231 = 0$.

8. Find an equation of the hyperbola with vertices at $(\pm 3, 2)$ and foci at $(\pm 5, 2)$.

9. Rotate the axes to eliminate the xy-term. Sketch the graph of the resulting equation, showing both sets of axes.
$5x^2 + 2xy + 5y^2 - 10 = 0$

10. Use the discriminant to determine whether the graph of the equation is a parabola, ellipse, or hyperbola.
(a) $6x^2 - 2xy + y^2 = 0$
(b) $x^2 + 4xy + 4y^2 - x - y + 17 = 0$

11. Convert the polar point $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ to rectangular coordinates.

12. Convert the rectangular point $\left(\sqrt{3}, -1\right)$ to polar coordinates.

13. Convert the rectangular equation $4x - 3y = 12$ to polar form.

14. Convert the polar equation $r = 5 \cos \theta$ to rectangular form.

15. Sketch the graph of $r = 1 - \cos \theta$.

16. Sketch the graph of $r = 5 \sin 2\theta$.

17. Sketch the graph of $r = \dfrac{3}{6 - \cos \theta}$.

18. Find a polar equation of the parabola with its vertex at $\left(6, \frac{\pi}{2}\right)$ and focus at $(0, 0)$.

For Exercises 19 and 20, eliminate the parameter and write the corresponding rectangular equation.

19. $x = 3 - 2 \sin \theta,\ y = 1 + 5 \cos \theta$

20. $x = e^{2t},\ y = e^{4t}$

CHAPTER 11

Analytic Geometry in Three Dimensions

Section 11.1 The Three-Dimensional Coordinate System **510**

Section 11.2 Vectors in Space **513**

Section 11.3 The Cross Product of Two Vectors **515**

Section 11.4 Lines and Planes in Space **519**

Review Exercises **523**

Problem Solving **528**

Practice Test **532**

C H A P T E R 1 1
Analytic Geometry in Three Dimensions

Section 11.1 The Three-Dimensional Coordinate System

1. three-dimensional

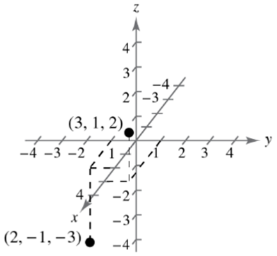
3. surface; space

5. The coordinate planes are the xy-plane, the xz-plane, and the yz-plane.

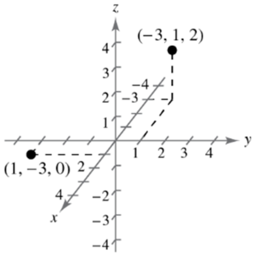
7. The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the Distance Formula in Space:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

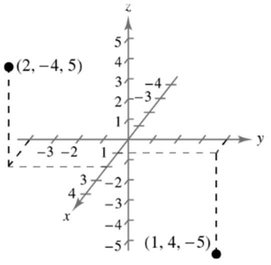
9.



11.



13.



15. $x = -2, y = 4, z = 3 \Rightarrow (-2, 4, 3)$

17. $x = 0, y = -3, z = 5 \Rightarrow (0, -3, 5)$

19. Octant IV

21. Octants II, IV, VI, and VIII

23.
$$\begin{aligned} d &= \sqrt{(4-0)^2 + (1-0)^2 + (6-2)^2} \\ &= \sqrt{16 + 1 + 16} \\ &= \sqrt{33} \text{ units} \end{aligned}$$

25.
$$\begin{aligned} d &= \sqrt{(7-3)^2 + (4-2)^2 + (8-5)^2} \\ &= \sqrt{4^2 + 2^2 + 3^2} \\ &= \sqrt{16 + 4 + 9} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

27.
$$\begin{aligned} d &= \sqrt{(1-0)^2 + [0-(-3)]^2 + (-10-0)^2} \\ &= \sqrt{1 + 9 + 100} \\ &= \sqrt{110} \text{ units} \end{aligned}$$

29.
$$\begin{aligned} d &= \sqrt{(-2-6)^2 + [-1-(-9)]^2 + (5-1)^2} \\ &= \sqrt{64 + 64 + 16} \\ &= \sqrt{144} \\ &= 12 \text{ units} \end{aligned}$$

31.
$$\begin{aligned} d &= \sqrt{[6-(-1)]^2 + [0-4]^2 + [-9-(-2)]^2} \\ &= \sqrt{7^2 + 4^2 + 7^2} \\ &= \sqrt{49 + 16 + 49} \\ &= \sqrt{114} \text{ units} \end{aligned}$$

33.
$$\begin{aligned} d_1 &= \sqrt{(0-0)^2 + (0-4)^2 + (2-0)^2} = \sqrt{20} = 2\sqrt{5} \\ d_2 &= \sqrt{(0-(-2))^2 + (0-5)^2 + (2-2)^2} = \sqrt{29} \\ d_3 &= \sqrt{(-2-0)^2 + (5-4)^2 + (2-0)^2} = 3 \\ d_1^2 + d_3^2 &= 20 + 9 = 29 = d_2^2 \end{aligned}$$

35. $d_1 = \sqrt{(5-1)^2 + (-1+3)^2 + (2+2)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$

$d_2 = \sqrt{(5+1)^2 + (-1-1)^2 + (2-2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$d_3 = \sqrt{(-1-1)^2 + (1+3)^2 + (2+2)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$d_1 = d_3$

Isosceles triangle

37. $d_1 = \sqrt{(8-4)^2 + [1-(-1)]^2 + [2-(-2)]^2} = \sqrt{16+4+16} = \sqrt{36} = 6$

$d_2 = \sqrt{(2-8)^2 + (3-1)^2 + (2-2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$d_3 = \sqrt{(2-4)^2 + [3-(-1)]^2 + [2-(-2)]^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$d_1 = d_3$

Isosceles triangle

39. Midpoint: $\left(\frac{3+2}{2}, \frac{0+(-6)}{2}, \frac{1+3}{2}\right) = \left(\frac{5}{2}, -3, 2\right)$

41. Midpoint: $\left(\frac{3-3}{2}, \frac{-6+4}{2}, \frac{10+4}{2}\right) = (0, -1, 7)$

43. Midpoint: $\left(\frac{-5+6}{2}, \frac{-2+3}{2}, \frac{5-7}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, -1\right)$

45. Midpoint: $\left(\frac{-2+7}{2}, \frac{8-4}{2}, \frac{10+2}{2}\right) = \left(\frac{5}{2}, 2, 6\right)$

47. $(x-3)^2 + (y-2)^2 + (z-4)^2 = 16$

49. Center: $\left(\frac{3+0}{2}, \frac{0+0}{2}, \frac{0+6}{2}\right) = \left(\frac{3}{2}, 0, 3\right)$

Radius: $\sqrt{\left(3-\frac{3}{2}\right)^2 + (0-0)^2 + (0-3)^2} = \sqrt{\frac{9}{4}+9} = \sqrt{\frac{45}{4}}$

Sphere: $\left(x-\frac{3}{2}\right)^2 + y^2 + (z-3)^2 = \frac{45}{4}$

51. $(x^2 - 6x + 9) + y^2 + z^2 = 9$

$(x-3)^2 + y^2 + z^2 = 9$

Center: $(3, 0, 0)$

Radius: 3

53. $(x^2 - 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$

$(x-2)^2 + (y+1)^2 + (z-3)^2 = 4$

Center: $(2, -1, 3)$

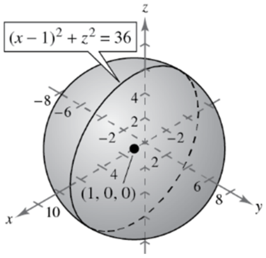
Radius: 2

55.
$$9x^2 - 6x + 9y^2 + 18y + 9z^2 = -1$$
$$x^2 - \tfrac{2}{3}x + \tfrac{1}{9} + y^2 + 2y + 1 + z^2 = -\tfrac{1}{9} + \tfrac{1}{9} + 1$$
$$\left(x - \tfrac{1}{3}\right)^2 + \left(y + 1\right)^2 + z^2 = 1$$

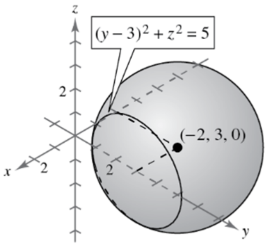
Center: $\left(\frac{1}{3}, -1, 0\right)$

Radius: 1

57.

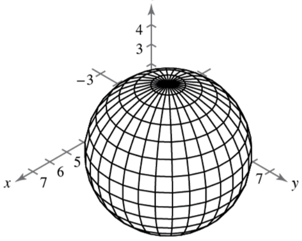


59.



61. $4x^2 + 4y^2 + 4z^2 - 8x - 16y + 8x - 25 = 0$
$$x^2 + y^2 + z^2 - 2x - 4y + 2z = \frac{25}{4}$$
$$x^2 - 2x + y^2 - 4y + \left(z^2 + 2z + 1\right) = \frac{25}{4} + 1$$
$$x^2 - 2x + y^2 - 4y + \left(z + 1\right)^2 = \frac{29}{4}$$

$$z_1 = -1 + \sqrt{\frac{29}{4} - x^2 + 2x - y^2 + 4y}$$
$$z_2 = 1 - \sqrt{\frac{29}{4} - x^2 + 2x - y^2 + 4y}$$



63. $d = 205 \Rightarrow r = \dfrac{205}{2}$
$$x^2 + y^2 + z^2 = \frac{42{,}025}{4}$$

65. False. z is the directed distance from the xy-plane to P.

67. In the xy-plane, the z-coordinate is 0.
In the xz-plane, the y-coordinate is 0.
In the yz-plane, the x-coordinate is 0.

69. $x_m = \dfrac{x_2 + x_1}{2} \Rightarrow x_2 = 2x_m - x_1$

Similarly for y_2 and z_2,

$\left(x_2, y_2, z_2\right) = \left(2x_m - x_1, 2y_m - y_1, 2z_m - z_1\right)$.

71. $\mathbf{v} = -6\mathbf{i} + 8\mathbf{j}$; $\mathbf{v}$ lies in Quadrant II.

Magnitude: $\sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

Direction angle: $\tan\theta = \dfrac{8}{(-6)} = -\dfrac{4}{3} \Rightarrow \theta \approx 126.87^\circ$

73. $\mathbf{v} = 2\mathbf{i} + \sqrt{12}\mathbf{j}$; $\mathbf{v}$ lies in Quadrant I.

Magnitude: $\sqrt{2^2 + \left(\sqrt{12}\right)^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

Direction angle: $\tan\theta = \dfrac{\sqrt{12}}{2} = \sqrt{3} \Rightarrow \theta = 60^\circ$

75.
$$\begin{aligned}(3\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= \left(3\langle 4, -3\rangle \cdot \langle 5, -1\rangle\right)\langle 1, 2\rangle \\ &= \left(\langle 12, -9\rangle \cdot \langle 5, -1\rangle\right)\langle 1, 2\rangle \\ &= (60 + 9)\langle 1, 2\rangle \\ &= 69\langle 1, 2\rangle \\ &= \langle 69, 138\rangle\end{aligned}$$

This is a vector.

77.
$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \left(\langle 4, -3\rangle + \langle 5, -1\rangle\right) \cdot \langle 1, 2\rangle \\ &= \langle 9, -4\rangle \cdot \langle 1, 2\rangle \\ &= 9 - 8 = 1\end{aligned}$$

This is a scalar.

Section 11.2 Vectors in Space

1. zero

3. component form

5. The magnitude (or length) of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

7. $\mathbf{v} = \langle 5-5, 3-(-3), 1-1\rangle = \langle 0, 6, 0\rangle$

9. $\mathbf{v} = \langle 2-(-5), -1-4, 4-(-3)\rangle = \langle 7, -5, 7\rangle$

11.
$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{2^2 + 5^2 + 4^2}\\ &= \sqrt{4+25+16}\\ &= \sqrt{45}\\ &= 3\sqrt{5}\end{aligned}$$

13.
$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{1^2 + 3^2 + (-1)^2}\\ &= \sqrt{11}\end{aligned}$$

15.
$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{4^2 + (-3)^2 + (-7)^2}\\ &= \sqrt{16+9+49}\\ &= \sqrt{74}\end{aligned}$$

17. $\mathbf{v} = \langle 1-1, 0-(-3), -1-4\rangle = \langle 0, 3, -5\rangle$
$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{0 + 3^2 + (-5)^2}\\ &= \sqrt{34}\end{aligned}$$

19. (a) $\|\mathbf{v}\| = \sqrt{4+0+25} = \sqrt{29}$
$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|} &= \frac{\langle 2, 0, 5\rangle}{\sqrt{29}}\\ &= \frac{1}{\sqrt{29}}\langle 2, 0, 5\rangle\\ &= \frac{\sqrt{29}}{29}\langle 2, 0, 5\rangle\end{aligned}$$

(b) $-\mathbf{v} = -\frac{\sqrt{29}}{29}\langle 2, 0, 5\rangle$

21. (a)
$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|} &= \frac{\langle 8, 3, -1\rangle}{\sqrt{74}}\\ &= \frac{1}{\sqrt{74}}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k})\\ &= \frac{\sqrt{74}}{74}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k})\end{aligned}$$

(b) $-\mathbf{v} = -\frac{1}{\sqrt{74}}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -\frac{\sqrt{74}}{74}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

23.
$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 4, 4, -1\rangle \cdot \langle 2, -5, -8\rangle\\ &= 4(2) + 4(-5) + (-1)(-8)\\ &= 8 - 20 + 8 = -4\end{aligned}$$

25.
$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 2, -5, 3\rangle \cdot \langle 9, 3, -1\rangle\\ &= 2(9) + (-5)(3) + 3(-1)\\ &= 18 - 15 - 3 = 0\end{aligned}$$

27. $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-8}{\sqrt{8}\sqrt{25}} \Rightarrow \theta \approx 124.45^\circ$

29. $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-120}{\sqrt{1700}\sqrt{73}} \Rightarrow \theta \approx 109.92^\circ$

31. $-\frac{3}{2}\langle 8, -4, -10\rangle = \langle -12, 6, 15\rangle \Rightarrow$ parallel

33. $\mathbf{u}\cdot\mathbf{v} = -2 + 3 - 1 = 0 \Rightarrow$ orthogonal

35. $\mathbf{v} = \langle 7-5, 3-4, -1-1\rangle = \langle 2, -1, -2\rangle$

$\mathbf{u} = \langle 4-7, 5-3, 3-(-1)\rangle = \langle -3, 2, 4\rangle$

Because **u** and **v** are not parallel, the points are not collinear.

37. $\mathbf{v} = \langle -1-1, 2-3, 5-2\rangle = \langle -2, -1, 3\rangle$

$\mathbf{u} = \langle 3-(-1), 4-2, -1-5\rangle = \langle 4, 2, -6\rangle$

Because $\mathbf{u} = -2\mathbf{v}$, the points are collinear.

39. $\mathbf{v} = \langle 2, -4, 7\rangle = \langle q_1 - 1, q_2 - 5, q_3 - 0\rangle$
$$\left.\begin{aligned} 2 &= q_1 - 1\\ -4 &= q_2 - 5\\ 7 &= q_3\end{aligned}\right\} \Rightarrow \left.\begin{aligned} q_1 &= 3\\ q_2 &= 1\\ q_3 &= 7\end{aligned}\right\}$$

Terminal point is $(3, 1, 7)$.

41. $c\mathbf{u} = c\mathbf{i} + 2c\mathbf{j} + 3c\mathbf{k}$

$\|c\mathbf{u}\| = \sqrt{c^2 + 4c^2 + 9c^2} = |c|\sqrt{14} = 3$

$\Rightarrow c = \pm\frac{3}{\sqrt{14}} = \frac{\pm 3\sqrt{14}}{\sqrt{14}}$

43. $\mathbf{v} = \langle q_1, q_2, q_3\rangle$

Because **v** lies in the *yz*-plane, $q_1 = 0$. Because **v** makes an angle of 45°, $q_2 = q_3$. Finally, $\|\mathbf{v}\| = 4$ implies that $q_2^2 + q_3^2 = 16$. So, $q_2 = q_3 = 2\sqrt{2}$ and $\mathbf{v} = \langle 0, 2\sqrt{2}, 2\sqrt{2}\rangle$, or $q_2 = 2\sqrt{2}$ and $q_3 = -2\sqrt{2}$ and $\mathbf{v} = \langle 0, 2\sqrt{2}, -2\sqrt{2}\rangle$.

45. $\overrightarrow{AB} = \langle 0, 70, 115\rangle. F_B = C_1\langle 0, 70, 115\rangle$

$\overrightarrow{AC} = \langle -60, 0, 115\rangle. F_C = C_2\langle -60, 0, 115\rangle$

$\overrightarrow{AD} = \langle 45, -65, 115\rangle. F_D = C_3\langle 45, -65, 115\rangle$

$F_B + F_C + F_D = \langle 0, 0, -500\rangle.$ So

$$\begin{aligned} -60C_2 + 45C_3 &= 0 \\ 70C_1 \qquad - 65C_3 &= 0 \\ 115C_1 + 115C_2 + 115C_3 &= -500 \end{aligned}$$

Solving this system yields $C_1 = \frac{-104}{69}, C_2 = \frac{-28}{23}, C_3 = \frac{-112}{69}.$

So,

$\|F_B\| \approx 202.92$ N

$\|F_C\| \approx 157.91$ N

$\|F_D\| \approx 226.52$ N.

47. True. $\cos\theta = 0 \Rightarrow \theta = 90°$

49. Because $-3\mathbf{u} = -3\langle 2, -2, 4\rangle = \langle -6, 6, -12\rangle = \mathbf{v}$, $\mathbf{u}$, and $\mathbf{v}$ are parallel. So, the points are collinear.

51. If $\mathbf{u}\cdot\mathbf{v} < 0$, then $\cos\theta < 0$ and the angle between **u** and **v** is obtuse, $180° > \theta > 90°$.

53. Expand along Row 1.

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 6 & -3 \\ -7 & 5 & -1 \end{vmatrix} = 1\begin{vmatrix} 6 & -3 \\ 5 & -1 \end{vmatrix} - 0\begin{vmatrix} -2 & -3 \\ -7 & -1 \end{vmatrix} + 0\begin{vmatrix} -2 & 6 \\ -7 & 5 \end{vmatrix} = -6 + 15 = 9$$

55. Expand along Row 2.

$$\begin{aligned}\begin{vmatrix} 3 & -4 & 5 \\ -1 & 0 & 6 \\ 4 & -9 & 10 \end{vmatrix} &= -(-1)\begin{vmatrix} -4 & 5 \\ -9 & 10 \end{vmatrix} + 0\begin{vmatrix} 3 & 5 \\ 4 & 10 \end{vmatrix} - 6\begin{vmatrix} 3 & -4 \\ 4 & -9 \end{vmatrix} \\ &= (-40 + 45) - 6(-27 + 16)\end{aligned}$$

57. Vertices: $(-1, 0), (3, -2), (-3, -4)$

$$\begin{aligned}\text{Area} &= \pm\frac{1}{2}\begin{vmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \\ -3 & -4 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(-1\begin{vmatrix} -2 & 1 \\ -4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ -3 & -4 \end{vmatrix}\right) \\ &= \pm\frac{1}{2}[-1(-2 + 4) + (-12 - 6)] \\ &= \pm\frac{1}{2}(-2 - 18) \\ &= \pm\frac{1}{2}(-20) \\ &= 10 \text{ square units}\end{aligned}$$

59. Vertices: $(5, 2), (-5, 2), (-5, -3)$

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} 5 & 2 & 1 \\ -5 & 2 & 1 \\ -5 & -3 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(5\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} -5 & 1 \\ -5 & 1 \end{vmatrix} + 1\begin{vmatrix} -5 & 2 \\ -5 & -3 \end{vmatrix}\right)$$

$$= \pm\frac{1}{2}\big[5(2 + 3) - 2(-5 + 5) + 1(15 + 10)\big]$$

$$= \pm\frac{1}{2}(25 + 25)$$

$$= \pm\frac{1}{2}(50) = 25 \text{ square units}$$

61. The area of the parallelogram with vertices: $(0, 0), (5, 0), (1, 2), (6, 2) \Rightarrow a = 5, b = 0, c = 1, d = 2.$

$$A = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\text{Area} = \left|\det(A)\right| = \left|5(2) - 0\right| = \left|10\right| = 10 \text{ square units}$$

63. The area of the parallelogram with vertices: $(0, 0), (-1, -1), (-2, 0), (-3, -1) \Rightarrow a = -1, b = -1, c = -2, d = 0.$

$$A = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix}$$

$$\text{Area} = \left|\det(A)\right| = \left|0 - (-2)(-1)\right| = \left|-2\right| = 2 \text{ square units}$$

Section 11.3 The Cross Product of Two Vectors

1. cross product

3. If $\mathbf{u} \times \mathbf{v} = \mathbf{o}$, then $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

5. $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

7. $$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0$$

9. $2\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$$(2\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 4 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & 4 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -2\mathbf{i} + 14\mathbf{j} + 10\mathbf{k}$$

11. $$\mathbf{u} \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}\mathbf{k} = \mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$$

13. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$

$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (3)(-1) + (-1)(7) + (2)(5) = 0$

15. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle 1, 1, 1 \rangle \cdot \langle 1, -1, 0 \rangle = 0$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle 1, 1, 1 \rangle \cdot \langle 0, 1, -1 \rangle = 0$

17. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle -2, 3, 3 \rangle \cdot \langle 3, -2, 4 \rangle = (-2)(3) + (3)(-2) + (3)(4) = 0$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle -2, 3, 3 \rangle \cdot \langle 0, 1, -1 \rangle = (-2)(0) + (3)(1) + (3)(-1) = 0$

19. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ -1 & 3 & -2 \end{vmatrix} = -17\mathbf{i} + \mathbf{j} + 10\mathbf{k}$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle -17, 1, 10 \rangle \cdot \langle 2, 4, 3 \rangle = 0$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle -17, 1, 10 \rangle \cdot \langle -1, 3, -2 \rangle = 0$

21. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \langle -7, 13, 16 \rangle = -7\mathbf{i} + 13\mathbf{j} + 16\mathbf{k}$

23. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \langle -1, -2, -1 \rangle = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

25. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{3}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} - \dfrac{2}{3}\mathbf{k}$

27. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{19}$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{\sqrt{19}}(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= \dfrac{\sqrt{19}}{19}(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

29. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j}$

$\|\mathbf{u} \times \mathbf{v}\| = 2\sqrt{2}$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{j})$

$= \dfrac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

31. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ -8 & 4 & 6 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{vmatrix} = 4\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{16 + 25 + 4} = \sqrt{45} = 3\sqrt{5}$

$\text{Unit vector} = \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{3\sqrt{5}}(4\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = \dfrac{\sqrt{5}}{15}(4\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

33. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & -6 \\ 0 & 4 & 6 \end{vmatrix} = 48\mathbf{i} - 24\mathbf{j} + 16\mathbf{k}$

$\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{48^2 + (-24)^2 + 16^2} = \sqrt{3136} = 56$ square units

35. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{j}$

$\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{j}\| = 1$ square unit

37. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 7 \\ 2 & -4 & 6 \end{vmatrix} = 46\mathbf{i} - 10\mathbf{j} - 22\mathbf{k}$

$\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{46^2 + (-10)^2 + (-22)^2}$

$= \sqrt{2700} = 30\sqrt{3}$ square units

39. (a) $\overrightarrow{AB} = \langle 3 - 2, 1 - (-1), 2 - 4 \rangle = \langle 1, 2, -2 \rangle$ is parallel to

$\overrightarrow{DC} = \langle 0 - (-1), 5 - 3, 6 - 8 \rangle = \langle 1, 2, -2 \rangle.$

$\overrightarrow{AD} = \langle -3, 4, 4 \rangle$ is parallel to $\overrightarrow{BC} = \langle -3, 4, 4 \rangle.$

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -3 & 4 & 4 \end{vmatrix} = \langle 16, 2, 10 \rangle$

$\text{Area} = \left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\| = \sqrt{16^2 + 2^2 + 10^2} = \sqrt{360} = 6\sqrt{10}$ square units

41. (a) $\overrightarrow{AB} = \langle -5, 0, -2 \rangle$ is parallel to $\overrightarrow{CD} = \langle -5, 0, -2 \rangle.$

$\overrightarrow{AC} = \langle 0, 3, -1 \rangle$ is parallel to $\overrightarrow{BD} = \langle 0, 3, -1 \rangle.$

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 2 \\ 0 & 3 & -1 \end{vmatrix} = -6\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}$

$\text{Area} = \left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \sqrt{(-6)^2 + (-5)^2 + (15)^2} = \sqrt{286}$ square units

43. $\mathbf{u} = \langle 1 - 0, 2 - 0, 3 - 0\rangle = \langle 1, 2, 3\rangle$

$\mathbf{v} = \langle 3 - 0, 0 - 0, 0 - 0\rangle = \langle 3, 0, 0\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 0 & 0 \end{vmatrix} = \langle 0, 9, -6\rangle$$

$$\text{Area} = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\sqrt{0 + 9^2 + (-6)^2}$$

$$= \frac{1}{2}\sqrt{117} = \frac{3\sqrt{13}}{2} \text{ square units}$$

45. $\mathbf{u} = \langle -2 - 2, -2 - 3, 0 - (-5)\rangle = \langle -4, -5, 5\rangle$

$\mathbf{v} = \langle 3 - 2, 0 - 3, 6 - (-5)\rangle = \langle 1, -3, 11\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -5 & 5 \\ 1 & -3 & 11 \end{vmatrix} = \langle -40, 49, 17\rangle$$

$$\text{Area} = \tfrac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \tfrac{1}{2}\sqrt{(-40)^2 + 49^2 + 17^2}$$

$$= \tfrac{1}{2}\sqrt{4290} \text{ square units}$$

47. $$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 3 & 4 & 4 \\ 2 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 3(18) - 4(12) + 4(0) = 6$$

49. $$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 2(-1) - 3(1) + 1(7) = 2$$

51. $$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \left|1(1 - 0) - 1(0 - 1) + 0\right| = 2 \text{ cubic units}$$

53. $$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 0 & -2 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= \left|0 - 2(0 + 6) + 2(0)\right| = 12 \text{ cubic units}$$

55. $\mathbf{u} = \langle 4, 0, 0\rangle, \mathbf{v} = \langle 0, -2, 3\rangle, \mathbf{w} = \langle 0, 5, 3\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 5 & 3 \end{vmatrix} = 4(-21) - 0 + 0 = -84$$

$$\text{Volume} = |-84| = 84 \text{ cubic units}$$

57. $\mathbf{V} = \frac{1}{2}(-\cos 40^\circ\,\mathbf{j} - \sin 40^\circ\,\mathbf{k})$

$\mathbf{F} = -p\mathbf{k}$

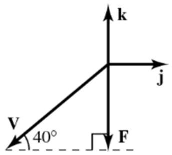
(a) $$\mathbf{V}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{1}{2}\cos 40^\circ & -\frac{1}{2}\sin 40^\circ \\ 0 & 0 & -p \end{vmatrix}$$

$$= \frac{1}{2}p\cos 40^\circ\,\mathbf{i}$$

$$T = \|\mathbf{V}\times\mathbf{F}\| = \frac{p}{2}\cos 40^\circ$$

(b)

p	T
15	5.75
20	7.66
25	9.58
30	11.49
35	13.41
40	15.32
45	17.24



59. True. The cross product is not defined for two-dimensional vectors.

61. $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} \\ &= \begin{vmatrix} \sin\alpha & 0 \\ \sin\beta & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \cos\alpha & 0 \\ \cos\beta & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \cos\alpha & \sin\alpha \\ \cos\beta & \sin\beta \end{vmatrix}\mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + (\cos\alpha\sin\beta - \sin\alpha\cos\beta)\mathbf{k} \\ &= \langle 0, 0, \cos\alpha\sin\beta - \sin\alpha\cos\beta\rangle\end{aligned}$$

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\| &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\beta-\alpha), \text{ where } \alpha > \beta \\ &= \sqrt{(\cos\alpha)^2 + (\sin\alpha)^2}\cdot\sqrt{(\cos\beta)^2 + (\sin\beta)^2}\cdot\sin(\beta-\alpha) \\ &= \sqrt{1}\cdot\sqrt{1}\cdot\sin(\beta-\alpha) \\ &= -\sin(\alpha-\beta)\end{aligned}$$

Because $\mathbf{u}\times\mathbf{v} = \langle 0, 0, \cos\alpha\sin\beta - \sin\alpha\cos\beta\rangle$ and $\|\mathbf{u}\times\mathbf{v}\| = -\sin(\alpha-\beta)$, this implies that

$$-\sin(\alpha, \beta) = \cos\alpha\sin\beta - \sin\alpha\cos\beta$$

$$\Rightarrow \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

63. $(x_1, y_1) = (1, -2)$

$y = 3x - 6 \Rightarrow 3x - y - 6 = 0$

$$d = \frac{|(3)(1) + (-1)(-2) + (-6)|}{\sqrt{3^2 + (-1)^2}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} \approx 0.3162$$

65. $x = -t + 1, y = -3t$

$t = -x + 1$

$y = -3(-x + 1)$

$y = 3x - 3$

67. $x = 4t - 7, y = 8t + 5$

$4t = x + 7$

$t = \frac{1}{4}(x + 7)$

$y = 8\left(\frac{1}{4}\right)(x + 7) + 5$

$y = 2x + 19$

Section 11.4 Lines and Planes in Space

1. direction; $\dfrac{\overline{PQ}}{t}$

3. For two distinct planes is three-space with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. The planes are perpendicular.

5. $x = x_1 + at = 4 + 5t$

$y = y_1 + bt = 1 + 3t$

$z = z_1 + ct = 2 + 6t$

(a) Parametric equations:
$x = 4 + 5t, y = 1 + 3t, z = 2 + 6t$

(b) Symmetric equations: $\dfrac{x-4}{5} = \dfrac{y-1}{3} = \dfrac{z-2}{6}$

7. $x = x_1 + at = 2 + 4t$

$y = y_1 + bt = -5 + 3t$

$z = z_1 + ct = 0 - t$

(a) Parametric equations:
$x = 2 + 4t, y = -5 + 3t, z = -t$

(b) Symmetric equations: $\dfrac{x-2}{4} = \dfrac{y+5}{3} = -z$

9. $x = x_1 + at = 2 + 2t,\ y = y_1 + bt = -3 - 3t,\ z = z_1 + ct = 5 + t$

(a) Parametric equations: $x = 2 + 2t,\ y = -3 - 3t,\ z = 5 + t$

(b) Symmetric equations: $\dfrac{x-2}{2} = \dfrac{y+3}{-3} = z - 5$

11. (a) $\mathbf{v} = \langle 1 - 2, 4 - 0, -3 - 2\rangle = \langle -1, 4, -5\rangle$

Point: $(2, 0, 2)$

$x = 2 - t,\ y = 4t,\ z = 2 - 5t$

(b) $\dfrac{x-2}{-1} = \dfrac{y}{4} = \dfrac{z-2}{-5}$

13. (a) $\mathbf{v} = \langle 1 - (-3), -2 - 8, 16 - 15\rangle = \langle 4, -10, 1\rangle$

Point: $(-3, 8, 15)$

$x = -3 + 4t,\ y = 8 - 10t,\ z = 15 + t$

(b) $\dfrac{x+3}{4} = \dfrac{y-8}{-10} = z - 15$

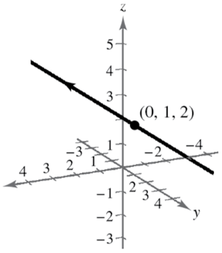
15. $(3, 1, 2), (-1, 1, 5)$

(a) $\mathbf{v} = \langle -1 - 3, 1 - 1, 5 - 2\rangle = \langle -4, 0, 3\rangle$

Parametric: $x = 3 - 4t,\ y = 1,\ z = 2 + 3t$

(b) Because $b = 0$, there are no symmetric equations.

17. $x = 3t,\ y = 1 + t,\ z = 2 + 2t$



19. $$\begin{aligned} -2(x-5) + 1(y-6) - 2(z-3) &= 0 \\ -2x + y - 2z + 10 &= 0 \end{aligned}$$

21. $$\begin{aligned} -2(x-2) + (y-1) + 2(z-3) &= 0 \\ -2x + y + 2z - 3 &= 0 \end{aligned}$$

23. $\mathbf{u} = \langle 2 - 3, 3 - 1, 4 - 2\rangle = \langle -1, 2, 2\rangle$

$\mathbf{v} = \langle -3 - 3, 4 - 1, 2 - 2\rangle = \langle -6, 3, 0\rangle$

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ -6 & 3 & 0 \end{vmatrix} = \langle -6, -12, 9\rangle$$

$$\begin{aligned} \text{Plane: } -6(x-3) - 12(y-1) + 9(z-2) &= 0 \\ -6x - 12y + 9z + 12 &= 0 \\ -2x - 4y + 3z + 4 &= 0 \end{aligned}$$

25. $\mathbf{u} = \langle 3 - 2, 4 - 3, 2 + 2\rangle = \langle 1, 1, 4\rangle$

$\mathbf{v} = \langle 1 - 2, -1 - 3, 0 + 2\rangle = \langle -1, -4, 2\rangle$

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \langle 18, -6, -3\rangle$$

$$\begin{aligned} \text{Plane: } 18(x-2) - 6(y-3) - 3(z+2) &= 0 \\ 18x - 6y - 3z - 24 &= 0 \\ 6x - 2y - z - 8 &= 0 \end{aligned}$$

27. $$\begin{aligned} \mathbf{n} = \mathbf{j}\text{: } 0(x-2) + 1(y-5) + 0(z-3) &= 0 \\ y - 5 &= 0 \end{aligned}$$

29. $\langle -1 - 0, -2 - 2, 0 - 4\rangle = \langle -1, -4, -4\rangle$ and $\langle 1, 0, 0\rangle$ are parallel to plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & -4 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, -4, 4\rangle$$

$$\begin{aligned} 0(x-0) - 4(y-2) + 4(z-4) &= 0 \\ -4y + 4z - 8 &= 0 \\ y - z + 2 &= 0 \end{aligned}$$

31. $\mathbf{n}_1 = \langle 5, -3, 1\rangle, \mathbf{n}_2 = \langle 1, 4, 7\rangle$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = 5 - 12 + 7 = 0$

So, the planes are perpendicular.

33. $\mathbf{n}_1 = \langle 1, -5, -1\rangle$

$\mathbf{n}_2 = \langle 5, -25, -5\rangle = 5\mathbf{n}_1$

So, the planes are parallel.

35. $\mathbf{n}_1 = \langle 3, 0, 2\rangle, \mathbf{n}_2 = \langle 6, -2, -3\rangle$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(6) + (0)(-2) + (2)(-3) = 12$

So, the planes are neither parallel nor perpendicular.

$\|\mathbf{n}_1\| = \sqrt{9 + 0 + 4} = \sqrt{13}$

$\|\mathbf{n}_2\| = \sqrt{36 + 4 + 9} = 7$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{12}{7\sqrt{13}}$$

$\theta \approx 61.6°$

37. $\mathbf{v} = \langle 0, 0, 1\rangle$ and $P = (2, 3, 4)$

$x = 2$

$y = 3$

$z = 4 + t$

39. (a) $\mathbf{n}_1 = \langle 1, 1, -2\rangle, \mathbf{n}_2 = \langle 2, -1, 3\rangle$; normal vectors to planes

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|-5|}{\sqrt{6}\sqrt{14}} = \frac{5}{\sqrt{84}} \Rightarrow \theta \approx 56.9°$$

(b) $x + y - 2z = 0$ Equation 1

$2x - y + 3z = 0$ Equation 2

Add the equations.

$$3x + z = 0$$

$$x = -\frac{1}{3}z$$

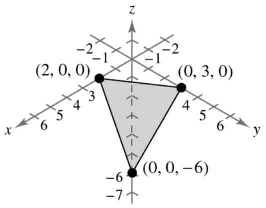
Substituting back into Equation 1 yields the following.

$$-\frac{1}{3}z + y - 2z = 0$$

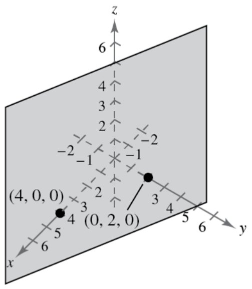
$$y = \frac{7}{3}z$$

Let $t = -\frac{1}{3}z$. $x = t, y = -7t, z = -3t$

41. $3x + 2y - z = 6$



43. $x + 2y = 4$



45. $P = (0, 0, 3)$ on plane, $Q = (1, 3, 4), \mathbf{n} = \langle 4, -5, 2\rangle$

$\overrightarrow{PQ} = \langle 1, 3, 1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle 1, 3, 1\rangle \cdot \langle 4, -5, 2\rangle|}{\sqrt{16 + 25 + 4}} = \frac{|-9|}{\sqrt{45}} = \frac{9}{3\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

47. $P = (2, 0, 0)$ on plane, $Q = (-2, 4, 3), \mathbf{n} = \langle 2, 3, 2\rangle$

$\overrightarrow{PQ} = \langle -4, 4, 3\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -4, 4, 3\rangle \cdot \langle 2, 3, 2\rangle|}{\sqrt{4 + 9 + 4}} = \frac{10}{\sqrt{17}} = \frac{10\sqrt{17}}{17}$$

49. The normal vector to the plane STP: $(0, 0, 0)$, $(-1, -1, 8)$, and $(6, 0, 0)$ is given by $\overline{ST} = \mathbf{v}_1 = \langle -1, -1, 8\rangle$ and $\overline{SP} = \mathbf{v}_2 = \langle 6, 0, 0\rangle$.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 8 \\ 6 & 0 & 0 \end{vmatrix} = \langle 0, -48, 6\rangle$$

$$\mathbf{n}_1 = \langle 0, -8, 1\rangle$$

The normal vector to the plane PQR: $(6, 0, 0)$, $(6, 6, 0)$, and $(7, 7, 8)$ is given by $\overline{PQ} = \mathbf{u}_1 = \langle 0, -6, 0\rangle$ and $\overline{QR} = \mathbf{u}_2 = \langle 1, 1, 8\rangle$.

$$\mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -6 & 0 \\ 1 & 1 & 8 \end{vmatrix} = \langle -48, 0, 6\rangle$$

$$\mathbf{n}_2 = \langle -8, 0, 1\rangle$$

The angle θ between two adjacent sides is given by

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|1|}{\sqrt{65}\sqrt{65}} = \frac{1}{65}$$

$$\theta \approx 89.1^\circ.$$

51. False. Lines that do not intersect and are not in the same plane may not be parallel.

53. The parametric equations should be $x = -2 - 3t$, $y = 5 - 4t$, and $z = 3 + 2t$.

55. $f(x) = x^5$

(a) $f(3) = 3^5 = 243$

(b) $f(-1) = (-1)^5 = -1$

57. $f(x) = \dfrac{x - 5}{x + 3}$

Domain: All real numbers x such that $x \neq -3$

$f(x) \to \infty$ as $x \to -3^-$.

$f(x) \to -\infty$ as $x \to -3^+$.

59. $f(x) = \dfrac{3x^2 + 12}{x^4 - 16} = \dfrac{3(x^2 + 4)}{(x^2 + 4)(x^2 - 4)} = \dfrac{3}{(x + 2)(x - 2)}$

Domain: All real numbers x such that $x \neq \pm 2$

$f(x) \to \infty$ as $x \to -2^-$.

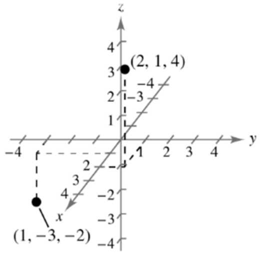
$f(x) \to -\infty$ as $x \to -2^+$.

$f(x) \to \infty$ as $x \to 2^+$.

$f(x) \to -\infty$ as $x \to 2^-$.

Review Exercises for Chapter 11

1.



3. $z = 0, y = 4, x = -5 \Rightarrow (-5, 4, 0)$

5.
$$\begin{aligned} d &= \sqrt{(5-4)^2 + (2-0)^2 + (1-7)^2} \\ &= \sqrt{1+4+36} \\ &= \sqrt{41} \end{aligned}$$

7.
$$\begin{aligned} d &= \sqrt{[2-(-1)]^2 + [3-(-3)]^2 + (-4-0)^2} \\ &= \sqrt{9+36+16} \\ &= \sqrt{61} \end{aligned}$$

9.
$$\begin{aligned} d_1 &= \sqrt{(3-0)^2 + (-2-3)^2 + (0-2)^2} = \sqrt{9+25+4} = \sqrt{38} \\ d_2 &= \sqrt{(0-0)^2 + (5-3)^2 + (-3-2)^2} = \sqrt{4+25} = \sqrt{29} \\ d_3 &= \sqrt{(0-3)^2 + (5-(-2))^2 + (-3-0)^2} = \sqrt{9+49+9} = \sqrt{67} \\ d_1^2 + d_2^2 &= \left(\sqrt{38}\right)^2 + \left(\sqrt{29}\right)^2 = \left(\sqrt{67}\right)^2 \end{aligned}$$

11. Midpoint: $\left(\frac{3+7}{2}, \frac{6+2}{2}, \frac{5+5}{2}\right) = (5, 4, 5)$

13. Midpoint: $\left(\frac{-8+4}{2}, \frac{5+-1}{2}, \frac{2+2}{2}\right) = (-2, 2, 2)$

15.
$$\begin{aligned} (x-h)^2 + (y-k)^2 + (z-j)^2 &= r^2 \\ (x-1)^2 + (y-5)^2 + (z-4)^2 &= 4 \end{aligned}$$

17. Diameter: 14 $\Rightarrow$ radius $= \frac{14}{2} = 7$
$$\begin{aligned} (x-h)^2 + (y-k)^2 + (z-j)^2 &= r^2 \\ (x-1)^2 + (y-(-4))^2 + (z-2)^2 &= 49 \\ (x-1)^2 + (y+4)^2 + (z-2)^2 &= 49 \end{aligned}$$

19. Center: $\left(\frac{-2+2}{2}, \frac{-2+2}{2}, \frac{-2+2}{2}\right) = (0, 0, 0)$

Radius: $\sqrt{(2-0)^2 + (2-0)^2 + (2-0)^2} = \sqrt{4+4+4} = \sqrt{12}$

Sphere: $(x-0)^2 + (y-0)^2 + (z-0)^2 = 12$

$x^2 + y^2 + z^2 = 12$

21.
$$\begin{aligned} x^2 + y^2 + \left(z^2 - 8z + 16\right) &= 16 \\ x^2 + y^2 + (z-4)^2 &= 16 \end{aligned}$$

Center: $(0, 0, 4)$

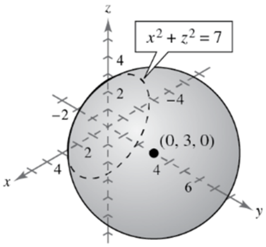
Radius: 4

23.
$$\begin{aligned} \left(x^2 - 10x + 25\right) + \left(y^2 + 6y + 9\right) + \left(z^2 - 4z + 4\right) &= -34 + 25 + 9 + 4 \\ (x-5)^2 + (y+3)^2 + (z-2)^2 &= 4 \end{aligned}$$

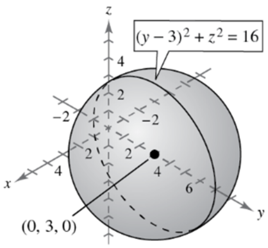
Center: $(5, -3, 2)$

Radius: 2

25. (a) xz-trace $(y = 0)$: $x^2 + z^2 = 7$, circle



(b) yz-trace $(x = 0)$: $(y - 3)^2 + z^2 = 16$, circle



27. Initial point: $(3, -2, 1)$

Terminal point: $(4, 4, 0)$

(a) $\mathbf{v} = \langle 4 - 3, 4 - (-2), 0 - 1\rangle = \langle 1, 6, -1\rangle$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + 6^2 + (-1)^2} = \sqrt{38}$

(c) $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \dfrac{1}{\sqrt{38}}, \dfrac{6}{\sqrt{38}}, \dfrac{-1}{\sqrt{38}}\right\rangle = \left\langle \dfrac{\sqrt{38}}{38}, \dfrac{3\sqrt{38}}{19}, -\dfrac{\sqrt{38}}{38}\right\rangle$

29. Initial point: $(7, -4, 3)$

Terminal point: $(-3, 2, 10)$

(a) $\mathbf{v} = \langle -3 - 7, 2 - (-4), 10 - 3\rangle = \langle -10, 6, 7\rangle$

(b) $\|\mathbf{v}\| = \sqrt{(-10)^2 + (6)^2 + (7)^2} = \sqrt{185}$

(c) $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle -\dfrac{10}{\sqrt{185}}, \dfrac{6}{\sqrt{185}}, \dfrac{7}{\sqrt{185}}\right\rangle = \left\langle -\dfrac{2\sqrt{185}}{37}, \dfrac{6\sqrt{185}}{185}, \dfrac{7\sqrt{185}}{185}\right\rangle$

31. $\mathbf{u} \cdot \mathbf{v} = -1(0) + 4(-6) + 3(5) = -9$

33. $\mathbf{u} \cdot \mathbf{v} = 2(1) - 1(0) + 1(-1) = 1$

35. $\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{|2 - 4 + 0|}{\sqrt{5}\sqrt{18}}$

$= \dfrac{2}{\sqrt{5}\sqrt{18}} \Rightarrow \theta \approx 77.83°$

37. Because $-\frac{2}{3}\langle 39, -12, 21\rangle = \langle -26, 8, -14\rangle$, the vectors are parallel.

39. Because $\mathbf{u} \cdot \mathbf{v} = 30 + 15 - 45 = 0$, the vectors are orthogonal.

41. First two points: $\langle -1, 5, 4\rangle$

Last two points: $\langle 2, -10, -8\rangle$

Because, $\langle 2, -10, -8\rangle = -2\langle -1, 5, 4\rangle$, the 3 points are collinear.

43. First two points: $\mathbf{u} = \langle 3, -1, -2\rangle$

Last two points: $\mathbf{v} = \langle 3, 11, -2\rangle$

Because, $\mathbf{u} \neq c\mathbf{v}$, the points are not collinear.

45. Let **a**, **b**, and **c** be the three force vectors determined by $A(0, 10, 10)$, $B(-4, -6, 10)$ and $C(4, -6, 10)$.

$$\mathbf{a} = \|\mathbf{a}\|\frac{\langle 0, 10, 10\rangle}{10\sqrt{2}} = \|\mathbf{a}\|\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$\mathbf{b} = \|\mathbf{b}\|\frac{\langle -4, -6, 10\rangle}{\sqrt{152}} = \|\mathbf{b}\|\left\langle \frac{-2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right\rangle$$

$$\mathbf{c} = \|\mathbf{c}\|\frac{\langle 4, -6, 10\rangle}{\sqrt{152}} = \|\mathbf{c}\|\left\langle \frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right\rangle$$

Must have $\mathbf{a} + \mathbf{b} + \mathbf{c} = 300\mathbf{k}$. So,

$$\frac{-2}{\sqrt{38}}\|\mathbf{b}\| + \frac{2}{38}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| - \frac{3}{\sqrt{38}}\|\mathbf{b}\| - \frac{3}{\sqrt{38}}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| + \frac{5}{\sqrt{38}}\|\mathbf{b}\| + \frac{5}{\sqrt{38}}\|\mathbf{c}\| = 300.$$

From the first equation $\|\mathbf{b}\| = \|\mathbf{c}\|$. From the second equation, $\frac{1}{\sqrt{2}}\|\mathbf{a}\| = \frac{6}{\sqrt{38}}\|\mathbf{b}\|$.

From the third equation, $\frac{1}{\sqrt{2}}\|\mathbf{a}\| = 300 - \frac{10}{\sqrt{38}}\|\mathbf{b}\|$. So,

$\frac{6}{\sqrt{38}}\|\mathbf{b}\| = 300 - \frac{10}{\sqrt{38}}\|\mathbf{b}\| \Rightarrow \frac{16}{\sqrt{38}}\|\mathbf{b}\| = 300$ and $\|\mathbf{b}\| = \|\mathbf{c}\| = \frac{75\sqrt{38}}{4} \approx 115.6.$

Finally, $\|\mathbf{a}\| = \sqrt{2}\left(\frac{6}{\sqrt{38}}\right)\left(\frac{75\sqrt{38}}{4}\right) = \frac{225\sqrt{2}}{2} \approx 159.1.$

So, the tensions are 159.1 pounds, 115.6 pounds, and 115.6 pounds.

47. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 8 & 2 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 8 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 8 \\ 1 & 1 \end{vmatrix}\mathbf{k} = \langle -10, 0, -10\rangle$

$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-2)(-10) + (8)(0) + (2)(-10) = 0$

$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (1)(-10) + (1)(0) + (-1)(-10) = 0$

49. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}\mathbf{k} = 4\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$

$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (2)(4) + (3)(2) + (2)(-7) = 0$

$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (3)(4) + (1)(2) + (2)(-7) = 0$

51. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$

Unit vector $\frac{1}{\sqrt{11}}\mathbf{i} + \frac{3}{\sqrt{11}}\mathbf{j} + \frac{1}{\sqrt{11}}\mathbf{k} = \frac{\sqrt{11}}{11}\mathbf{i} + \frac{3\sqrt{11}}{11}\mathbf{j} + \frac{\sqrt{11}}{11}\mathbf{k}$

53. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -5 \\ 10 & -15 & 2 \end{vmatrix} = \langle -71, -44, 25 \rangle$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{7602}$

Unit vector: $\frac{1}{\sqrt{7602}}\langle -71, -44, 25 \rangle = -\frac{71\sqrt{7602}}{7602}\mathbf{i} - \frac{22\sqrt{7602}}{3801}\mathbf{j} + \frac{25\sqrt{7602}}{7602}\mathbf{k}$

55. (a) A: $(0, 1, 1)$, B: $(2, -1, 1)$, C: $(5, 1, 4)$, D: $(3, 3, 4)$

$\overrightarrow{AD} = \langle 3, 2, 3 \rangle, \overrightarrow{BC} = \langle 3, 2, 3 \rangle, \overrightarrow{AB} = \langle 2, -2, 0 \rangle, \overrightarrow{CD} = \langle 2, -2, 0 \rangle$

Opposite sides parallel and equal length.

Adjacent sides: $\overrightarrow{AD} = \langle 3, 2, 3 \rangle, \overrightarrow{AB} = \langle 2, -2, 0 \rangle$

$\overrightarrow{\mathbf{AD}} \times \overrightarrow{\mathbf{AB}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 3 \\ 2 & -2 & 0 \end{vmatrix} = \langle 6, 6, -10 \rangle$

(b) Area $= \left|\overrightarrow{\mathbf{AD}} \times \overrightarrow{\mathbf{AB}}\right| = \sqrt{36 + 36 + 100} = \sqrt{172} = 2\sqrt{43}$ square units

57. $\mathbf{u} = \langle 3, 0, 0 \rangle, \mathbf{v} = \langle 2, 0, 5 \rangle, \mathbf{w} = \langle 0, 5, 1 \rangle$

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 0 & 5 \\ 0 & 5 & 1 \end{vmatrix} = -75$

Volume $= |-75| = 75$ cubic units

59. Passes through the point $(0, 0, 0)$ and is parallel to $\mathbf{v} = \langle -2, 5, 1 \rangle$.

(a) Parametric equations: $x = 0 - 2t, y = 0 + 5t, z = 0 + t$

$x = -2t, y = 5t, z = t$

(b) Symmetric equations: $\frac{x}{-2} = \frac{y}{5} = \frac{z}{1}$

61. Passes through the points: $(-1, 3, 5)$ and $(3, 6, -1)$.

$\mathbf{v} = \langle 3 + 1, 6 - 3, -1 - 5 \rangle = \langle 4, 3, -6 \rangle,$

(a) Parametric equations: $x = -1 + 4t, y = 3 + 3t, z = 5 - 6t$

(b) Symmetric equations: $\frac{x + 1}{4} = \frac{y - 3}{3} = \frac{z - 5}{-6}$

63. Passes through $(0, 0, 0)$, $(5, 0, 2)$, and $(2, 3, 8)$.

$$\mathbf{u} = \langle 5, 0, 2\rangle, \mathbf{v} = \langle 2, 3, 8\rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 2 \\ 2 & 3 & 8 \end{vmatrix} = \langle -6, -36, 15\rangle$$

$$\mathbf{n} = \langle 2, 12, -5\rangle$$

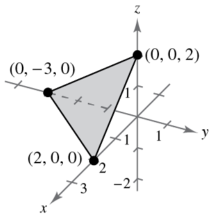
$$\begin{aligned} a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ 2(x - 0) + 12(y - 0) - 5(z - 0) &= 0 \\ 2x + 12y - 5z &= 0 \end{aligned}$$

65. Passes through $(5, 3, 2)$ and is parallel to the xy-plane.

$\mathbf{n} = \mathbf{k}$, normal vector

$$\begin{aligned} \text{Plane: } 0(x - 5) + 0(y - 3) + 1(z - 2) &= 0 \\ z - 2 &= 0 \end{aligned}$$

67. $3x - 2y + 3z = 6$



69. $D = \dfrac{\left|\overrightarrow{PQ} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}$

$Q = (1, 2, 3)$, $P = (2, 0, 0)$ on plane. $\overrightarrow{PQ} = \langle -1, 2, 3\rangle$, $\mathbf{n} = \langle 2, -1, 1\rangle$

$$D = \frac{\left|\langle -1, 2, 3\rangle \cdot \langle 2, -1, 1\rangle\right|}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

71. $D = \dfrac{\left|\overrightarrow{PQ} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}$

$Q = (-1, 3, -5)$, $P = (0, 0, 4)$ on plane. $\overrightarrow{PQ} = \langle -1, 3, -9\rangle$, $\mathbf{n} = \langle 3, -2, 2\rangle$

$$D = \frac{\left|\langle -1, 3, -9\rangle \cdot \langle 3, -2, 2\rangle\right|}{\sqrt{9 + 4 + 4}} = \frac{|-27|}{\sqrt{17}} = \frac{27\sqrt{17}}{17}$$

73. False.

$$c(\mathbf{u} \times \mathbf{v}) \neq c\mathbf{u} \times c\mathbf{v}$$

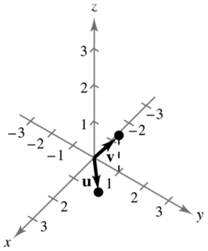
$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

75. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$.

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3\rangle \cdot \left(\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle\right) \\ &= \langle u_1, u_2, u_3\rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle \\ &= \langle u_1(v_1 + w_1), u_2(v_2 + w_2), u_3(v_3 + w_3)\rangle \\ &= \langle u_1v_1 + u_1w_1, u_2v_2 + u_2w_2, u_3v_3 + u_3w_3\rangle \\ &= \langle u_1v_1, u_2v_2, u_3v_3\rangle + \langle u_1w_1, u_2w_2, u_3w_3\rangle \\ &= \langle u_1, u_2, u_3\rangle\langle v_1, v_2, v_3\rangle + \langle u_1, u_2, u_3\rangle\langle w_1, w_2, w_3\rangle \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \end{aligned}$$

Problem Solving for Chapter 11

1. (a)



(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$\mathbf{0} = \langle a, a + b, b\rangle \Rightarrow a = b = 0$

(c) $\mathbf{w} = \langle 1, 2, 1\rangle = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$1 = a$

$2 = a + b$

$1 = b$

So, $a = b = 1$.

(d) $\mathbf{w} = \langle 1, 2, 3\rangle = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$1 = a$

$2 = a + b$

$3 = b$

Impossible

3. The largest angle in a triangle is always opposite the longest side of the triangle. First, determine the lengths of the three sides. Then, once the largest angle has been identified, use the fact that $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, where **u** and **v** are defined to be the vectors that form θ. If $\mathbf{u}\cdot\mathbf{v} = 0$, the angle is a right angle. If $\mathbf{u}\cdot\mathbf{v} > 0$, the angle is acute. If $\mathbf{u}\cdot\mathbf{v} < 0$, the angle is obtuse.

(a) A: $(1, 2, 0)$

B: $(0, 0, 0)$

C: $(-2, 1, 0)$

$d(AB) = \sqrt{5}, d(AC) = \sqrt{10}, d(BC) = \sqrt{5}$

Angle B is largest.

$\overrightarrow{BA} = \langle 1, 2, 0\rangle, \overrightarrow{BC} = \langle -2, 1, 0\rangle$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = 0 \Rightarrow$ The triangle is a right triangle.

(b) A: $(-3, 0, 0)$

B: $(0, 0, 0)$

C: $(1, 2, 3)$

$d(AB) = 3, d(AC) = \sqrt{29}, d(BC) = \sqrt{14}$

Angle B is largest.

$\overrightarrow{BA} = \langle -3, 0, 0\rangle, \overrightarrow{BC} = \langle 1, 2, 3\rangle$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = -3 < 0 \Rightarrow$ The triangle is an obtuse triangle.

(c) A: $(2, -3, 4)$

B: $(0, 1, 2)$

C: $(-1, 2, 0)$

$d(AB) = \sqrt{24}, d(AC) = \sqrt{50}, d(BC) = \sqrt{6}$

Angle B is largest.

$\overrightarrow{BA} = \langle 2, -4, 2\rangle, \overrightarrow{BC} = \langle -1, 1, -2\rangle$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = -10 < 0 \Rightarrow$ The triangle is an obtuse triangle.

(d) $A\colon (2, -7, 3)$

$B\colon (-1, 5, 8)$

$C\colon (4, 6, -1)$

$d(AB) = \sqrt{178}, d(AC) = \sqrt{189}, d(BC) = \sqrt{107}$

Angle B is largest.

$\overrightarrow{BA} = \langle 3, -12, -5\rangle, \overrightarrow{BC} = \langle 5, 1, -9\rangle$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 48 \Rightarrow$ The triangle is an acute triangle.

5. Let A lie on the y-axis and the wall on the x-axis. Then, $A = (0, 10, 0), B = (8, 0, 6), C = (-10, 0, 6)$ and $\overrightarrow{AB} = \langle 8, -10, 6\rangle, \overrightarrow{AC} = \langle -10, -10, 6\rangle$.

$$\left\|\overrightarrow{AB}\right\| = \sqrt{8^2 + (-10)^2 + 6^2} = 10\sqrt{2}$$

$$\left\|\overrightarrow{AC}\right\| = \sqrt{(-10)^2 + (-10)^2 + 6^2} = 2\sqrt{59}$$

$$\mathbf{F}_1 = 420\frac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|} = \frac{420}{10\sqrt{2}}\langle 8, -10, 6\rangle = \frac{84}{\sqrt{2}}\langle 4, -5, 3\rangle$$

$$\mathbf{F}_2 = 650\frac{\overrightarrow{AC}}{\left\|\overrightarrow{AC}\right\|} = \frac{650}{2\sqrt{59}}\langle -10, -10, 6\rangle = \frac{650}{\sqrt{59}}\langle -5, -5, 3\rangle$$

$$\mathbf{F}_1 = \mathbf{F}_1 + \mathbf{F}_2 = \left\langle \frac{(4)(84)}{\sqrt{2}} + \frac{(-5)(650)}{\sqrt{59}}, \frac{(-5)(84)}{\sqrt{2}} + \frac{(-5)(650)}{\sqrt{59}}, \frac{(3)(84)}{\sqrt{2}} + \frac{(3)(650)}{\sqrt{59}} \right\rangle$$

$$\approx \langle -185.526, -720.099, 432.059\rangle$$

$\|\mathbf{F}\| \approx 860.0$ lb

7. (a) $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2\rangle$

(b) $\|\mathbf{u} + \mathbf{v}\| \approx 8.7321$

(c) $\|\mathbf{u}\| = \sqrt{26} \approx 5.0990$

(d) $\|\mathbf{v}\| \approx 9.0139$

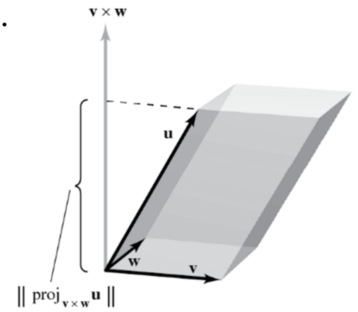
9. $\mathbf{u} = \langle a_1, b_1, c_1\rangle, \mathbf{v} = \langle a_2, b_2, c_2\rangle, \mathbf{w} = \langle a_3, b_3, c_3\rangle$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix}$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \left[b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)\right]\mathbf{i} - \left[a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)\right]\mathbf{j} \\ &\quad + \left[a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)\right]\mathbf{k} \\ &= \left[a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{i} \\ &\quad + \left[b_2(a_1a_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{j} \\ &\quad + \left[c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)\right]\mathbf{k} \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned}$$

11.

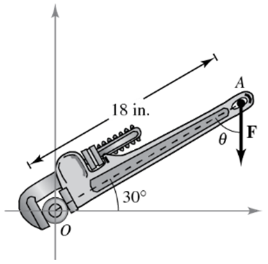


$\|\mathbf{v} \times \mathbf{w}\| =$ area of base and $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| =$ height of parallelepiped

So, the volume is

$$V = (\text{height})(\text{area of base}) = \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|\,\|\mathbf{v} \times \mathbf{w}\| = \left|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|}\right|\|\mathbf{v} \times \mathbf{w}\| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

13.



(a) $O = (0, 0, 0)$

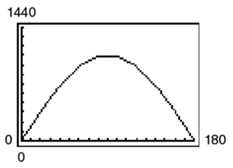
$A = (18 \cos 30°, 18 \sin 30°, 0) = \left(9\sqrt{3}, 9, 0\right)$

$\overrightarrow{OA} = \left\langle 9\sqrt{3}, 9, 0\right\rangle$

$\|\mathbf{M}\| = \|\overrightarrow{OA} \times \mathbf{F}\| = \|\overrightarrow{OA}\|\|\mathbf{F}\| \sin\theta$

$= \left(\sqrt{\left(9\sqrt{3}\right)^2 + 9^2 + 0^2}\right)(60) \sin\theta$

$= 1080 \sin\theta$



(b) $\|\overrightarrow{M}\| = 1080 \sin(45°) = (1080)\left(\frac{\sqrt{2}}{2}\right) = 540\sqrt{2} \approx 763.68$ in-lb

(c) $\|\overrightarrow{M}\| = 1080 \sin\theta$ has its maximum value at $\theta = 90°$. In order to generate the maximum torque, the force should be applied in a direction perpendicular to the wrench handle.

15. The area of the triangle is one-half of the area of any of the 3 parallelograms having the following adjacent sides: **b** and **c**, −**b** and **a**, −**c** and −**a**

So,

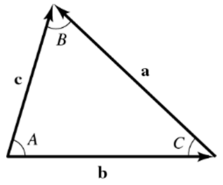
$$\text{Area} = \frac{\|\mathbf{b} \times \mathbf{c}\|}{2} = \frac{\|(-\mathbf{a}) \times (-\mathbf{c})\|}{2} = \frac{\|\mathbf{a} \times (-\mathbf{b})\|}{2}$$

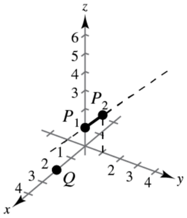
$$\|\mathbf{b} \times \mathbf{c}\| = \|(-\mathbf{a}) \times (-\mathbf{c})\| = \|\mathbf{a} \times (-\mathbf{b})\|$$

$$\|\mathbf{b}\|\|\mathbf{c}\| \sin A = \|\mathbf{a}\|\|\mathbf{c}\| \sin B = \|\mathbf{a}\|\|\mathbf{b}\| \sin C$$

Divide by $\|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|$:

$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|} = \frac{\sin C}{\|\mathbf{c}\|}$$





17. (a) $\mathbf{u} = \langle 0, 1, 1\rangle$ direction vector of line determined by P_1 and P_2

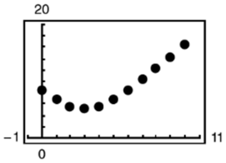
$$D = \frac{\left\|\overrightarrow{P_1Q} \times \mathbf{u}\right\|}{\|\mathbf{u}\|} = \frac{\left\|\langle 2, 0, -1\rangle \times \langle 0, 1, 1\rangle\right\|}{\sqrt{2}}$$

$$= \frac{\left\|\langle 1, -2, 2\rangle\right\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

(b) The shortest distance to the line **segment** is $\left\|\overrightarrow{P_1Q}\right\| = \left\|\langle 2, 0, -1\rangle\right\| = \sqrt{5}$.

19. First insect: $x = 6 + t, y = 8 - t, z = 3 + t$

Second insect: $x = 1 + t, y = 2 + t, z = 2t$



(a) When $t = 0$ the first insect is located at $(6, 8, 3)$ and the second insect is located at $(1, 2, 0)$.

$$d = \sqrt{(1-6)^2 + (2-8)^2 + (0-3)^2} = \sqrt{70} \text{ inches}$$

(b) $$d = \sqrt{\left[(1+t) - (6+t)\right]^2 + \left[(2+t) - (8-t)\right]^2 + \left[2t - (3+t)\right]^2}$$

$$= \sqrt{(-5)^2 + (2t-6)^2 + (t-3)^2}$$

$$= \sqrt{5t^2 - 30t + 70}$$

t	0	1	2	3	4	5	6	7	8	9	10
d	$\sqrt{70}$	$\sqrt{45}$	$\sqrt{30}$	5	$\sqrt{30}$	$\sqrt{45}$	$\sqrt{70}$	$\sqrt{105}$	$\sqrt{150}$	$\sqrt{205}$	$\sqrt{270}$

(c) The distance between the two insects appears to lessen in the first 3 seconds, but then begins to increase with time.

(d) When $t = 3$, the insects get within 5 inches of each other.

Practice Test for Chapter 11

1. Find the lengths of the sides of the triangle with vertices $(0, 0, 0), (1, 2, -4),$ and $(0, -2, -1).$ Show that the triangle is a right triangle.

2. Find the standard form of the equation of a sphere having center $(0, 4, 1)$ and radius 5.

3. Find the center and radius of the sphere $x^2 + y^2 + z^2 + 2x - 4z - 11 = 0.$

4. Find the vector $\mathbf{u} - 3\mathbf{v}$ given $\mathbf{u} = \langle 1, 0, -1 \rangle$ and $\mathbf{v} = \langle 4, 3, -6 \rangle.$

5. Find the length of $\frac{1}{2}\mathbf{v}$ if $\mathbf{v} = \langle 2, 4, -6 \rangle.$

6. Find the dot product of $\mathbf{u} = \langle 2, 1, -3 \rangle$ and $\mathbf{v} = \langle 1, 1, -2 \rangle.$

7. Determine whether $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle -3, -3, 3 \rangle$ are orthogonal, parallel, or neither.

8. Find the cross product of $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, -1, 3 \rangle.$ What is $\mathbf{v} \times \mathbf{u}$?

9. Use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle 0, -1, 1 \rangle,$ and $\mathbf{w} = \langle 1, 0, 4 \rangle.$

10. Find a set of parametric equations for the line through the points $(0, -3, 3)$ and $(2, -3, 4).$

11. Find an equation of the plane passing through $(1, 2, 3)$ and perpendicular to the vector $\mathbf{n} = \langle 1, -1, 0 \rangle.$

12. Find an equation of the plane passing through the three points $A = (0, 0, 0), B = (1, 1, 1),$ and $C = (1, 2, 3).$

13. Determine whether the planes $x + y - z = 12$ and $3x - 4y - z = 9$ are parallel, orthogonal or neither.

14. Find the distance between the point $(1, 1, 1)$ and the plane $x + 2y + z = 6.$